# THE GEOMETRY OF NUMBERS ON A QUADRIC SURFACE

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Abstract. Several problems are treated about minimizing the absolute value of a real ternary quadratic form and of a real ternary cubic form, when restricted to the integer points on a quadric surface. Among the results are an estimate for the minimum that holds for all ternary quadratic forms and, for certain decomposable ternary cubics, a best possible result that supplements a theorem of Davenport.

## 1. Introduction

One basic problem in the classical geometry of numbers is to estimate the minimal absolute value of a real homogeneous polynomial $P$ on $\mathbb{R}^n$ when evaluated at non-zero points of $\mathbb{Z}^n$. Typically, such an estimate is given in terms of $\mathrm{SL}(n, \mathbb{R})$ invariants of the polynomial. A major refinement is to understand the set of minima of all of those polynomials in the $\mathrm{SL}(n, \mathbb{R})$-orbit of $P$. Equivalently, we may fix $P$ and vary the lattice. After the case when $P$ is an indefinite binary quadratic form, which was made famous by Markoff, this set of minima may be called the spectrum of $P$.

In his recent lectures [31], Sarnak explains how these problems are part of a much more general program that studies the "bass note" spectra of certain linear differential operators. These operators act on functions or forms that are defined on the quotient of a non-compact symmetric space by one of a family of discrete groups of motions. The spectrum records the size of the smallest eigenvalue (the bass note) over the family. The spectra in the classical geometry of numbers arise from ordinary lattices in Euclidean space, where $P$ corresponds to a differential operator, for instance $x_1^2 + \cdots + x_n^2$ to the Laplacian.

A different kind of generalization of the classical geometry of numbers occurs when we replace the lattice $\mathbb{Z}^n$ by the set of integer points on an affine homogeneous variety and seek to minimize the absolute value of a polynomial restricted to these integers. This topic is a natural counterpart to that of counting asymptotically the integer points on the variety in a large ball [16]. In one of the simplest interesting cases the integer points are those on the (affine) quadric surface in $\mathbb{R}^3$ determined by

$$J(x, y, z) := y^2 - 4xz = 1. \tag{1.1}$$

Here we replace $\mathrm{SL}(3, \mathbb{R})$ by the orthogonal group $\mathrm{SO}^+(J, \mathbb{R})$. The main goal of this paper is to prove analogues of some well-known results about the minima of quadratic and cubic forms and their spectra, when we restrict them to the integer points on this quadric surface.

In each of the following two sections I will briefly describe the classical results and then state some new ones. Their proofs rely heavily on work of Hermite, Mahler, Davenport and Barnes.

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## 2. Ternary quadratic forms

Recall the problem of minimizing the absolute value of a real ternary quadratic form $Q(x, y, z)$ evaluated at non-zero $(x, y, z) \in \mathbb{Z}^3$. Let the symmetric matrix associated to $Q$ have determinant $D$. A result going back to Gauss [17] asserts that, when $D \neq 0$, there exists a nonzero integral vector $(x, y, z)$ with

$$|Q(x, y, z)| \leq (2|D|)^{\frac{1}{3}}. \tag{2.1}$$

The constant $2^{\frac{1}{3}}$ is best possible in that equality holds for some $(x, y, z)$ when $Q(x, y, z)$ is integrally equivalent to a non-zero multiple of

$$x^2 + y^2 + z^2 + xy + yz + xz.$$

The first new problem I consider is that of minimizing $|Q(x, y, z)|$ for *any* real ternary quadratic form $Q$ when we require that the integral vector $(x, y, z)$ lie on the quadric surface (1.1). What is sought is a bound that holds for all $Q$ and is compatible with the quadric condition, meaning that it is an invariant of $G = \mathrm{SO}^+(J, \mathbb{R})$. The bound will be defined in terms of the roots $\lambda_j$ of the characteristic-type polynomial

$$\chi_Q(t) = \det(Q - tJ) = 4(t - \lambda_1)(t - \lambda_2)(t - \lambda_3), \tag{2.2}$$

which are such invariants.

**Theorem 1.** *Let $Q(x, y, z)$ be any real ternary quadratic form. For some $(x, y, z) \in \mathbb{Z}^3$ with $y^2 - 4xz = 1$ we have*

$$|Q(x, y, z)| \leq \tfrac{1}{3}\Big(\sum_{i \neq j} |\lambda_i - \lambda_j| + |\lambda_1 + \lambda_2 + \lambda_3|\Big).$$

Equality holds for $(x, y, z) = (0, 1, 0)$ when $Q = kJ$ for any fixed $k$. Thus the constant $\frac{1}{3}$ is best possible for a bound of this type. To compare the inequality of Theorem 1 with (2.1), note that $D = -4\lambda_1\lambda_2\lambda_3$ and the RHS of (2.1) can be written as $2|\lambda_1\lambda_2\lambda_3|^{1/3}$. Of course,

$$2|\lambda_1\lambda_2\lambda_3|^{\frac{1}{3}} \leq \tfrac{2}{3}(|\lambda_1| + |\lambda_2| + |\lambda_3|).$$

The proof of Theorem 1 makes use of the Hermite's original method of reduction of a binary quartic form, which allows us to bound effectively the middle coefficient of a reduced form. Most other approaches to reduction theory avoid estimating the middle coefficients.

