Math 131AH - Week 9
Textbook pages: 120-133.
Topics covered:

- Partitions
- Piecewise constant functions
- Upper and lower Riemann sums
- Riemann integrability of continuous functions
- Riemann integrability of monotone functions
- Piecewise continuous functions
- Other properties of the Riemann integral

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Integration

- In the last week's notes we reviewed differentiation - one of the two pillars of single variable calculus. The other pillar is, of course, integration, which we will turn to next. (Strictly speaking, we will turn to the *definite integral*, the integral of a function on a fixed interval, as opposed to the *indefinite integral*, otherwise known as the antiderivative. These two are of course linked by the *Fundamental theorem of calculus*, of which more will be said later).

- For us, the definite integral will start with an interval I which could be open, closed, or half-open, and a function $f : I \to \mathbf{R}$, and give us a number $\int_I f$; we can expand this integral as $\int_I f(x)\ dx$ (of course, we could replace x by any other dummy variable), or if I has endpoints a and b, we shall also write this integral as $\int_a^b f$ or $\int_a^b f(x)\ dx$.

- To actually *define* this integral $\int_I f$ is somewhat delicate (especially if one does not want to assume any axioms concerning geometric notions such as area), and not all functions f are integrable. It turns out that there are in fact two ways to define this integral: the *Riemann integral*, which we will do here and which suffices for most applications, and the *Lebesgue integral*, which supercedes the Riemann integral and works for a much larger class of functions. (There is also the *Riemann-Steiltjes integral*, which generalizes the Riemann integral $\int_I f(x)\ dx$ with a more general type of integral $\int_I f(x)\ dg(x)$, but we will only briefly discuss that generalization here, and refer the reader to the textbook for more detail).

- Our strategy in defining the Riemann integral is as follows. We begin by first defining a notion of integration on a very simple class of functions - the *piecewise constant* functions. These functions are quite primitive, but their advantage is that integration is very easy for these functions, as is verifying all the usual properties. Then, we handle more general functions by approximating them by piecewise constant functions.

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Partitions

- We need a certain amount of machinery to begin with.

- **Definition** Let X be a subset of $\mathbf{R}$. We say that X is *connected* iff the following property is true: whenever x, y are elements in X such that $x < y$, the interval $[x, y]$ is a subset of X (i.e. every number between x and y is also in X).

- (Note: In Math 121 (Introduction to topology) you will find another, much more powerful, notion of connectedness, which generalizes this one-dimensional notion, but the definition here will suffice for this course).

- **Examples.** The set $[1, 2]$ is connected, because if $x < y$ both lie in $[1, 2]$, then $1 \leq x < y \leq 2$, and so every element between x and y also lies in $[1, 2]$. A similar argument shows that the set $(1, 2)$ is connected. However, the set $[1, 2] \cup [3, 4]$ is not connected (why?). The real line is connected (why?). The empty set, as well as singleton sets such as $\{3\}$, are connected, but for rather trivial reasons (these sets do not contain two elements x, y for which $x < y$).

- **Definition** A *generalized interval* is a subset I of $\mathbf{R}$ which is either an interval (i.e. a set of the form $[a, b]$, (a, b), $[a, b)$, or $(a, b]$); a point $\{a\}$; or the empty set $\emptyset$.

- **Lemma 1** Let X be a subset of the real line. Then the following two statements are equivalent:

- (a) X is bounded and connected.

- (b) X is a generalized interval.

- **Proof.** See Week 9 homework. □

- **Corollary 2** If I and J are generalized intervals, then the intersection $I \cap J$ is also a generalized interval.

- **Proof.** See Week 9 homework. □

- For instance, the intersection of the generalized intervals $[2, 4]$ and $[4, 6]$ is $\{4\}$, which is also a generalized interval. The intersection of $(2, 4)$ and $(4, 6)$ is $\emptyset$.

- We now give each generalized interval a length.

- **Definition** If I is a generalized interval, we define the *length* of I, denoted $|I|$ as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $|I| := b - a$. Otherwise, if I is a point or the empty set, we define $|I| = 0$.

- For instance, the length of $[3, 5]$ is 2, as is the length of $(3, 5)$; meanwhile, the length of $\{5\}$ or the empty set is 0.

- **Definition** Let I be a generalized interval. A *partition* of I is a finite set $\mathbf{P}$ of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in $\mathbf{P}$.

- Note that a partition is a set of generalized intervals, while each generalized interval is itself a set of real numbers. Thus a partition is a set consisting of other sets.

