Homework 3 for Math 215A Commutative Algebra

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Due: Monday, October 22, 2018

Rings are understood to be commutative, unless stated otherwise.

(1) Show that the kernel of the $\mathbf{C}$-algebra homomorphism $\mathbf{C}[x,y] \to \mathbf{C}[t]$ given by $x \mapsto t^2$ and $y \mapsto t^3$ is the ideal $(x^3 - y^2)$. (One possible approach is to show first that every element of the quotient ring $\mathbf{C}[x,y]/(x^3 - y^2)$ can be written as $f(x) + g(x)y$ for some polynomials f and g.) Deduce that the ideal $(x^3 - y^2)$ in $\mathbf{C}[x,y]$ is prime.

(2) For any commutative ring R, show that $\mathrm{Spec}(R)$ is quasi-compact. (That is, if $\mathrm{Spec}(R)$ is the union of some collection of open subsets, then it is the union of finitely many of them. In point-set topology this would just be called "compact". The word "quasi-compact" is meant to emphasize that these topological spaces are not necessarily Hausdorff.)

(3) Let R be a nonzero commutative ring. Let I and J be sets of different cardinalities. Show that the free R-modules $R^{\oplus I}$ and $R^{\oplus J}$ are not isomorphic. (Hint: this is true when R is a field.)

(4) (a) Let $\mathfrak{p}$ be a prime ideal in a ring R. Describe (in both directions) the 1-1 correspondence between the prime ideals I of the localization $R_{\mathfrak{p}}$ and the prime ideals of R contained in $\mathfrak{p}$. Using this, describe $\mathrm{Spec}(R_{\mathfrak{p}})$ in terms of $\mathrm{Spec}(R)$.

(b) That 1-1 correspondence does not work for ideals I that are not prime. In particular, give an example of a prime ideal $\mathfrak{p}$ in a ring R and two different ideals in R contained in $\mathfrak{p}$ that generate the same ideal in $R_{\mathfrak{p}}$.