In the original problem where we minimize over all non-zero integer vectors, if we restrict to indefinite $Q$, then the RHS of (2.1) can be replaced by $(\frac{2}{3}|D|)^{\frac{1}{3}}$, with equality holding for some $(x, y, z)$ if $Q$ is equivalent to a non-zero multiple of

$$x^2 + y^2 - z^2 + xz + yz.$$

This was shown by Markoff [26] and a simple proof was given by Davenport [12]. To go further, for any nonsingular indefinite $Q$ and $g \in \mathrm{SL}(n, \mathbb{R})$ set

$$\mu(g) = \inf_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ (x,y,z)\neq 0}} |Q((x, y, z)g)|.$$

Note that we can write any indefinite ternary with the same determinant and signature as $Q$ in the form $Q((x, y, z)g)$. Define the spectrum to be

$$\sigma = \{\mu(g);\ g \in \mathrm{SL}(n, \mathbb{R})\}.$$

The result of Markoff and Davenport implies that

$$\sup \sigma = (\tfrac{2}{3}|D|)^{\frac{1}{3}} \in \sigma.$$

If we normalize so that $|D| = 1$, the four largest elements of $\sigma$ are given by (see [15, Thm. 84 p.112] and [34] for seven more):

$$\{\dots, (\tfrac{8}{25})^{\frac{1}{3}}, (\tfrac{1}{3})^{\frac{1}{3}}, (\tfrac{2}{5})^{\frac{1}{3}}, (\tfrac{2}{3})^{\frac{1}{3}}\}.$$

After the work of many people, culminating in the proof by Margulis [25] of the Oppenheim conjecture, it is known that 0 is the only limit point of the spectrum. In the terminology of [31], the spectrum is rigid. Also, the forms with positive minima come from classes with rational representatives.[1]

To define the spectrum of minima for a *fixed* $Q$ over the quadric surface, set for $g \in G = \mathrm{SO}^+(J, \mathbb{R})$

$$\mu_J(g) = \inf_{\substack{(x,y,z)\in\mathbb{Z}^3\\ J(x,y,z)=1}} |Q((x,y,z)g)| \quad \text{and let } \sigma_J = \{\mu_J(g);\ g \in G\}.$$

It follows from Theorem 1 that $\sup \sigma_J < \infty$. Since the integer points on our quadric are not a lattice we cannot immediately deduce this from Minkowski's well-known existence theorem. It is also not obvious that the supremum of elements in $\sigma_J$ is actually attained. The following result on this is, in this special situation, similar in form to Mahler's main theorem on the existence of critical lattices in the classical geometry of numbers [24].

**Theorem 2.** *For a fixed real ternary quadratic form $Q$ we have that* $\sup \sigma_J \in \sigma_J$.

For a general $Q$ we do not have an estimate beyond that implied by Theorem 1 for $\sup \sigma_J$. For some singular $Q$ we can say more. If $Q$ is singular it is known that it is decomposable:

$$Q(x, y, z) = (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z),$$

where in general the coefficients are complex. The set of all such $Q$ where $a_j, b_j$ are real and satisfy $b_1^2 = a_1c_1$ and $b_2^2 = a_2c_2$ is closed under the usual action of $G$. For such $Q$, when normalized, we can deduce from a result of Barnes [2] the following.

**Theorem 3.** *Suppose that $\alpha, \beta$ are unequal real numbers with $|\alpha - \beta| = 1$. For*

$$Q(x, y, z) = (x + \alpha y + \alpha^2 z)(x + \beta y + \beta^2 z) \tag{2.3}$$

*we have that* $\sup \sigma_J = \frac{1}{5}$. *The spectrum has a limit point at*

$$\tfrac{-18+78\sqrt{10}}{1681} = 0.136025\dots$$

*arising as the limit of the spectrum above it, which is an explicit sequence of rationals* $\{\dots, \frac{1333}{9797}, \frac{3}{20}, \frac{1}{5}\}$.

Theorem 1 implies that $\sup \sigma_J \le \frac{1}{2}$ for these $Q$. For results on singular forms in the corresponding classical problem see [28].

## 3. Ternary cubic forms

A second set of problems concerns minima of ternary cubic forms over our quadric surface. First recall the classical case. Suppose that a real ternary cubic form is equivalent under $\mathrm{SL}(3, \mathbb{R})$ to one of the form

$$K(x, y, z) = n(x^3 + y^3 + z^3) + 6mxyz.$$

[1]Some remarkable results about the structure of this spectrum and related issues have been obtained recently by Gamburd, Ghosh, Sarnak and Whang [18].

This includes all nonsingular forms when $n \neq 0$ as well as decomposable ones, where either $n = 0$ or $n^3 + 8m^2 = 0$. Not much is known about the spectrum $\sigma$ of minima

$$\mu(g) = \inf_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ (x,y,z)\neq 0}} |K((x,y,z)g)|$$

in the non-singular case other than the fact that it contains its supremum, and by Minkowski we have that

$$\sup \sigma \leq \tfrac{1}{\Gamma^3(\frac{4}{3})}(|n| + 2|m|).$$

There are improvements of this estimate by Mordell [27], and by Davenport [9] when $m = 0$ but even there the value of the supremum is not known and the spectrum is mysterious.