- **Examples** The set $\mathbf{P} = \{\{1\}, (1,3), [3,5), \{5\}, (5,8], \emptyset\}$ of generalized intervals is a partition of $[1,8]$, because all the generalized intervals in $\mathbf{P}$ lie in $[1,8]$, and each element of $[1,8]$ lies in exactly one generalized interval in $\mathbf{P}$. Note that one could have removed the empty set from $\mathbf{P}$ and still obtain a partition. However, the set $\{[1,4],[3,5]\}$ is not a partition of $[1,5]$ because some elements of $[1,5]$ are included in more than one generalized interval in the set. The set $\{(1,3),(3,5)\}$ is not a partition of $(1,5)$ because some elements of $(1,5)$ are not icluded in any generalized interval in the set. The set $\{(0,3),[3,5)\}$ is not a partition of $(1,5)$ because some intervals in the set are not contained in $(1,5)$.

- Now we come to a basic property about length: it is *finitely additive.*

- **Theorem 3.** Let I be a generalized interval, n be a natural number, and let $\mathbf{P}$ be a partition of I of cardinality n. Then

$$|I| = \sum_{J \in \mathbf{P}} |J|.$$

- **Proof.** We prove this by induction on n. More precisely, we let $P(n)$ be the property that whenever I is a generalized interval, and whenever $\mathbf{P}$ is a partition of I with cardinality n, that $|I| = \sum_{J \in \mathbf{P}} |J|$.

- The base case $P(0)$ is trivial; the only way that I can be partitioned into an empty partition is if I is itself empty (why?), at which point the claim is easy. The case $P(1)$ is also very easy; the only way that I can be partitioned into a singleton set $\{J\}$ is if $J = I$ (why?), at which point the claim is again very easy.

- Now suppose inductively that $P(n)$ is true for some $n \geq 1$, and now we prove $P(n++)$. Let I be a generalized interval, and let $\mathbf{P}$ be a partition of I of cardinality $n+1$.

- If I is the empty set or a point, then all the intervals in $\mathbf{P}$ must also be either the empty set or a point (why?), and so everybody has length zero and the claim is trivial. Thus we will assume that I is an interval of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$.

- Let us first suppose that $b \in I$, i.e. I is either $(a, b]$ or $[a, b]$. Since $b \in I$, we know that one of the intervals K in $\mathbf{P}$ contains b. Since K is contained in I, it must therefore be of the form $(c, b]$, $[c, b]$, or $\{b\}$ for some real number c, with $a \leq c \leq b$ (in the latter case $K = \{b\}$, we set $c := b$). In particular, this means that the set $I - K$ is also a generalized interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. Either way, we easily see that

  $$|I| = |K| + |I - K|.$$

  On the other hand, since $\mathbf{P}$ form a partition of I, then $\mathbf{P} - \{K\}$ forms a partition of $I - K$ (why?). By the induction hypothesis, we thus have

  $$|I - K| = \sum_{J \in \mathbf{P} - \{K\}} |J|.$$

  Combining these two identities (using the laws of addition for finite sets, see Proposition 10 of Week 5 notes) we obtain

  $$|I| = \sum_{J \in \mathbf{P}} |J|$$

  as desired.

- Now suppose that $b \notin I$, i.e. I is either (a, b) or $[a, b)$. Then one of the intervals K also is of the form (c, b) or $[c, b)$ (see homework). In particular, this means that the set $I - K$ is also a generalized interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. The rest of the argument then proceeds as above. $\square$

- There are two more things we need to do with partitions. One is to say when one partition is finer than another, and the other is to talk about the common refinement of two partitions.

- **Definition** Let I be a generalized interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We say that $\mathbf{P}'$ is *finer* than $\mathbf{P}$ (or equivalently, that $\mathbf{P}$ is *coarser* than $\mathbf{P}'$) if for every J in $\mathbf{P}'$, there exists a K in $\mathbf{P}$ such that $J \subseteq K$.

- **Example.** The partition $\{[1,2), \{2\}, (2,3), [3,4]\}$ is finer than $\{[1,2], (2,4]\}$ (why?). Both partitions are finer than $\{[1,4]\}$ (which is the coarsest partition of all). Note that there is no such thing as a "finest" partition, since all partitions are assumed to be finite.

- **Definition** Let I be a generalized interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We define the *common refinement* $\mathbf{P}\#\mathbf{P}'$ of $\mathbf{P}$ and $\mathbf{P}'$ to be the set

$$\mathbf{P}\#\mathbf{P}' := \{K \cap J : K \in \mathbf{P} \text{ and } J \in \mathbf{P}'\}.$$

- **Example.** Let $\mathbf{P} := \{[1,3), [3,4]\}$ and $\mathbf{P}' := \{[1,2], (2,4]\}$ be two partitions of $[1,4]$. Then $\mathbf{P}\#\mathbf{P}'$ is the set $\{[1,2], (2,3), [3,4], \emptyset\}$ (why?).

