Homework 9 for Math 215A Commutative Algebra

Burt Totaro

Due: Monday, December 3, 2018

Rings are understood to be commutative, unless stated otherwise.

(1) Compute $\dim(\mathbf{C}[x, y, z]/(x^2 + y^2 + z^2 - 1, xy - z^3))$.

(2) Let A and B be algebras of finite type over a field k. Show that $\dim(A \otimes_k B) = \dim(A) + \dim(B)$.

(3) Let R be an algebra of finite type over a field k. Assume that R is a domain. Let M be a finitely generated module over R. Show that M is projective (or equivalently, flat or locally free) if and only if the $R/\mathfrak{m}$-vector spaces $M/\mathfrak{m}M$ all have the same dimension. (Hint: If these vector spaces have dimension n, construct a surjection $R_{\mathfrak{m}}^{\oplus n} \to M_{\mathfrak{m}}$ at any maximal ideal $\mathfrak{m}$. Show that this comes from a surjection $R[1/s]^{\oplus n} \to M[1/s]$ for some $s \in R - \mathfrak{m}$. Show that any element of the kernel must be zero.)

As an application, show that the ring $k[x, y]/(x^2, xy)$ is finite but not flat over $k[y]$.

Give an example to show that this characterization of projective modules can fail if R is not a domain.