To treat the analogous problem on the quadric we need to define some invariants and a covariant for a general ternary cubic form transforming by $g \in G = \mathrm{SO}^+(J, \mathbb{R})$. Write[2]

$$K(x,y,z) = ax^3 + by^3 + cz^3 + 3dx^2y + 3ey^2z + 3fz^2x + 3kxy^2 + 3hyz^2 + 3izx^2 + 6jxyz.$$

A linear covariant is given by

$$L_K(x,y,z) = (k-i)x + (b-j)y + (e-f)z,$$

meaning that for $g \in G$ we have

$$L_K((x,y,z)g) = L_{K'}(x,y,z) \quad \text{where} \quad K'(x,y,z) = K((x,y,z)g)$$

(see §7). In addition to the usual invariants for a ternary cubic

$$S = abcj + cd^2e + \cdots \quad \text{and} \quad T = a^2b^2c^2 + 4bc^2d^3 + \cdots,$$

which are given in full in [30, p. 191,192], we have two other invariants

$$R = 5ac - 8b^2 - 24bj - 30dh + 48ek + 12ei + 12fk + 3fi - 18j^2$$

and

$$\Delta = -\tfrac{1}{46656}\mathrm{disc}\, K(x^2, 2x, 1) = a^5c^5 - 4096a^2b^6c^2 + \text{nearly 3700 terms}. \tag{3.1}$$

For $g \in G = \mathrm{SO}^+(J, \mathbb{R})$ set

$$\mu_J(g) = \inf_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ J(x,y,z)=1}} |K((x,y,z)g)| \tag{3.2}$$

and let $\sigma_J = \{\mu_J(g);\ g \in G\}$. As was true for a general ternary quadratic form, for a general ternary cubic form over our quadric, the question of the supremum of the spectrum $\sigma$ being finite and contained in $\sigma$ is non-trivial.

**Theorem 4.** *For a fixed real ternary cubic form $K$ with $L = 0$ and $\Delta \neq 0$ we have that $\sup \sigma_J < \infty$ and $\sup \sigma_J \in \sigma_J$.*

For the classical problem much more is known for decomposable forms, which are of course singular. In a series of papers ending with [8], Davenport obtained several results about them. For $K(x,y,z) = xyz$ he showed that the two largest elements of the spectrum are $\frac{1}{9}$ and $\frac{1}{7}$ and that $\frac{1}{9}$ is isolated. Swinnerton-Dyer [33] (see also [6]) extended Davenport's method and found 15 more isolated elements $\{\frac{1}{\sqrt{148}}, \frac{5}{63}, \dots\}$ and has suggested that 0 is the only limit point of the spectrum, i.e that it is rigid. It remains a very attractive open problem to show this. Davenport also studied the form $K(x,y,z) = x(y^2 + z^2)$ and showed that the largest element of its spectrum is $\frac{2}{\sqrt{23}}$ and that it is not isolated. For references and more on this spectrum see [14].

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[2] I have replaced the expected "$g$" by "$k$" to avoid a notational conflict.

For the corresponding problem over the quadric, we are able to obtain a satisfactory result for the class of decomposable forms

$$K(x,y,z) = (\alpha_1^2 x + \alpha_1\alpha_2 y + \alpha_2^2 z)(\beta_1^2 x + \beta_1\beta_2 y + \beta_2^2 z)(\gamma_1^2 x + \gamma_1\gamma_2 y + \gamma_2^2 z), \tag{3.3}$$

where $\alpha_j, \beta_j, \gamma_j$ are real for $j = 1, 2, 3$. This class is closed under the action of $g \in G$. An invariant under this action is

$$D = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_2\gamma_1 - \alpha_1\gamma_2)(\beta_1\gamma_2 - \beta_2\gamma_1).$$

When $\alpha_1 = \beta_1 = \gamma_1 = 1$ and $\alpha_2, \beta_2, \gamma_2$ are conjugates in a totally real cubic number field, $K$ is a rational norm form. For this class of decomposable forms we obtain the same value as Davenport for the largest minimum, even when we only minimize over the quadric. However, for us the value is not isolated.

**Theorem 5.** *For $K$ in (3.3) with $D \neq 0$ we have*

$$\sup \sigma_J = \tfrac{1}{7}|D| \in \sigma_J.$$

*This value of the spectrum is not isolated.*

## 4. Proof of Theorem 1

Hermite's original reduction theory of a general binary form in [19] applied to a binary quartic form with real coefficients, say

$$F(x,y) = (a,b,c,d,e) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

is the main tool needed for the proof of Theorem 1. [3] He associated to $F$ a positive definite binary quadratic form whose discriminant is minimal in a certain sense and to which he applied the classical Lagrange reduction. His method naturally leads directly to bounds for $|ae|, |bd|$ and $|c|^2$ in terms of the roots of $F(x,1) = 0$. For the proof of Theorem 1, it turns out that what is needed is precisely an upper bound for $|c|$, and Hermite's method is very effective for this. Hermite only worked out details when all zeros of $F(x,1)$ are real (this is given in [20]), but fortunately the other cases also work out nicely.[4] After applying some classical identities relating the zeros of $F(x,1)$ to those of $\chi_Q$ we can derive the simple bound of Theorem 1, which is uniform across signatures.