- **Lemma 4.** Let I be a generalized interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. Then $\mathbf{P}\#\mathbf{P}'$ is also a partition of I, and is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$.

- **Proof.** See Week 9 homework. □

Piecewise constant functions

- We can now describe the class of "simple" functions for which we can integrate very easily.

- **Definition** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *constant* iff there exists a real number c such that $f(x) = c$ for all $x \in X$. If E is a subset of X, we say that f is *constant on* E if the restriction $f|_E$ of f to E is constant, in other words there exists a real number c such that $f(x) = c$ for all $x \in E$. We refer to c as the *constant value* of f on E.

- Note that if E is a non-empty set, then a function f which is constant on f can have only one constant value; it is not possible for a function to always equal 3 on E while simultaneously always equaling 4. However, if E is empty, every real number c is a constant value for f on E (why?).

- **Definition** Let I be a generalized interval, let $f : I \to \mathbf{R}$ be a function, and let $\mathbf{P}$ be a partition of I. We say that f is *piecewise constant with respect to* $\mathbf{P}$ if for every $J \in \mathbf{P}$, f is constant on J.

- **Example.** The function $f : [1,6] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 7 & \text{if } 1 \leq x < 3 \\ 4 & \text{if } x = 3 \\ 5 & \text{if } 3 < x < 6 \\ 2 & \text{if } x = 6 \end{cases}$$

  is piecewise constant with respect to the partition $\{[1,3), \{3\}, (3,6), \{6\}\}$ of $[1,6]$. Note that it is also piecewise constant with respect to some other partitions as well; for instance, it is piecewise constant with respect to the partition $\{[1,2), \{2\}, (2,3), \{3\}, (3,5), [5,6), \{6\}, \emptyset\}$.

- **Definition** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a function. We say that f is *piecewise constant on I* if there exists a partition $\mathbf{P}$ of I such that f is piecewise constant with respect to $\mathbf{P}$.

- **Example** The function used in the previous example is piecewise constant on $[1,6]$. Also, every constant function on a generalized interval I is automatically piecewise constant also (why?).

- **Lemma 5.** Let I be a generalized interval, let $\mathbf{P}$ be a partition of I, and let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Let $\mathbf{P}'$ be a partition of I which is finer than $\mathbf{P}$. Then f is also piecewise constant with respect to $\mathbf{P}'$.

- **Proof.** See Week 9 homework. □

- The space of piecewise constant functions is closed under algebraic operations:

- **Lemma 6.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be piecewise constant functions on I. Then the functions $f + g$, $f - g$, $\max(f, g)$ and fg are also piecewise constant functions on I. (Note: $\max(f,g) : I \to \mathbf{R}$ is the function $\max(f,g)(x) := \max(f(x), g(x))$). If g does not vanish anywhere on I (i.e. $g(x) \neq 0$ for all $x \in I$) then f/g is also a piecewise constant function on I.

- **Proof.** See Week 9 homework. □

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Integration of piecewise constant functions

- We are now ready to integrate piecewise constant functions. We begin with a temporary definition of an integral with respect to a partition.

- **Definition.** Let I be a generalized interval, let $\mathbf{P}$ be a partition of I. Let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Then we define the *piecewise constant integral p.c.* $\int_{[\mathbf{P}]} f$ of f with respect to the partition $\mathbf{P}$ by the formula

$$p.c. \int_{[\mathbf{P}]} f := \sum_{J \in \mathbf{P}} c_J |J|,$$

  where for each J in $\mathbf{P}$, we let c_J be the constant value of f on J.

- Note that this definition may seem potentially ill-defined, because if J is empty then every number c_J can be the constant value of f on J, but fortunately in such cases $|J|$ is zero and so the choice of c_J is irrelevant. The notation $p.c. \int_{[\mathbf{P}]} f$ is rather artificial, but we shall only need it temporarily, en route to a more useful definition. Note that since $\mathbf{P}$ is finite, the sum $\sum_{J \in \mathbf{P}} c_J |J|$ is always well-defined (it is never divergent or infinite).

