T. Liggett Mathematics 170B – Midterm 1 Solutions April 25, 2012

(10) 1. Suppose X has the Cauchy density

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty.$$

Find the density of $Y = 2X + 1$.

Solution:

$$P(Y \le y) = P\left(X \le \frac{y-1}{2}\right) = \int_{-\infty}^{\frac{y-1}{2}} \frac{1}{\pi(1+x^2)} dx,$$

so Y has density

$$\frac{2}{\pi} \frac{1}{4+(y-1)^2}.$$

(15) 2. Suppose X, Y, Z are independent random variables with mean zero and variance 1, and let $U = 3X + 4Y$ and $V = X - 2Y + 2Z$.

(a) Find the variances of U and V.

Solution: $\text{var}(U) = 9 + 16 = 25$ and $\text{var}(V) = 1 + 4 + 4 = 9$.

(b) Find the covariance of U and V.

Solution: $\text{cov}(U, V) = 3 - 8 = -5$.

(c) Find the correlation coefficient of U and V.

Solution: $\rho(U, V) = \frac{-5}{15} = -\frac{1}{3}$.

(20) 3. Suppose $X_1, X_2, X_3, \ldots$ are independent Bernoulli random variables with parameter p, and let $N = \min\{i : X_i = 1\}$ be the time that the first 1 is observed.

(a) Express $E(N \mid X_1)$ and $E(N^2 \mid X_1)$ in terms of EN and EN^2.

Solution: $(N \mid X_1 = 1) =^d 1$ and $(N \mid X_1 = 0) =^d 1 + N$. Therefore $E(N \mid X_1) = 1 + (1 - X_1)EN$ and $E(N^2 \mid X_1) = 1 + (1 - X_1)(2EN + EN^2)$.

(b) Use the result of part (a) to compute EN and $\text{var}(N)$ without using the distribution of N.

Solution: By the law of iterated expectations,

$$EN = 1 + EN(1-p), \qquad EN^2 = 1 + (2EN + EN^2)(1-p).$$

Solving gives $EN = \frac{1}{p}$, $EN^2 = \frac{2-p}{p^2}$, and $\text{var}(N) = \frac{1-p}{p^2}$.

(15) 4. A box contains n balls, numbered $1, 2, \ldots, n$. A collection of m balls is drawn WITH replacement. Let Z be the largest number drawn.

(a) Find the distribution function of Z.

Solution: For $1 \le k \le n$,

$$F(k) = P(\text{all balls drawn are } \le k) = \left(\frac{k}{n}\right)^m.$$

(b) Find the PMF of Z.

Solution:

$$p(k) = \left(\frac{k}{n}\right)^m - \left(\frac{k-1}{n}\right)^m.$$

(20) 5. Suppose $X_1, X_2, X_3 \ldots$ are i.i.d. Bernoulli random variables with parameter p, and $S_n = X_1 + \cdots + X_n$ be their partial sums. For $k \geq 1$, let $N_k = \min\{n \geq 1 : S_n = k\}$ be the time at which the kth 1 is observed. Let $M_k(s)$ be the moment generating function of N_k.

(a) Compute $M_1(s)$.

Solution:

$$\begin{aligned} M_1(s) = Ee^{sN_1} &= \sum_{k=1}^{\infty} P(N_1 = k)e^{sk} = \sum_{k=1}^{\infty} p(1-p)^{k-1}e^{sk} \\ &= pe^s \sum_{k=0}^{\infty} [(1-p)e^s]^k = \frac{pe^s}{1-(1-p)e^s}. \end{aligned}$$

(b) Find a relation that expresses $M_k(s)$ in terms of $M_1(s)$.

Solution: $M_k(s) = [M_1(s)]^k$.

(c) Use the result of part (b) to compute the mean of N_k.

Solution: $EN_k = \frac{d}{ds}M_k(s)|_{s=0} = k[M_1(0)]M_1'(0) = \frac{k}{p}$.

(20) 6. A fair coin is tossed $n \geq 4$ times. Let X be the number of times that the pattern HTH is observed. Using indicators,

(a) compute EX, and

Solution: Let X_i be the indicator of the event that HTH is observed on tosses $i, i+1, i+2$. Then $X = \sum_{i=1}^{n-2} X_i$. It follows that $EX = \frac{n-2}{8}$.

(b) compute $Var(X)$.

Solution:

$$\begin{aligned} Var(X) = Cov(X, X) &= \sum_{i,j=1}^{n-2} Cov(X_i, X_j) \\ &= (n-2)Var(X_1) + 2(n-3)Cov(X_1, X_2) + 2(n-4)Cov(X_1, X_3) \\ &= \frac{7}{64}(n-2) - \frac{1}{64}(2n-6) + \frac{1}{64}(2n-8) = \frac{7n-16}{64}. \end{aligned}$$