Turning to the proof, write

$$Q = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

Then

$$\chi_Q(t) = \det(Q - tJ) = 4t^3 + 4(c-d)t^2 + (c^2 - 4cd + 4be - af)t + D. \tag{4.1}$$

Clearly Theorem 1 holds for $Q = kJ$ for any $k \in \mathbb{R}$. For $Q$ not of this form we next show that we may assume $a > 0$. We use a well-known isomorphism

$$\mathrm{SL}(2,\mathbb{R})/\{\pm 1\} \to G = \mathrm{SO}^+(J,\mathbb{R}),$$

where $G$ is the connected component of the identity of the special orthogonal group of $J$. The isomorphism is given by

$$\begin{pmatrix} m & n \\ r & s \end{pmatrix} \mapsto g = g_{\begin{pmatrix} m & n \\ r & s \end{pmatrix}} = \begin{pmatrix} m^2 & 2mn & n^2 \\ mr & ms+nr & ns \\ r^2 & 2rs & s^2 \end{pmatrix}. \tag{4.2}$$

[3]Here and below I denote by $(a_1, a_2, \ldots, a_n)$ the binary form of degree $n$ with binomial coefficients attached.

[4]In his thesis Julia [22] extended Hermite's work and gave geometric interpretations of reduced forms. He computed what I call cases ii) and iii) of Lemma 1 but his computation of iii) is only partial and that of ii) needs to be corrected. He obtained (4.5) with $\alpha_3$ and $\alpha_4$ interchanged, which results in a different bound.

Note that $J = gJg^t$. The image of $\mathrm{SL}(2,\mathbb{Z})/\{\pm 1\}$ is the subgroup $\Gamma = \mathrm{SO}^+(J,\mathbb{Z})$ of $G$. For $g \in G$ we have $\chi_{gQg^t} = \chi_Q$ and for $\gamma \in \Gamma$ the form $Q' = \gamma Q \gamma^t$ represents the same numbers as does $Q$. It is now easy to check that unless $Q = kJ$ we can find $\gamma \in \Gamma$ so that the $(1,1)$ entry of $Q'$ is nonzero. Thus $\pm Q'$ works for us.

Next we will relate the problem of minimizing $Q$ on the quadric to that of minimizing the middle coefficient of a representative in an associated class of binary quartic forms. We know that $(x,y,z) = (mr, ms+nr, ns)$ runs over all integral solutions of $J(x,y,z) = 1$ as $m,n,r,s$ run over integers with $ms - nr = 1$. Therefore

$$\inf_{\substack{J(x,y,z)=1 \\ (x,y,z)\in\mathbb{Z}^3}} |Q(x,y,z)| = \inf_{\gamma\in\Gamma} |(0,1,0)\gamma Q \gamma^t (0,1,0)^t)|. \tag{4.3}$$

Associated to $Q$ is the quartic polynomial

$$F(x) = (x^2, 2x, 1) Q (x^2, 2x, 1)^t = ax^4 + 4bx^3 + (2c+4d)x^2 + 4ex + f.$$

A computation shows that if

$$(rx+s)^4 F\left(\tfrac{mx+n}{rx+s}\right) = a_1 x^4 + 4b_1 x^3 + 6c_1 x^2 + 4d_1 x + e_1$$

for $\left(\begin{smallmatrix} m & n \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ then for $h = g_{\left(\begin{smallmatrix} m & r \\ n & s \end{smallmatrix}\right)}$ from (4.2)

$$(0,1,0) h Q h^t (0,1,0)^t = c_1 + \tfrac{1}{3}(d-c).$$

Thus by (4.3)

$$\inf_{\substack{J(x,y,z)=1 \\ (x,y,z)\in\mathbb{Z}^3}} |Q(x,y,z)| \le |c_1| + \tfrac{1}{3}|d-c|. \tag{4.4}$$

Here we use that $h$ runs over $\Gamma$ as $\left(\begin{smallmatrix} m & n \\ r & s \end{smallmatrix}\right)$ runs over $\mathrm{SL}(2,\mathbb{Z})$.

As already mentioned, we now apply Hermite's method of continual reduction from [19] to the binary quartic form $F(x,y)$ associated to $F(x)$. The binary quadratic form associated to $F$ depends on the nature of the zeros of $F(x)$ and will be given in (4.7).

**Lemma 1.** *Let*

$$F(x,y) = (a,b,c,d,e) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$$

*for $a,b,c,d,e \in \mathbb{R}$ with $a > 0$. Define $\alpha_j$ to be the zeros of*

$$F(x,1) = a(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4).$$

*i) Assume that either each $\alpha_j$ is real with $\alpha_1 \ge \alpha_2 \ge \alpha_3 \ge \alpha_4$ or*

*ii) that none of the $\alpha_j$ are real and that $\alpha_2 = \bar\alpha_1, \alpha_4 = \bar\alpha_3$ with $\operatorname{Im}\alpha_1 > 0, \operatorname{Im}\alpha_3 < 0$.*

*Then $F$ is equivalent under $\mathrm{SL}(2,\mathbb{Z})$ to a form $F_1(x) = (a_1,b_1,c_1,d_1,e_1)$ with*

$$|c_1| < \tfrac{1}{3}|a(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)|. \tag{4.5}$$

