# Homework 7 for Math 215A Commutative Algebra

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Due: Monday, November 19, 2018

Rings are understood to be commutative, unless stated otherwise.

(1) Let $M$ be a module over a ring $R$. Show that the following are equivalent:
(a) $M$ is flat;
(b) $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ in $R$;
(c) $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ in $R$.
Using an earlier homework problem, deduce that for *finitely generated* modules over a *noetherian* ring, "flat = locally free".

(2) Let $R$ be an algebra of finite type over a field. Show that $R$ is a Jacobson ring, which means that every prime ideal in $R$ is an intersection of maximal ideals. Which prime ideals in $R$ can be written as a finite intersection of maximal ideals?

(3) Let $f\colon X \to Y$ be a finite morphism of affine schemes. (That means that $X = \mathrm{Spec}\,(R)$ and $Y = \mathrm{Spec}\,(S)$ for some commutative rings $R$ and $S$, $f$ is the mapping associated to a ring homomorphism $S \to R$, and $R$ is a finite $S$-algebra.) Show that the inverse image of each point in $Y$ is a finite subset of $X$ (that is, a finite morphism is *quasi-finite*). Show that the image of any closed subset of $X$ is closed in $Y$ (that is, a finite morphism is *closed*). Give an example of a morphism of affine schemes which is (a) quasi-finite but not closed; (b) closed but not quasi-finite.