Homework 4 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1) Let I be an ideal in a commutative ring R, $I \neq R$. Show that there is a minimal prime ideal containing I. (That means: there is a prime ideal containing I which contains no other prime ideal containing I.) What does this mean geometrically, in terms of $\mathrm{Spec}(R)$?

(2) A *finite* algebra over a ring A means an A-algebra which is finitely generated as an A-module. Show that a finite flat $\mathbf{Z}$-algebra is free as a $\mathbf{Z}$-module, and give an example of a flat $\mathbf{Z}$-algebra which is not free as a $\mathbf{Z}$-module. Is a flat $\mathbf{Z}$-algebra of finite type always free as a $\mathbf{Z}$-module?

(3) Let k be a field. Show that every ideal in the polynomial ring $k[x]$ is generated by one element. But show that the ideal $(x^n, x^{n-1}y, \ldots, y^n)$ in $k[x, y]$ cannot be generated by fewer than $n+1$ elements. Thus there is no upper bound for the number of elements needed to generate an ideal in $k[x, y]$. (Hint: try to reduce this to a question about the dimension of a vector space over a field.)

(4) (a) Let M be a module over a ring R. Define the *support* of M to be the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ such that the localization $M_{\mathfrak{p}}$ is not zero. If M is a finitely generated R-module, show that the support of M is the closed subset of $\mathrm{Spec}(R)$ defined by the annihilator ideal $\mathrm{Ann}_R(M)$. Conversely, for every closed subset Y in $\mathrm{Spec}(R)$, give an example of a finitely generated R-module with support equal to Y.

(b) Give an example of a ring R and an R-module M such that the support of M is not closed in $\mathrm{Spec}(R)$. (Some authors define the support of a module as the closure of the subset defined here.)