*iii) When only $\alpha_1, \alpha_2$ are real and $\alpha_4 = \bar\alpha_3$ then $F$ is equivalent to $(a_1,b_1,c_1,d_1,e_1)$ with*

$$|c_1| < \tfrac{1}{6}a(|(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)| + 2|(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)|). \tag{4.6}$$

*Proof.* For real variables $t_j, u_j$ define the positive definite binary quadratic form

$$Px^2 + 2Q'xy + Ry^2 = \begin{cases} \sum_{j=1}^4 t_j^2 (x+\alpha_j y)^2, & \text{in case i)} \\ 2u_1^2(x+\alpha_1 y)(x+\bar\alpha_1 y) + 2u_2^2(x+\alpha_3 y)(x+\bar\alpha_3 y), & \text{in case ii)} \\ t_1^2(x+\alpha_1 y)^2 + t_2^2(x+\alpha_2 y)^2 + 2u_1^2(x+\alpha_3 y)(x+\bar\alpha_3 y), & \text{in case iii).} \end{cases} \tag{4.7}$$

In case i) the idea is to minimize

$$T = a^2(PR - Q'^2)^2/(t_1t_2t_3t_4)^2$$

with respect to real $t_j$. After computing $PR - Q'^2$ in terms of the $t_j$, this is basically a calculus problem. The result found by Hermite is that the minimal $T$ is

$$16a^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_4)^2. \tag{4.8}$$

After an $\mathrm{SL}(2,\mathbb{Z})$ transformation the resulting binary quadratic form can be made to satisfy $PR < \frac{4}{3}(PR - Q'^2)$. From this one concludes (see [19, p.174] for the method) that

$$|a_1 e_1|, |b_1 d_1|, |c_1|^2 < \tfrac{1}{144} T.$$

Using (4.8), (4.5) follows.

Similarly, in case ii) we must minimize

$$T = a^2(PR - Q'^2)^2/(u_1^2 u_2^2)^2 \tag{4.9}$$

with respect to $u_1, u_2$ (it is enough to minimize $\sqrt{T}$). If we write $\alpha_1 = u + iv$ and $\alpha_3 = s + it$ then

$$PR - Q'^2 = 4(v^2 u_1^4 + (u^2 + v^2 - 2us + s^2 + t^2) u_1^2 u_2^2 + t^2 u_2^4)$$

and the minimization of $T$ leads to (4.5) when $v > 0$ and $t < 0$. Case iii) is similar where $t_1 t_2 u_1^2$ is used in place of $u_1^2 u_2^2$ in (4.9) and now

$$PR - Q'^2 = (\alpha_1 - \alpha_2)^2 t_1^2 t_2^2 + 2(s^2 + t^2 - 2s\alpha_1 + \alpha_2^2) t_1^2 u_1^2 + 2(s^2 + t^2 - 2s\alpha_2 + \alpha_1^2) t_2^2 u_1^2 + 4t^2 u_1^4.$$

Note that the bounds in (4.5) and (4.6) are invariants of $F$ under $\mathrm{SL}(2,\mathbb{Z})$. □

We apply this lemma to $F$ with

$$F(x) = F(x,1) = (x^2, 2x, 1) Q (x^2, 2x, 1)^t = ax^4 + 4bx^3 + (2c + 4d)x^2 + 4ex + f.$$

We have that $\chi_Q(t - \frac{c-d}{3})$ is the resolvent cubic of $F$, where $\chi_Q$ was defined in (2.2). It is classical that $\chi_Q$ has all of its roots real if $F$ has either all roots real or none of them real. Otherwise, $\chi_Q$ has exactly one real root. Furthermore, the differences between roots of $\chi_Q$ satisfy

$$\begin{aligned} 4(\lambda_3 - \lambda_2) &= a(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4) \\ 4(\lambda_3 - \lambda_1) &= a(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4) \\ 4(\lambda_2 - \lambda_1) &= a(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4). \end{aligned}$$

For these statements see [5, p.124]. In case i) or ii) of Lemma 1 it follows that $\lambda_3 \geq \lambda_2 \geq \lambda_1$ and so

$$|c_1| < \tfrac{1}{3}|a(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_4)| = \tfrac{4}{3}|\lambda_3 - \lambda_1| = \tfrac{1}{3}\sum_{i \neq j} |\lambda_i - \lambda_j|.$$

In case iii) we have the same bound for $|c_1|$. Since by (4.1)

$$\lambda_1 + \lambda_2 + \lambda_3 = d - c$$

we conclude from (4.4) that for $Q \neq kJ$ we have

$$\inf_{\substack{J(x,y,z)=1 \\ (x,y,z) \in \mathbb{Z}^3}} |Q(x,y,z)| < \tfrac{1}{3}\Big(\sum_{i \neq j} |\lambda_i - \lambda_j| + |\lambda_1 + \lambda_2 + \lambda_3|\Big). \quad \square$$