- **Example.** Let $f : [1, 4] \to \mathbf{R}$ be the function

$$f(x) = \begin{cases} 2 & \text{if } 1 \leq x < 3 \\ 4 & \text{if } x = 3 \\ 6 & \text{if } 3 < x \leq 4 \end{cases}$$

  and let $\mathbf{P} := \{[1, 3), \{3\}, (3, 4]\}$. Then

$$p.c. \int_{[\mathbf{P}]} f = c_{[1,3)}|[1, 3)| + c_{\{3\}}|\{3\}| + c_{(3,4]}|(3, 4]| = 2 \times 2 + 4 \times 0 + 6 \times 1 = 10.$$

  Alternatively, if we let $\mathbf{P}' := \{[1, 2), [2, 3), \{3\}, (3, 4], \emptyset\}$ then

$$p.c. \int_{[\mathbf{P}']} f = c_{[1,2)}|[1, 2)| + c_{[2,3)}|[2, 3)| + c_{\{3\}}|\{3\}| + c_{(3,4]}|(3, 4]| + c_{\emptyset}|\emptyset|$$

$$= 2 \times 1 + 2 \times 1 + 4 \times 0 + 6 \times 1 + c_{\emptyset} \times 0 = 10.$$

- Note that the piecewise constant integral corresponds intuitively to one's notion of area, given that the area of a rectangle ought to be the product of the lengths of the sides. (Of course, if f is negative somewhere, then the "area" $c_J|J|$ would also be negative).

- This example suggests that this integral does not really depend on what partition you pick, so long as your function is piecewise constant with respect to that partition. That is indeed true:

- **Proposition 7.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a function. Suppose that $\mathbf{P}$ and $\mathbf{P}'$ are partitions of I such that f is piecewise constant both with respect to $\mathbf{P}$ and with respect to $\mathbf{P}'$. Then $p.c. \int_{[\mathbf{P}]} f = p.c. \int_{[\mathbf{P}']} f$.

- **Proof.** See Week 9 homework. □

- Because of this proposition, we can now make the following definition:

- **Definition.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a piecewise constant function on I. We define the *piecewise constant integral* $p.c. \int_I f$ by the formula

  $$p.c. \int_I f := p.c. \int_{[\mathbf{P}]} f,$$

  where $\mathbf{P}$ is any partition of I with respect to which f is piecewise constant. (Note that Proposition 7 tells us that the precise choice of this partition is irrelevant).

- **Example** If f is the function given in the previous example, then $p.c. \int_{[1,4]} f = 10$.

- We now give some basic properties of the piecewise constant integral.

- **Theorem 8 (Laws of integration, piecewise constant version).** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be piecewise constant functions on I.

- (a) We have $p.c. \int_I (f + g) = p.c. \int_I f + p.c. \int_I g$.

- (b) For any real number c, we have $p.c. \int_I (cf) = c(p.c. \int_I f)$.
- (c) We have $p.c. \int_I (f - g) = p.c. \int_I f - p.c. \int_I g$.
- (d) If $f(x) \geq 0$ for all $x \in I$, then $p.c. \int_I f \geq 0$.
- (e) If $f(x) \geq g(x)$ for all $x \in I$, then $p.c. \int_I f \geq p.c. \int_I g$.
- (f) If f is the constant function $f(x) = c$ for all x in I, then $p.c. \int_I f = c|I|$.
- (g) Let J be a generalized interval containing I (i.e. $I \subseteq J$), and let $F : J \to \mathbf{R}$ be the function

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

  Then F is piecewise constant on J, and $p.c. \int_J F = p.c. \int_I f$.
- (h) Suppose that $\{J, K\}$ is a partition of I into two generalized interval J and K. Then the functions $f|_J : J \to \mathbf{R}$ and $f|_K : K \to \mathbf{R}$ are piecewise constant on J and K respectively, and we have

$$p.c. \int_I f = p.c. \int_J f|_J + p.c. \int_K f|_K.$$

- **Proof.** See Week 9 homework. □
- This concludes our integration of piecewise constant functions. We now turn to the question of how to integrate bounded functions.

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Upper and lower Riemann integrals

- Now let $f : I \to \mathbf{R}$ be a bounded function defined on a generalized interval I. We want to define the Riemann integral $\int_I f$. To do this we first need to define the notion of upper and lower Riemann integrals $\overline{\int}_I f$ and $\underline{\int}_I f$. These notions are related to the Riemann integral in much the same way that the lim sup and lim inf of a sequence are related to the limit of that sequence.

- **Definition.** Let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$. We say that g *majorizes* f *on* I if we have $g(x) \geq f(x)$ for all $x \in I$, and that g *minorizes of* f *on* I if $g(x) \leq f(x)$ for all $x \in I$.

- The idea of the Riemann integral is to try to integrate a function by first majorizing or minorizing that function by a piecewise constant function (which we already know how to integrate).

