Homework 2 for Math 215A Commutative Algebra

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Rings are understood to be commutative, unless stated otherwise.

(1)(a) Let R be a domain. Show that the polynomial ring $R[x]$ is a domain and that the group of units $R[x]^*$ is equal to R^* (viewed as constant polynomials). Give an example showing that this description of $R[x]^*$ fails for R not a domain, and say where your proof fails. By induction on n, it follows that the polynomial ring $A = k[x_1, \ldots, x_n]$ over a field k is a domain, and that $A^* = k^*$.

(1)(b) Show that the power series ring $B = k[[x_1, \ldots, x_n]]$ over a field is also a domain, and find the group of units B^*.

(2) Let A and B be commutative rings. The product ring $A \times B$ (not to be confused with a tensor product) is the product set, with ring structure $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$. State and prove a universal property that characterizes $A \times B$ in the category of commutative rings. Show that $\mathrm{Spec}(A \times B)$ is the disjoint union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, as a set. (In fact, it is the disjoint union as a topological space, but you need not prove that.)

(3) Let k be a field. Since $R = k[x_1, \ldots, x_n]$ is a UFD, the ideal (f) is prime for every irreducible polynomial f in R. The following exercise gives some practice in finding irreducible polynomials.

Show that a polynomial in $k[x_1, \ldots, x_n]$ of the form $x_n - f(x_1, \ldots, x_{n-1})$ is irreducible over k. Show that a polynomial of the form $x_n^2 - f(x_1, \ldots, x_{n-1})$ is irreducible over k if and only if f is not a square in $k[x_1, \ldots, x_{n-1}]$.