## 5. Proof of Theorem 2

The ideas behind the (rest of the) proof of Theorem 2 are from Mahler's paper [24], where they are expressed in the language of lattices and star bodies. For $g \in G$ we have

$$\mu_J(g) = \inf_{\gamma \in \Gamma} |Q((0,1,0)\gamma g)|,$$

where again $\Gamma = \mathrm{SO}^+(J,\mathbb{Z})$. Clearly $\mu_J(\gamma g) = \mu_J(g)$ for $\gamma \in \Gamma$. It is easy to show that for $T \in \mathbb{R}$ the subset of $\Gamma\backslash G$ given by

$$S_T = \{g \in \Gamma\backslash G; \mu_J(g) \geq T\}$$

is closed. Thus $\mu_J$ is upper semi-continuous on $\Gamma\backslash G$ and since $\mu_J$ is bounded by Theorem 1, to prove Theorem 2 we need only show that $S_T$ is compact for any $T > 0$. For this it is enough to show sequential compactness. The following simple reduction lemma is needed.

**Lemma 2.** *There is an absolute constant $C > 0$ such that each $g \in \Gamma\backslash G$ has a representative $g$ with*

$$\prod_{1\leq j\leq 3} \|(g)_j\| \leq C,$$

*where $(g)_j$ is the $j^{th}$ row of $g$ and $\|\cdot\|$ is the Euclidean norm.*

*Proof.* For $g$ from (4.2) we have

$$\prod_{1\leq j\leq 3} \|(g)_j\|^2 = \left(m^4+4m^2n^2+n^4\right)\left(r^4+4r^2s^2+s^4\right)\left(m^2r^2+m^2s^2+2mnrs+n^2r^2+n^2s^2\right)$$
$$\leq C_1(m^2+n^2)^3(r^2+s^2)^3.$$

Now choose $\left(\begin{smallmatrix} m & n \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ to be reduced in the usual sense:

$$(m^2+n^2)(r^2+s^2) < C_2.$$

□

The proof of sequential compactness proceeds as in the proof of Theorem 2 in [24]. The key idea is that a lower bound for each $\|(g)_j\|$ together with the upper bound for $\prod_{1\leq j\leq 3}\|(g)_j\|$ in Lemma 2 gives an upper bound for each $\|(g)_j\|$. Then in a sequence $\{g_k\}$ in $S_T$ we can extract a subsequence that converges to a point $g \in S_T$. Here we use that $g_k \in S_T$ implies that for some constant $c_T > 0$ we have $\|(g_k)_j\| > c_T$ for each $j$. □

## 6. Proof of Theorem 3

Theorem 3 is a nearly immediate consequence of a result of Barnes on the minimum of the absolute value of the product of two related values of an indefinite binary quadratic form. His result and its proof have a lot in common with those of Markoff on the minimum of the absolute value of such a quadratic form.

For the homogeneous version of $Q(x,y,z)$ from (2.3), namely

$$Q(x,y,z) = (\alpha_1^2 x + \alpha_1\alpha_2 y + \alpha_2^2 z)(\beta_1^2 x + \beta_1\beta_2 y + \beta_2^2 z),$$

the associated binary quadratic form

$$q(x,y) = (\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)$$

is indefinite and has discriminant $(\alpha_1\beta_2 - \beta_1\alpha_2)^2$. We have the identity

$$q(m,n)q(r,s) = Q(mr, ms+nr, ns).$$

Letting $m, n, r, s$ run over integers with $ms - nr = 1$, the result now follows from [2, Theorem 1]. □

## 7. Proof of Theorem 4

The arguments in the proof of Theorem 4 are similar to those for Theorems 1 and 2 so I will be brief. Again the main are devices used are Hermite's upper bound for the middle coefficient of a reduced binary form, this time a binary sextic, and Mahler's compactness argument. Recall that

$$K(x,y,z)=ax^3+by^3+cz^3+3dx^2y+3ey^2z+3fz^2x+3kxy^2+3hyz^2+3izx^2+6jxyz \tag{7.1}$$

and

$$L_K(x,y,z)=(k-i)x+(b-j)y+(e-f)z.$$

The fact that $L_K$ is a covariant follows from (4.2) after verifying the identity

$$L_K\Big((x,y,z)\left(\begin{smallmatrix} m^2 & 2mn & n^2\\ mr & ms+nr & ns\\ r^2 & 2rs & s^2\end{smallmatrix}\right)\Big)=L_{K'}(x,y,z),$$

where

$$K'(x,y,z)=K\Big((x,y,z)\left(\begin{smallmatrix} m^2 & 2mn & n^2\\ mr & ms+nr & ns\\ r^2 & 2rs & s^2\end{smallmatrix}\right)\Big).$$

The claim that $\sup\sigma_J<\infty$ where $\sigma_J$ is defined below (3.2) when $L(x,y,z)=0$ is a consequence of the following more general result.