- **Definition.** Let $f : I \to \mathbf{R}$ be a bounded function defined on a generalized interval I. We define the *upper Riemann integral* $\overline{\int}_I f$ by the formula

  $$\overline{\int}_I f := \inf\{p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which majorizes } f\}$$

  and the *lower Riemann integral* $\underline{\int}_I f$ by the formula

  $$\underline{\int}_I f := \sup\{p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which minorizes } f\}.$$

- **Lemma 9.** Let $f : I \to \mathbf{R}$ be a function on a generalized interval I which is bounded by some real number M, i.e. $-M \leq f(x) \leq M$ for all $x \in I$. Then we have

  $$-M|I| \leq \underline{\int}_I f \leq \overline{\int}_I f \leq M|I|.$$

  In particular, both the lower and upper Riemann integrals are real numbers.

- **Proof.** The function $g : I \to \mathbf{R}$ defined by $g(x) = M$ is constant, hence piecewise constant, and majorizes f; thus $\overline{\int}_I f \leq p.c. \int_I g = M|I|$ by definition of upper Riemann integral. A similar argument gives $-M|I| \leq \underline{\int}_I f$. Finally, we have to show that $\underline{\int}_I f \leq \overline{\int}_I f$. Let g be any piecewise constant function majorizing f, and let h be any piecewise constant function minorizing f. Then g majorizes h, and hence $p.c. \int_I h \leq p.c. \int_I g$. Taking suprema in h, we obtain that $\underline{\int}_I f \leq p.c. \int_I g$. Taking infima in g, we thus obtain $\underline{\int}_I f \leq \overline{\int}_I g$, as desired. $\square$

- We now know that the upper Riemann integral is always greater than or equal to the lower Riemann integral. If the two integrals match, then we can define the Riemann integral:

- **Definition.** Let $f : I \to \mathbf{R}$ be a bounded function on a generalized interval I. If $\underline{\int}_I f = \overline{\int}_I f$, then we say that f is *Riemann integrable on* I and define
$$\int_I f := \underline{\int}_I f = \overline{\int}_I f.$$
If the upper and lower Riemann integrals are unequal, we say that f is not Riemann integrable.

- Compare this to the relationship between the lim sup, lim inf, and limit of a sequence a_n; the lim sup is always greater than or equal to the lim inf, but they are only equal when the sequence converges, in which case they are both equal to the limit of that sequence.

- Note that we do not consider unbounded functions to be Riemann integrable; an integral involving such functions is known as an *improper integral*. It is possible to still evaluate such integrals using more sophisticated integration methods (such as the Lebesgue integral), but this will be deferred until Math 131B.

- The Riemann integral generalizes the piecewise constant integral:

- **Lemma 10.** Let $f : I \to \mathbf{R}$ be a piecewise constant function on a generalized integral I. Then f is Riemann integrable, and $\int_I f = p.c. \int_I f$.

- **Proof.** See Week 9 homework. □

- One special case of Lemma 10: if I is a point or the empty set, then $\int_I f = 0$ for all functions $f : I \to \mathbf{R}$ (note that all such functions are automatically constant).

- Thus every piecewise constant function is Riemann integrable. However, the Riemann integral is more general, and can integrate a wider class of functions; we shall see this shortly.

- The above definition may seem strange, compared to the Riemann sum definition you may have been exposed to in lower-division. However, the two definitions turn out to be equivalent; this is the purpose of the next section.

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Connection with Riemann sums

- In this section we relate the above definition to Riemann sums.

- **Definition** Let $f : I \to \mathbf{R}$ be a bounded function on a generalized integral I, and let $\mathbf{P}$ be a partition of I. We define the *upper Riemann sum* $U(f, \mathbf{P})$ and the *lower Riemann sum* $L(f, \mathbf{P})$ by

$$U(f, \mathbf{P}) := \sum_{J \in \mathbf{P}: J \neq \emptyset} (\sup_{x \in J} f(x))|J|$$

  and

$$L(f, \mathbf{P}) := \sum_{J \in \mathbf{P}: J \neq \emptyset} (\inf_{x \in J} f(x))|J|.$$

- The restriction $J \neq \emptyset$ is required because the quantities $\inf_{x \in J} f(x)$ and $\sup_{x \in J} f(x)$ are infinite (or negative infinite) if J is empty.

- We now connect these Riemann sums to the upper and lower Riemann integral.

- **Lemma 11.** Let $f : I \to \mathbf{R}$ be a bounded function on a generalized interval I, and let g be a function which majorizes f and which is piecewise constant with respect to some partition $\mathbf{P}$ of I. Then

$$p.c. \int_I g \geq U(f, \mathbf{P}).$$

  Similarly, if h is a function which minorizes f and is piecewise constant with respect to $\mathbf{P}$, then

$$p.c. \int_I h \leq L(f, \mathbf{P}).$$

- **Proof.** See Week 9 homework. □

- **Proposition 12.** Let $f : I \to \mathbf{R}$ be a bounded function on a generalized interval I. Then

$$\overline{\int}_I f = \inf\{U(f, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

and

$$\underline{\int}_I f = \sup\{L(f, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

- **Proof.** See Week 9 homework. □

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Basic properties of the Riemann integral

- **Theorem 13 (Laws of integration).** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ and $g : I \to \mathbf{R}$ be Riemann integrable functions on I.

