Math 131AH - Weeks 7-8

Textbook pages: 89-93, 97-98, 103-106, 109-110 (Optional reading: 94-97).

Topics covered:

- Maximum principle
- Intermediate value theorem
- Uniform continuity
- Differentiability
- Properties of differentiable functions
- Mean-value theorem
- Inverse function theorem

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The maximum principle

- In last week's notes we introduced the notion of *continuity*. Recall that if X is a subset of $\mathbf{R}$ and $x_0 \in X$, a function $f : X \to \mathbf{R}$ is said to be *continuous* at x_0 iff we have $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$. Equivalently, a function f is continuous at x_0 iff whenever $(x_n)_{n=1}^\infty$ is a sequence of numbers in X converging to x_0, then $(f(x_n))_{n=1}^\infty$ also converges to $f(x_0)$. Another equivalent definition is that a function f is continuous at x_0 iff, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ is ε-close to $f(x_0)$ whenever $x \in X$ is δ-close to x_0. (See Proposition 10 from last week's notes).
- We say that a function $f : X \to \mathbf{R}$ is *continuous* iff it is continuous at every point x_0 in X. As we saw in last week's notes, a large number of functions are continuous. We now show that continuous functions enjoy a number of other useful properties, especially if their domain is a closed interval. This shall mainly be because of the Bolzano-Weierstrass theorem proven in last week's notes.

- **Definition.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *bounded from above* if there exists a real number M such that $f(x) \leq M$ for all $x \in X$. We say that f is *bounded from below* if there exists a real number M such that $f(x) \geq -M$ for all $x \in X$. We say that f is *bounded* if there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$.

- Note that a function is bounded if and only if it is bounded both from above and below. (Why? Note that one part of the "if and only if" is slightly trickier than the other).

- Not all continuous functions are bounded. For instance, the function $f(x) := x$ on the domain $\mathbf{R}$ is continuous but unbounded (why?), although it is bounded on some smaller domains, such as $[1, 2]$. The function $f(x) := 1/x$ is continuous but unbounded on $(0, 1)$ (why?), though it is continuous and bounded on $[1, 2]$ (why?).

- However, if the domain of the continuous function is a closed and bounded interval, then we do have boundedness:

- **Lemma 1.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function continuous on $[a, b]$. Then f is bounded.

- **Proof.** Suppose for contradiction that f is not bounded. Thus for every real number M there exists an element $x \in [a, b]$ such that $|f(x)| > M$.

- In particular, for every natural number n, we can find an element $x_n \in [a, b]$ for which $|f(x_n)| > n$. Fix such a sequence $(x_n)_{n=0}^{\infty}$. (Note: strictly speaking, fixing such a sequence requires the *axiom of choice* in set theory. It is possible to prove this lemma without the axiom of choice; however we will not go into this somewhat delicate issue, and just use the axiom of choice in all our arguments.) This sequence lies in $[a, b]$, and so by the Bolzano-Weierstrass theorem (see last week's notes) there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ which converges to some limit L, where $n_0 < n_1 < n_2 < \ldots$ is an increasing sequence of natural numbers. In particular, we see that $n_j \geq j$ for all $j \in \mathbf{N}$ (why? use induction).

  Since all the x_{n_j} lie in $[a, b]$, and L is the limit of the x_{n_j}, we see that L is adherent to $[a, b]$ (Lemma 5 from last week's notes), and thus must also lie in $[a, b]$ (see Lemma 4 from last week's notes). In particular, L lies in the domain of f. Since f is continuous, it is continuous at L, and in particular we see that

$$\lim_{j \to \infty} f(x_{n_j}) = f(L).$$

In particular, the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is convergent, hence bounded. On the other hand, we know from construction that $|f(x_{n_j})| \geq n_j \geq j$ for all j, and hence this sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is not bounded, contradiction. $\square$

- Note two things about this proof. Firstly, it shows how useful the Bolzano-Weierstrass theorem is. Secondly, it is an indirect proof; it doesn't say *how* to find the bound for f, but it shows that having f unbounded leads to a contradiction.