**Lemma 3.** *For $K$ from (7.1) with $\Delta$ from (3.1) nonzero, there exists a constant $C$ depending only on $\Delta$ such that*

$$\inf_{\substack{(x,y,z)\in\mathbb{Z}^3\\ J(x,y,z)=1}}|K(x,y,z)-\tfrac{3}{5}L(x,y,z)|\le C.$$

*Proof.* A computation verifies the following identity for $m,n,r,s\in\mathbb{Z}$ with $ms-nr=1$:

$$K(mn,ms{+}nr,rs)-\tfrac{3}{5}L(mn,ms{+}nr,rs)=\tfrac{1}{20}\big(\text{coefficient of } x^3y^3 \text{ in } F_K(mx+ny,rx+sy)\big),$$

where

$$F_K=(a,d,\tfrac{4}{5}k+\tfrac{1}{5}i,\tfrac{2}{5}b+\tfrac{3}{5}j,\tfrac{4}{5}e+\tfrac{1}{5}f,h,c)=K(x^2,2xy,y^2).$$

Now we may apply Hermite's upper bound for the coefficient of $x^3y^3$ in the binary sextic $F_K(x,y)$ when it is reduced. Here we make use of the assumption that $\Delta=\operatorname{disc}F_K\neq0$ (c.f. [4, §5]). □

The proof that $\sup\sigma_J\in\sigma_J$ proceeds in the same way as that of Theorem 2.

## 8. Proof of Theorem 5

The main tools used in the proof of Theorem 5 are two theorems of Davenport. The first is recorded as Lemma 4 below. This result, whose proof is intricate, gives a sharp upper bound for the minimum among various products of special values of a reduced binary cubic form with a positive discriminant. It does not appear to have been applied before now. Also, as Davenport shows in [11], the corresponding result for a binary cubic with a negative discriminant is not true.

The second result of Davenport, given as Lemma 5, shows that the minimum of a binary cubic with a positive discriminant is not isolated.

Recall the notation from around (3.3). Associate to $K$ the binary cubic form defined (up to sign) by

$$f_K(x,y)=\pm(\alpha_1x+\alpha_2y)(\beta_1x+\beta_2y)(\gamma_1x+\gamma_2y). \tag{8.1}$$

The discriminant of $f_K$ is well-defined and given by $D^2$. We have for any $m,n,r,s$ the identity

$$|K(mr,ms+nr,ns)|=|f_K(m,n)f_K(r,s)|. \tag{8.2}$$

For integers $m_1, n_1, r_1, s_1$ with $m_1 n_1 - r_1 s_1 = \pm 1$ let

$$f(x,y) = f_K(m_1 x + r_1 y, n_1 x + s_1 y). \tag{8.3}$$

A calculation shows

$$f(m,n)f(r,s) = f_K(m_2,n_2)f_K(r_2,s_2) \text{ where } \left(\begin{smallmatrix} m_2 & n_2 \\ r_2 & s_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} m & n \\ r & s \end{smallmatrix}\right)\left(\begin{smallmatrix} m_1 & n_1 \\ r_1 & s_1 \end{smallmatrix}\right).$$

Now using this and (8.2) we have

$$\inf_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ y^2-4xz=1}} |K(x,y,z)| = \inf_{\substack{(m,n,r,s)\in\mathbb{Z}^4 \\ mn-rs=\pm1}} |f(n,m)f(r,s)|. \tag{8.4}$$

We may choose $f(x,y) = \pm(ax^3 + bx^2y + cxy^2 + dy^3)$ to be reduced in the sense of Hermite. Explicitly, this means that the covariant binary quadratic form $Ax^2+Bxy+Cy^2$, where

$$A = b^2 - 3ac, \quad B = bc - 9ad, \quad C = c^2 - 3bd,$$

satisfies

$$0 \le B \le A \le C.$$

The binary quadratic is positive definite with $B^2 - 4AC = -3\Delta(f) < 0$ (see [20] and [10]).

The following result is contained in Theorem 2 of [10], which after (8.4) finishes the proof of the first statement of Theorem 5.

**Lemma 4** (Davenport)**.** *Let $f(x,y)$ be a reduced binary cubic form with discriminant $D^2$. Then*

$$\min\{|f(1,0)f(0,1)|, |f(1,0)f(1,1)|, |f(1,0)f(1,-1)|, |f(0,1)f(1,1)|, |f(0,1)f(1,-1)|\} \le \tfrac{|D|}{7}.$$

The last statement of Theorem 5 is an immediate consequence of (8.4) and the following result, which is Theorem A of [7]. Here, given $f$, we define $K$ using (3.3).

**Lemma 5** (Davenport)**.** *Let $\epsilon > 0$. Then there exists a binary cubic form $f(x,y)$ with positive discriminant $D^2$ such that none of the roots of $f(x,1) = 0$ is equivalent to any root of $x^3 + x^2 - 2x - 1 = 0$, and with the property that*

$$|f(x,y)| > \left(\tfrac{|D|}{7+\epsilon}\right)^{\frac{1}{2}}$$

*for all integers $(x,y) \ne (0,0)$.*

□

*Remarks.* i) If instead of restricting to the the quadric surface $y^2 - 4xz = 1$ we restrict to the cone $y^2 = 4xz$, then minimizing a ternary form reduces directly to minimizing a binary form of twice its degree over nonzero integral pairs. Some striking new results about the spectra of higher degree binary forms have recently been found [23] (see also [31]).

ii) The papers [32],[13],[3],[1] study the minima of the absolute values of certain real bilinear forms, when they are restricted to the integral points of the affine hypersurface $xy - zw = 1$.

## References

[1] Andersen, N.; Duke, W. Markov spectra for modular billiards. Math. Ann. 373 (2019), no. 3-4, 1151–1175.