- (a) The function $f + g$ is Riemann integrable, and we have $\int_I (f + g) = \int_I f + \int_I g$.

- (b) For any real number c, the function cf is Riemann integrable, and we have $\int_I (cf) = c(\int_I f)$.

- (c) The function $f - g$ is Riemann integrable, and we have $\int_I (f - g) = \int_I f - \int_I g$.

- (d) If $f(x) \geq 0$ for all $x \in I$, then $\int_I f \geq 0$.

- (e) If $f(x) \geq g(x)$ for all $x \in I$, then $\int_I f \geq \int_I g$.

- (f) If f is the constant function $f(x) = c$ for all x in I, then $\int_I f = c|I|$.

- (g) Let J be a generalized interval containing I (i.e. $I \subseteq J$), and let $F : J \to \mathbf{R}$ be the function

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then F is Riemann integrable on J, and $\int_J F = \int_I f$.

- (h) Suppose that $\{J, K\}$ is a partition of I into two generalized interval J and K. Then the functions $f|_J : J \to \mathbf{R}$ and $f|_K : K \to \mathbf{R}$ are Riemann integrable on J and K respectively, and we have

$$\int_I f = \int_J f|_J + \int_K f|_K.$$

- **Proof.** See Week 9 homework. $\square$

- A remark on Lemma 10: Because of this proposition, we will not refer to the piecewise constant integral *p.c.* $\int_I$ again, and just use the Riemann integral $\int_I$.

- A remark on Theorem 13(h): We often abbreviate $\int_J f|_J$ as $\int_J f$, even though f is really defined on a larger domain than just J.

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Riemann integrability of continuous functions

- We have already said a lot about Riemann integrable functions, but we have not yet actually produced any such functions other than the piecewise constant ones. Now we rectify this by showing that a large class of useful functions are Riemann integrable.

- **Theorem 14.** Let I be a generalized interval, and let f be a function which is uniformly continuous on I. Then f is Riemann integrable.

- **Proof.** From Proposition 8 of Week 7/8 notes we see that f is bounded. Now we have to show that $\underline{\int}_I f = \overline{\int}_I f$.

- If I is a point or the empty set then the Theorem is trivial, so let us assume that I is an interval, say one of the four intervals $[a, b]$, (a, b), $(a, b]$, or $[a, b)$.

- Let $\varepsilon > 0$ be arbitrary. By uniform continuity, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I$ are such that $|x - y| < \delta$. By the Archimedean principle, there exists an integer $N > 0$ such that $(b - a)/N < \delta$.

- Note that we can partition I into N intervals $J_1, \ldots, J_N$, each of length $(b-a)/N$ (how? One has to treat each of the cases $[a,b]$, (a,b), $(a,b]$, $[a,b)$ slightly differently). By Proposition 12, we thus have

  $$\overline{\int}_I f \leq \sum_{k=1}^{N} (\sup_{x \in J_k} f(x)) |J_k|$$

  and

  $$\underline{\int}_I f \geq \sum_{k=1}^{N} (\inf_{x \in J_k} f(x)) |J_k|$$

  so in particular

  $$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{k=1}^{N} (\sup_{x \in J_k} f(x) - \inf_{x \in J_k} f(x)) |J_k|.$$

  However, we have $|f(x) - f(y)| < \varepsilon$ for all $x, y \in J_k$, since $|J_k| = (b-a)/N < \delta$. In particular we have

  $$f(x) < f(y) + \varepsilon \text{ for all } x, y \in J_k.$$

  Taking suprema in x, we obtain

  $$\sup_{x \in J_k} f(x) \leq f(y) + \varepsilon \text{ for all } y \in J_k,$$

  and then taking infima in y we obtain

  $$\sup_{x \in J_k} f(x) \leq \inf_{y \in J_k} f(y) + \varepsilon.$$

  Inserting this bound into our previous inequality, we obtain

  $$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{k=1}^{N} \varepsilon |J_k|,$$

  but by Theorem 3 we thus have

  $$\overline{\int}_I f - \underline{\int}_I f \leq \varepsilon (b-a).$$

  But $\varepsilon > 0$ was arbitrary, while $(b-a)$ is fixed. Thus $\overline{\int}_I f - \underline{\int}_I f$ cannot be positive. By Lemma 9 and the definition of Riemann integrability we thus have that f is Riemann integrable. □

- Combining Theorem 14 with Theorem 9 from Week 7/8 notes, we thus obtain

- **Corollary 15.** Let $[a, b]$ be a closed interval, and let $f : [a, b] \to \mathbf{R}$ be continuous. Then f is Riemann integrable.