- We now improve this lemma to say something more.

- **Definition.** Let $f : X \to \mathbf{R}$ be a function, and let $x_0 \in X$. We say that f *attains its maximum at* x_0 if we have $f(x_0) \geq f(x)$ for all $x \in X$ (i.e. the value of f at the point x_0 is larger than or equal to the value of f at any other point in X). We say that f *attains its minimum at* x_0 if we have $f(x_0) \leq f(x)$.

- Note that if a function attains its maximum somewhere, then it must be bounded from above (why?). Similarly if it attains its minimum somewhere, then it must be bounded from below.

- **Maximum principle.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function continuous on $[a, b]$. Then f attains its maximum at some point $x_{max} \in [a, b]$, and also attains its minimum at some point $x_{min} \in [a, b]$.

- (Strictly speaking, "maximum principle" is a misnomer, since the principle also concerns the minimum. Perhaps a more precise name would have been "extremum principle").

- **Proof.** We shall just show that f attains its maximum somewhere; the proof that it attains its minimum also is similar but is left to the reader.

- From Lemma 1 we know that f is bounded, thus there exists an M such that $-M \leq f(x) \leq M$ for each $x \in [a,b]$. Now let E denote the set
$$E := \{f(x) : x \in [a,b]\}.$$
(In other words, $E := f([a,b])$). By what we just said, this set is a subset of $[-M, M]$. It is also non-empty, since it contains for instance the point $f(a)$. Hence by the least upper bound principle, it has a supremum $\sup(E)$ which is a real number.

- Write $m := \sup(E)$. By definition of supremum, we know that $y \leq m$ for all $y \in E$; by definition of E, this means that $f(x) \leq m$ for all $x \in [a,b]$. Thus to show that f attains its maximum somewhere, it will suffice to find an $x_{max} \in [a,b]$ such that $f(x_{max}) = m$ (why will this suffice?).

- Let $n \geq 1$ be any integer. Then $m - \frac{1}{n} < m = \sup(E)$. Since $\sup(E)$ is the least upper bound for E, $m - \frac{1}{n}$ cannot be an upper bound for E, thus there exists a $y \in E$ such that $m - \frac{1}{n} < y$. By definition of E, this implies that there exists an $x \in [a,b]$ such that $m - \frac{1}{n} < f(x)$.

- We now choose a sequence $(x_n)_{n=1}^\infty$ by choosing, for each n, x_n to be an element of $[a,b]$ such that $m - \frac{1}{n} < f(x_n)$. (Again, this requires the axiom of choice; however it is possible to prove this principle without the axiom of choice. For instance, you will see a better proof of this proposition using the notion of *compactness* in Math 121). This is a sequence in $[a,b]$; by the Bolzano-Weierstrass theorem, we can thus find a subsequence $(x_{n_j})_{j=1}^\infty$, where $n_1 < n_2 < \ldots$, which converges to some limit x_{max}. As in Lemma 1, we know that x_{max} is adherent to $[a,b]$ and hence lies in $[a,b]$. Since $(x_{n_j})_{j=1}^\infty$ converges to x_{max}, and f is continuous at x_{max}, we have as before that
$$\lim_{j \to \infty} f(x_{n_j}) = f(x_{max}).$$
On the other hand, by construction we know that
$$f(x_{n_j}) > m - \frac{1}{n_j} \geq m - \frac{1}{j},$$

and so by taking limits of both sides we see that

$$f(x_{max}) = \lim_{j\to\infty} f(x_{n_j}) \geq \lim_{j\to\infty} m - \frac{1}{j} = m.$$

On the other hand, we know that $f(x) \leq m$ for all $x \in [a,b]$, so in particular $f(x_{max}) \leq m$. Combining these two inequalities we see that $f(x_{max}) = m$ as desired. $\square$

- Note that the maximum principle does not prevent a function from attaining its maximum or minimum at more than one point. For instance, the function $f(x) := x^2$ on the interval $[-2,2]$ attains its maximum at two different points, at -2 and at 2.