[2] Barnes, E. S. The minimum of the product of two values of a quadratic form. I. Proc. London Math. Soc. (3) 1 (1951), 257–283.

[3] Barnes, E. S. The minimum of a bilinear form. Acta Math. 88 (1952), 253–277.

[4] Birch, B. J.; Merriman, J. R. Finiteness theorems for binary forms with given discriminant. Proc. London Math. Soc. (3) 24 (1972), 385–394.

[5] Burnside, W. S ; Panton, A.W. The theory of equations: With an introduction to the theory of binary algebraic forms. Vol 1. 8th ed. Hodges Figgis And Company. (1924)

[6] Cassels, J. W. S.; Swinnerton-Dyer, H. P. F. On the product of three homogeneous linear forms and the indefinite ternary quadratic forms. Philos. Trans. Roy. Soc. London Ser. A 248 (1955), 73–96.

[7] Davenport, H. On a conjecture of Mordell concerning binary cubic forms. Proc. Cambridge Philos. Soc. 37 (1941), 325–330.

[8] Davenport, H. On the product of three homogeneous linear forms. IV. Proc. Cambridge Philos. Soc. 39 (1943), 1–21.

[9] Davenport, H. On the minimum of a ternary cubic form. J. London Math. Soc. 19 (1944), 13–18.

[10] Davenport, H. The reduction of a binary cubic form. I. J. London Math. Soc. 20 (1945), 14–22.

[11] Davenport, H. The reduction of a binary cubic form. II. J. London Math. Soc. 20 (1945), 139-147.

[12] Davenport, H. On a theorem of Markoff. J. London Math. Soc. 22 (1947), 96–99.

[13] Davenport, H.; Heilbronn, H. On the minimum of a bilinear form. Quart. J. Math. Oxford Ser. 18 (1947), 107–121.

[14] Davenport, H.; Rogers, C. A. Diophantine inequalities with an infinity of solutions. Philos. Trans. Roy. Soc. London Ser. A 242 (1950), 311–344.

[15] Dickson, L.E. Studies in the theory of numbers, (1930) Chelsea reprint.

[16] Duke, W.; Rudnick, Z.; Sarnak, P. Density of integer points on affine homogeneous varieties. Duke Math. J. 71 (1993), no. 1, 143–179.

[17] Gauss, C.F. Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber, Göttingische gelehrte Anzeigen, July 9, 1831. in Werke, Vol. 2, 188–196.

[18] Gamburd, A.; Ghosh,A.; Sarnak, P.; Whang, J.P. On indefinite integral ternary quadratic forms, Preprint 2026.

[19] Hermite, C. Sur l'introduction des variables continues dans la théorie des nombres. Journal für die reine und angewandte Mathematik 41 (1851): 191-216., in Oeuvres I, 171-178.

[20] Hermite, C. Sur la theorie des functions homogènes a deux indeterminées, Journal für die reine und angewandte Mathematik 52 (1856): 1–17, in Oeuvres I, 350–371.

[21] Hermite, C. Sur la reduction forems cubiques a deux indéterminées, C.R. XLVIII (1859), 351, in Oeuvres II, 93–99.

[22] Julia, G. Étude sur les formes binaires non quadratiques à indéterminées réelles, ou complexes, ou à indéterminées conjugées, Mem. Acad. Sci. l'Inst. France 55 (1917), 1-293, in Oeuvres VI.

[23] Kotsovolis, G. The spectrum of binary forms, Thesis Princeton (2024)

[24] Mahler, K. On lattice points in n-dimensional star bodies. I. Existence theorems. Proc. Roy. Soc. London Ser. A 187 (1946), 151–187.

[25] Margulis, G.A Formes quadratriques indéfinies et flots unipotents sur les espaces homogènes. C. R. Acad. Sci. Paris Sér. I Math. 304 (1987), no. 10, 249–253

[26] Markoff, A. Sur les formes quadratiques ternaires indéfinies. Math. Ann. 56 (1902), no. 2, 233–251.

[27] Mordell, L. J. The product of three homogeneous linear ternary forms. J. London Math. Soc. 17 (1942), 107–115.

[28] Mordell, L. J. The minimum of a singular ternary quadratic form. J. London Math. Soc. (2) 2 (1970), 393–394.

[29] Salmon, G. Modern Higher Algebra. 5th ed Chelsea Publishing Co., New York, xix+376 pp.

[30] Salmon, G. A treatise on the higher plane curves: intended as a sequel to "A treatise on conic sections". 3rd ed Chelsea Publishing Co., New York, 1960. xix+395 pp.

[31] Sarnak, P. Prescribing the spectra of locally uniform geometries, 2023 - Chern Lectures Berkeley available at `https://publications.ias.edu/node/2728`.

[32] Schur, Zur Theorie der indefiniten binarën quadratischen Formen, I. S.-B. Preuss. Akad. Wiss. (1913), 212–231, in Gesammelte Abh. II #22.

[33] Swinnerton-Dyer, H. P. F. On the product of three homogeneous linear forms. Acta Arith. 18 (1971), 371–385.

[34] Venkov, B. A. Sur le problème extrémale de Markoff pour les formes quadratiques ternaires indéfinies.(Russian. French summary) Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9 (1945), 429–494.

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