(20) 1. (a) State Markov's inequality.

Solution: If $X \geq 0$, then $P(X \geq a) \leq EX/a$ for $a > 0$.

(b) Prove Markov's inequality.

Solution: $a1_{\{X \geq a\}} \leq X$. Taking expected values gives $aP(X \geq a) \leq EX$.

(c) Suppose X is $N(0,1)$. What explicit upper bound for $P(X^2 \geq 3)$ does Markov's inequality provide?

Solution: $P(X^2 \geq 3) \leq \frac{1}{3}$, since $EX^2 = 1$.

(15) 2. (a) State Chebyshev's inequality.

Solution: If $EX^2 < \infty$, then

$$P(|X - EX| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}.$$

(b) Deduce Chebyshev's inequality from Markov's inequality.

Solution: Apply Markov to the random variable $Y = (X - EX)^2$:

$$P(|X - EX| \geq \epsilon) = P(Y \geq \epsilon^2) \leq \frac{EY}{\epsilon^2} = \frac{Var(X)}{\epsilon^2}.$$

(15) 3. (a) State the Weak Law of Large Numbers (WLLN).

Solution: If $X_1, X_2, \dots$ are i.i.d. with $E|X_1| < \infty$, and $S_n = X_1 + \cdots + X_n$, then

$$\frac{S_n}{n} \to EX_1 \quad \text{in probability.}$$

(b) Prove the WLLN under a second moment assumption.

Solution: Assume $EX_1^2 < \infty$. By Chebyshev, for $\epsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - EX_1\right| \geq \epsilon\right) \leq \frac{Var(S_n/n)}{\epsilon^2} = \frac{Var(X_1)}{n\epsilon^2} \to 0$$

as $n \to \infty$.

(10) 4. (a) Consider the three modes of convergence we discussed: A. Convergence in distribution, B. Convergence in probability, and C. Convergence a.s. (or with probability 1). Place the letters A, B, C in the spaces below so that the following is correct:

$$(C) \text{ implies } (B) \text{ implies } (A).$$

(b) Give a counterexample that shows that the reverse of the second implication above is false. No explanation is necessary.

Solution: Let $X_1, X_2, \ldots$ be i.i.d. non-constant random variables. Then $X_n \to X_1$ in distribution, but not in probability.

(20) 5. Suppose that X_n is $N(0, \frac{1}{n})$.

(a) Show that $X_n \to 0$ in probability.

Solution:

$$P(|X_n| \geq \epsilon) \leq \frac{EX_n^2}{\epsilon^2} = \frac{1}{n\epsilon^2} \to 0$$

as $n \to 0$.

(b) Show that $X_n \to 0$ a.s.

Solution: Since

$$X_n \text{ and } \frac{X_1}{\sqrt{n}} \quad \text{have the same distribution,}$$

$$EX_n^4 = \frac{EX_1^4}{n^2},$$

so $\sum EX_n^4 < \infty$. It follows that $X_n^4 \to 0$ a.s., and hence $X_n \to 0$ a.s.

(20) 6. (a) State the Central Limit Theorem.

Solution: If $X_1, X_2, \ldots$ are i.i.d. with $EX_1^2 < \infty$, and $S_n = X_1 + \cdots + X_n$, then

$$\frac{S_n - nEX_1}{n\sqrt{Var(X_1)}} \to N(0,1) \quad \text{in distribution.}$$

(b) A UCLA basketball player knows that he usually makes 60% of his free throw attempts. What is the (approximate) probability that in 24 attempts he will make more than half of his shots?

Solution: Let S be the number of free throws he makes. Then

$$\frac{S - 14.4}{2.4} \sim N(0,1) \text{ distributed.}$$

Therefore

$$P(S > 12) = P\left(\frac{S - 14.4}{2.4} > -1\right) \sim .8413.$$