- Note that this theorem is not true if $[a, b]$ is replaced by any other sort of interval, since it is not even guaranteed then that continuous functions are bounded. For instance, the function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) := 1/x$ is continuous but not Riemann integrable. However, if we assume that a function is both continuous *and* bounded, we can recover Riemann integrability:

- **Proposition 16.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be both continuous and bounded. Then f is Riemann integrable on I.

- **Proof.** If I is a point or an empty set then the claim is trivial; if I is a closed interval the claim follows from Corollary 15. So let us assume that I is of the form $(a, b]$, (a, b), or $[a, b)$ for some $a < b$.

- We have a bound M for f, so that $-M \leq f(x) \leq M$ for all $x \in I$. Now let $0 < \varepsilon < (b-a)/2$ be a small number. The function f when restricted to the interval $[a+\varepsilon, b-\varepsilon]$ is continuous, and hence Riemann integrable by Corollary 15. In particular, we can find a piecewise constant function $h : [a + \varepsilon, b - \varepsilon] \to \mathbf{R}$ which majorizes f on $[a + \varepsilon, b - \varepsilon]$ such that

$$\int_{[a+\varepsilon,b-\varepsilon]} h \leq \int_{[a+\varepsilon,b-\varepsilon]} f + \varepsilon.$$

  Defining $\tilde{h} : I \to \mathbf{R}$ by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \in [a+\varepsilon, b-\varepsilon] \\ M & \text{if } x \in I \backslash [a+\varepsilon, b-\varepsilon] \end{cases}$$

  Clearly $\tilde{h}$ is piecewise constant on I and majorizes f; by Theorem 8 we have

$$\int_I \tilde{h} = \varepsilon M + \int_{[a+\varepsilon,b-\varepsilon]} h + \varepsilon M \leq \int_{[a+\varepsilon,b-\varepsilon]} f + (2M+1)\varepsilon.$$

In particular we have

$$\overline{\int}_I f \leq \int_{[a+\varepsilon, b-\varepsilon]} f + (2M+1)\varepsilon.$$

A similar argument gives

$$\underline{\int}_I f \geq \int_{[a+\varepsilon, b-\varepsilon]} f - (2M+1)\varepsilon$$

and hence

$$\overline{\int}_I f - \underline{\int}_I f \leq (4M+2)\varepsilon.$$

But ε is arbitrary, and so we can argue as in the proof of Theorem 14 to conclude Riemann integrability. □

- This gives a large class of Riemann integrable functions already; the bounded continuous functions. But we can expand this class a little more.

- **Definition.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$. We say that f is *piecewise continuous on I* iff there exists a partition $\mathbf{P}$ of I such that $f|_J$ is continuous on J for all $J \in \mathbf{P}$.

- **Example** The function $f : [1,3] \to \mathbf{R}$ defined by

$$F(x) := \begin{cases} x^2 & \text{if } 1 \leq x < 2 \\ 7 & \text{if } x = 2 \\ x^3 & \text{if } 2 < x \leq 3 \end{cases}$$

  is not continuous on $[1,3]$, but it is piecewise continuous on $[1,3]$ (since it is continuous when restricted to $[1,2)$ or $\{2\}$ or $(2,3]$, and those three generalized intervals partition $[1,3]$).

- **Proposition 17.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be both piecewise continuous and bounded. Then f is Riemann integrable.

- **Proof.** By hypothesis, there exists a partition $\mathbf{P}$ of I such that $f|_J$ is continuous on J for all $J \in \mathbf{P}$. In particular $f|_J$ is both continuous and bounded on J, hence Riemann integrable on J. If we then define the function $F_J : I \to \mathbf{R}$ by

$$F_J(x) := \begin{cases} f(x) & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}$$

then F_J is Riemann integrable on I, by Theorem 13(g). But then observe that

$$f = \sum_{J \in \mathbf{P}} F_J$$

(i.e. $f(x) = \sum_{J \in \mathbf{P}} F_J(x)$ for all $x \in I$). By an induction argument using Theorem 13(a) we thus see that f is Riemann integrable on I as desired. □

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Riemann integrability of monotone functions

- In addition to piecewise continuous functions, another wide class of functions is Riemann integrable.

- **Proposition 18.** Let $[a, b]$ be a closed interval and let $f : [a, b] \to \mathbf{R}$ be a monotone function. Then f is Riemann integrable on $[a, b]$.

- Note from Week 8 homework that there exist monotone functions which are not piecewise continuous, so this Proposition is not subsumed by Proposition 17.