- Let us write $\sup_{x\in[a,b]} f(x)$ as short-hand for $\sup\{f(x) : x \in [a,b]\}$, and similarly define $\inf_{x\in[a,b]} f(x)$. The maximum principle thus asserts that $m := \sup_{x\in[a,b]} f(x)$ is a real number and is the *maximum value* of f on $[a,b]$, i.e. there is at least one point x_{max} in $[a,b]$ for which $f(x_{max}) = m$, and for every other $x \in [a,b]$, $f(x_{max})$ is less than or equal to m. Similarly $\inf_{x\in[a,b]} f(x)$ is the minimum value of f on $[a,b]$.

- We now know that on a closed interval, every continuous function is bounded and attains its maximum at least once and minimum at least once. The same is not true for open or infinite intervals; see Week 7 homework.

- A final remark; you may encounter a rather different "maximum principle" in complex analysis (Math 132) or partial differential equations (Math 136), involving analytic functions and harmonic functions respectively, instead of continuous functions. Those maximum principles are not directly related to this one (though they are also concerned with whether maxima exist, and where the maxima are located).

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The intermediate value theorem

- We have just shown that a continuous function attains both its maximum value $\sup_{x\in[a,b]} f(x)$ and its minimum value $\inf_{x\in[a,b]} f(x)$. We now show that f also attains every value in between. To do this, we first prove a very intuitive theorem:

- **Intermediate value theorem.** Let $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, i.e. either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = y$.

- **Proof.** We have two cases: $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. We will assume the former, that $f(a) \leq y \leq f(b)$; the latter is proven similarly and is left to the reader.

- If $y = f(a)$ or $y = f(b)$ then the claim is easy (just set $c = a$ or $c = b$), so we will in fact assume that $f(a) < y < f(b)$.

- Let E denote the set

$$E := \{x \in [a, b] : f(x) < y\}.$$

Clearly E is a subset of $[a, b]$, and is hence bounded. Also, since $f(a) < y$, we see that a is an element of E, so E is non-empty. By the least upper bound principle, the supremum

$$c := \sup(E)$$

is thus finite. Since E is bounded by b, we know that $c \leq b$; since E contains a, we know that $c \geq a$. Thus we have $c \in [a, b]$. To complete the proof we now show that $f(c) = y$. The idea is to work from the left of c to show that $f(c) \leq y$, and to work from the right of c to show that $f(c) \geq y$.

- Let $n \geq 1$ be an integer. The number $c - \frac{1}{n}$ is less than $c = \sup(E)$ and hence cannot be an upper bound for E. Thus there exists a point, call it x_n, which lies in E and which is greater than $c - \frac{1}{n}$. Also $x_n \leq c$ since c is an upper bound for E. Thus

$$c - \frac{1}{n} \leq x_n \leq c.$$

By the squeeze test we thus see that $\lim_{n\to\infty} x_n = c$. Since f is continuous at c, this implies that $\lim_{n\to\infty} f(x_n) = f(c)$. But since x_n lies in E for every n, we have $f(x_n) < y$ for every n. By the comparison principle we thus have $f(c) \leq y$. In particular we have $c \neq b$ since $f(b) > f(c)$.

