Homework 5 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) Let R be a noetherian ring. We have shown that $X = \mathrm{Spec}(R)$ can be written as the union of finitely many irreducible closed subsets, $X = X_1 \cup X_2 \cup \cdots \cup X_m$, such that X_i is not contained in X_j for any $i \neq j$. Show that such a decomposition of X is unique up to reordering the X_i's.

(2) Write $A^3_{\mathbf{C}}$ for affine 3-space over $\mathbf{C}$, meaning $\mathrm{Spec}(\mathbf{C}[x, y, z])$. Let X be the closed subset of $A^3_{\mathbf{C}}$ defined by $x^2 = yz$ and $xz = x$. Decompose X into its irreducible components.

(3) Show that the following are equivalent, for a module M over a ring R. (1) M is projective. (2) $\mathrm{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i > 0$. (3) $\mathrm{Ext}^1_R(M, N) = 0$ for all R-modules N. Likewise, show that the following are equivalent, for a module M over a ring R. (1) M is flat. (2) $\mathrm{Tor}^R_i(M, N) = 0$ for all R-modules N and all $i > 0$. (3) $\mathrm{Tor}^R_1(M, N) = 0$ for all R-modules N.