Mathematics 170B – HW6 – Due Tuesday, May 8, 2012.

Problem 5 on page 288.

F_1. (a) Use the Schwartz inequality to show that for any random variable X, $(EX^2)^2 \le EX^4$.

(b) Find a PDF for which the corresponding random variable X satisfies $EX^4 = \infty$ and $EX^2 < \infty$.

F_2. Suppose $X_1, X_2, \dots$ are i.i.d. random variables satisfying $EX_i = 0$ and $EX_i^4 = 1$. Letting $S_n = X_1 + \cdots + X_n$, use problems E_1 and $F_1(a)$ to show that

$$P\left(\left|\frac{S_n}{n}\right| > 1\right) \le \frac{6}{n^2}.$$

(Recall that if $EX^2 < \infty$, the proof of the WLLN shows that the above probability converges to zero at least at the rate c/n for some c. This problem shows that the stronger fourth moment assumption leads to a stronger conclusion on the rate of convergence.)

F_3. (a) Explain why if $a_n, a > 0$,

$$\lim_{n\to\infty} a_n = a \quad \text{is equivalent to} \quad \lim_{n\to\infty} \log a_n = \log a.$$

(b) Using part (a) and the fact that $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$, show that

$$\lim_{n\to\infty} a_n = a \quad \text{implies} \quad \lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n = e^a.$$

(You will find this useful in doing the next two problems.)

F_4. Suppose X_n is $B(n,p)$.

(a) Find the moment generating function $M_n(s)$ of $(X_n - np)/\sqrt{np(1-p)}$.

(b) Compute the limit

$$\lim_{n\to\infty} M_n(s),$$

directly, without using the central limit theorem.

F_5. Suppose X_n is Poisson with parameter n.

(a) Find the moment generating function $M_n(s)$ of $(X_n - n)/\sqrt{n}$.

(b) Compute the limit

$$\lim_{n\to\infty} M_n(s),$$

directly, without using the central limit theorem.