- Since $c \neq b$ and $c \in [a, b]$, we must have $c < b$. In particular there is an $N > 0$ such that $c + \frac{1}{n} < b$ for all $n > N$ (since $c + \frac{1}{n}$ converges to c as $n \to \infty$). Since c is the supremum of E and $c + \frac{1}{n} > c$, we thus have $c + \frac{1}{n} \notin E$ for all $n > N$. Since $c + \frac{1}{n} \in [a, b]$, we thus have $f(c + \frac{1}{n}) \geq y$ for all $n \geq N$. But $c + \frac{1}{n}$ converges to c, and f is continuous at c, thus $f(c) \geq y$. But we already knew that $f(c) \leq y$, thus $f(c) = y$, as desired. $\square$

- The intermediate value theorem says that if f takes the values $f(a)$ and $f(b)$, then it must also take all the values in between. Note that if f is not assumed to be continuous, then the intermediate value theorem no longer applies. For instance, if $f : [-1, 1] \to \mathbf{R}$ is the function

$$f(x) := \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

  then $f(-1) = -1$, and $f(1) = 1$, but there is no $c \in [-1, 1]$ for which $f(c) = 0$. Thus if a function is discontinuous, it can "jump" past intermediate values; however continuous functions cannot do so.

- Note that a continuous function may take an intermediate value multiple times. For instance, if $f : [-2, 2] \to \mathbf{R}$ is the function $f(x) := x^3 - x$, then $f(-2) = -6$ and $f(2) = 6$, so we know that there exists a $c \in [-2, 2]$ for which $f(c) = 0$. In fact, in this case there exists three such values of c: we have $f(-1) = f(0) = f(1) = 0$.

- The intermediate value theorem gives another way to show that one can take n^{th} roots of a number. For instance, to construct the square root of 2, consider the function $f : [0, 2] \to \mathbf{R}$ defined by $f(x) = x^2$. This function is continuous, with $f(0) = 0$ and $f(2) = 4$. Thus there exists a $c \in [0, 2]$ such that $f(c) = 2$, i.e. $c^2 = 2$. (This argument does not show that there is just one square root of 2, but it does prove that there is *at least* one square root of 2).

- **Corollary 2.** Let $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x \in [a,b]} f(x)$ be the maximum value of f, and let $m := \inf_{x \in [a,b]} f(x)$ be the minimum value. Let y be a real number between m and M (i.e. $m \leq y \leq M$). Then there exists a $c \in [a, b]$ such that $f(c) = y$. Furthermore, we have $f([a, b]) = [m, M]$.

- **Proof.** See Week 7 homework. $\square$

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Monotonic functions

- We now discuss a class of functions which is distinct from the class of continuous functions, but has somewhat similar properties: the class of monotone (or monotonic) functions.

- **Definition** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *monotone increasing* iff $f(y) \geq f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *strictly monotone increasing* iff $f(y) > f(x)$ whenever $x, y \in X$ and $y > x$. Similarly, we say f is *monotone decreasing* iff $f(y) \leq f(x)$ whenever $x, y \in X$ and $y > x$, and *strictly monotone decreasing* iff $f(y) < f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *monotone* if it is monotone increasing or monotone decreasing, and *strictly monotone* if it is strictly monotone increasing or strictly monotone decreasing.

- **Examples.** The function $f(x) := x^2$, when restricted to the domain $[0, \infty)$, is strictly monotone increasing (why?), but when restricted instead to the domain $(-\infty, 0]$, is strictly monotone decreasing (why?). Thus the function is strictly monotone on both $(-\infty, 0]$ and $[0, \infty)$, but is not strictly monotone (or monotone) on the full real line $(-\infty, \infty)$. Note that if a function is strictly monotone on a domain X, it is automatically monotone as well on the same domain X. The constant function $f(x) := 6$, when restricted to an arbitrary domain $X \subseteq \mathbf{R}$, is both monotone increasing and monotone decreasing, but is not strictly monotone (unless X consists of at most one point - why?).

- Continuous functions are not necessarily monotone (cf. the function $f(x) = x^2$ on $\mathbf{R}$), and monotone functions are not necessarily continuous; for instance, consider the function $f : [-1, 1] \to \mathbf{R}$ defined earlier by

$$f(x) := \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

  Monotone functions obey the Maximum principle, but not the intermediate value principle (See Week 7 homework).

