Homework 8 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) A morphism $f\colon X \to Y$ of affine schemes is said to be *dominant* if the image $f(X)$ is dense in Y. Show that f is dominant if and only if the kernel I of the pullback homomorphism $f^*\colon O(Y) \to O(X)$ is a *nil* ideal, meaning that every element of I is nilpotent. In particular, if A is a subring of a commutative ring B, then the associated morphism $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is dominant.

(2) (a) Let $\mathfrak{p}$ be a prime ideal in a noetherian ring R, and let M and N be R-modules. Give an example to show that the natural map

$$\mathrm{Hom}_R(M, N)_{\mathfrak{p}} \to \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

need not be an isomorphism. But show that it is an isomorphism if M is finitely generated.

(b) Let $\mathfrak{p}$ be a prime ideal in a noetherian ring R. Let M and N be R-modules with M finitely generated. Show that $\mathrm{Ext}^i_R(M, N)_{\mathfrak{p}} \cong \mathrm{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ for any $i \geq 0$.

(c) Show that a finitely generated module over a noetherian ring R which is locally free must be projective. Deduce that "flat = locally free = projective" for finitely generated modules over a noetherian ring. (In geometric language: finitely generated projective modules over a noetherian ring R are equivalent to vector bundles over the affine scheme $\mathrm{Spec}\,(R)$.)

(3) Let R be an algebra of finite type over a field k. Show that R is artinian if and only if R has finite dimension as a k-vector space. (Hint: consider the filtration of R constructed in the proof that "artinian" is equivalent to "noetherian of dimension zero".)

(4) Show that every finitely generated module M over a Dedekind domain R is the direct sum of a torsion module M_1 and a projective module M_2. Give an example to show that such a splitting is not unique (in the sense that the submodule M_2 of M is not uniquely determined).