- **Proof.** Without loss of generality we may take f to be monotone increasing (instead of monotone decreasing). From Assignment 7 we know that f is bounded. Now let $N > 0$ be an integer, and partition $[a, b]$ into N half-open intervals $\{[a + \frac{b-a}{N} j, a + \frac{b-a}{N}(j + 1)) : 0 \leq j \leq N - 1\}$ of length $(b - a)/N$, together with the point $\{b\}$. Then by Proposition 12 we have

$$\overline{\int}_I f \leq \sum_{j=0}^{N-1} \Big(\sup_{x \in [a + \frac{b-a}{N} j, a + \frac{b-a}{N}(j+1))} f(x) \Big) \frac{b - a}{N},$$

(the point $\{b\}$ clearly giving only a zero contribution). Since f is monotone increasing, we thus have

$$\overline{\int}_I f \leq \sum_{j=0}^{N-1} f(a + \frac{b-a}{N}(j+1))\frac{b-a}{N}.$$

Similarly we have

$$\underline{\int}_I f \geq \sum_{j=0}^{N-1} f(a + \frac{b-a}{N}j)\frac{b-a}{N}.$$

Thus we have

$$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{j=0}^{N-1}(f(a + \frac{b-a}{N}(j+1)) - f(a + \frac{b-a}{N}j))\frac{b-a}{N}.$$

Using telescoping series (see Midterm 2) we thus have

$$\overline{\int}_I f - \underline{\int}_I f \leq (f(a+\frac{b-a}{N}(N)) - f(a+\frac{b-a}{N}0))\frac{b-a}{N} = (f(b)-f(a))\frac{b-a}{N}.$$

But N was arbitrary, so we can conclude as in the proof of Theorem 14 to conclude that f is Riemann integrable. □

- By arguing as in Proposition 16 we thus have
- **Corollary 19.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be both monotone and bounded. Then f is Riemann integrable on I.
- One can then define piecewise monotone functions in analogy with piecewise continuous functions, and conclude that all bounded piecewise monotone functions are Riemann integrable, but we will not bother to spell this out in detail here.

* * * * *

A non-Riemann integrable function

- We conclude these notes with an example of a function which is bounded but not Riemann integrable.

- Let $f : [0,1] \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

As we have seen in earlier notes, this function is not continuous anywhere at $[0,1]$. It turns out that it is not Riemann integrable, either.

- Let $\mathbf{P}$ be any partition of $[0,1]$. For any $J \in \mathbf{P}$, observe that if J is not a point or the empty set, then

$$\sup_{x \in J} f(x) = 1$$

(by Proposition 25 of Week 2). In particular we have

$$(\sup_{x \in J} f(x))|J| = |J|$$

(note this is also true when J is a point, since both sides are zero). In particular we see that

$$U(f, \mathbf{P}) = \sum_{J \in \mathbf{P} : J \neq \emptyset} |J| = [0,1] = 1$$

by Theorem 3 (note that the empty set does not contribute anything to the total length). In particular we have $\overline{\int}_{[0,1]} f = 1$, by Proposition 12.

- A similar argument gives that

$$\sup_{x \in J} f(x) = 0$$

for all J (other than points or the empty set), and so

$$L(f, \mathbf{P}) = \sum_{J \in \mathbf{P} : J \neq \emptyset} 0 = 0.$$

In particular we have $\underline{\int}_{[0,1]} f = 0$, by Proposition 12. Thus the upper and lower Riemann integrals do not match, and so this function is not Riemann integrable.

- (Optional discussion) As you can see, it is only rather "artificial" bounded functions which are not Riemann integrable. Because of this, the Riemann integral is good enough for a large majority of cases. There are ways to generalize or improve this integral, though. One of these is the *Lebesgue integral*, which we will cover in 131B. Another is the *Riemann-Stieltjes integral* $\int_I f d\alpha$, where $\alpha : I \to \mathbf{R}$ is a monotone increasing function. This integral is defined just like the Riemann integral, but with one twist: instead of taking the length $|J|$ of generalized intervals J, we take the α-length $\alpha[J]$, defined as follows. If J is a point or the empty set, then $\alpha[J] := 0$. If J is an interval of the form $[a, b]$, (a, b), $(a, b]$, or $[a, b)$, then $\alpha[J] := \alpha(b) - \alpha(a)$. Note that in the special case where α is the identity function $\alpha(x) := x$, then $\alpha[J]$ is just the same as $|J|$. However, for more general monotone functions α, the α-length $\alpha[J]$ is a different quantity from $|J|$. Nevertheless, we can still do much of the above theory, but replacing $|J|$ by $\alpha[J]$ throughout; see the textbook for a more thorough discussion.