- It is possible for a monotone function to have many, many discontinuities. In Week 7 homework we construct an example of a function which has a discontinuity at every rational point.

- If a function is both strictly monotone and continuous, then it has many nice properties:

- **Proposition 3.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse $f^{-1} : [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing.

- **Proof.** See Week 7 homework. $\square$

- There is a similar Proposition for functions which are strictly monotone decreasing; we leave it to the reader to work out what it is.

- For example, let n be a positive integer and $R > 0$. Since the function $f(x) := x^n$ is strictly increasing on the interval $[0, R]$, we see from Proposition 3 that this function is a bijection from $[0, R]$ to $[0, R^n]$, and hence there is an inverse from $[0, R^n]$ to $[0, R]$. This can be used to give an alternate means to construct the n^{th} root $x^{1/n}$ of a number $x \in [0, R]$ than what was done in the Week 5 notes.

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Uniform continuity

- We know that a continuous function on a closed interval $[a, b]$ remains bounded (and in fact attains its maximum and minimum, by the Maximum principle). However, if we replace the closed interval by an open interval, then continuous functions need not be bounded any more. An example is the function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$. This function is continuous at every point in $(0, 2)$, and is hence continuous at $(0, 2)$, but is not bounded.

- Informally speaking, the problem here is that while the function is indeed continuous at every point in the open interval $(0, 2)$, it becomes "less and less" continuous as one approaches the endpoint 0.

- Let us analyze this phenomenon further, using the "epsilon-delta" definition of continuity - Proposition 10(c) from Week 6 notes. We know that if $f : X \to \mathbf{R}$ is continuous at a point x_0, then for every $\varepsilon > 0$ there exists a δ such that $f(x)$ will be ε-close to $f(x_0)$ whenever $x \in X$ is δ-close to x_0. In other words, we can force $f(x)$ to ε-close to $f(x_0)$ if we ensure that x is sufficiently close to x_0. One way of thinking about this is that around every point x_0 there is an "island of stability" $(x_0 - \delta, x_0 + \delta)$, where the function $f(x)$ doesn't stray by more than ε from $f(x_0)$.

- For instance, take the function $f(x) := 1/x$ mentioned above at the point $x_0 = 1$. In order to ensure that $f(x)$ is 0.1-close to $f(x_0)$, it suffices to take x to be 1/11-close to x_0, since if x is 1/11-close to x_0 then $10/11 < x < 12/11$, and so $11/12 < f(x) < 11/10$, and so $f(x)$ is 0.1-close to $f(x_0)$. Thus the "δ" one needs to make $f(x)$ 0.1-close to $f(x_0)$ is about 1/11 or so, at the point $x_0 = 1$.

- Now let us look instead at the point $x_0 = 0.1$. The function $f(x) = 1/x$ is still continuous here, but we shall see the continuity is much worse. In order to ensure that $f(x)$ is 0.1-close to $f(x_0)$, we need x to be 1/1010-close to x_0 (if x is 1/1010 close to x_0, then $10/101 < x < 102/1010$, and so $9.901 < f(x) < 10.1$, so $f(x)$ is 0.1-close to $f(x_0)$. Thus one needs a much smaller "δ" for the same value of ε - i.e. $f(x)$ is much more "unstable" near 0.1 than it is near 1, in the sense that there is a much smaller "island of stability" around 0.1 as there is around 1 (if one is interested in keeping $f(x)$ 0.1-stable).

- On the other hand, there are other continuous functions which do not exhibit this behavior. Consider the function $g : (0, 2) \to \mathbf{R}$ defined by $g(x) := 2x$. Let us fix $\varepsilon = 0.1$ as before, and investigate the island of stability around $x_0 = 1$. It is clear that if x is 0.05-close to x_0, then $g(x)$ is 0.1-close to $g(x_0)$; in this case we can take δ to be 0.05 at $x_0 = 1$. And if we move x_0 around, say if we set x_0 to 0.1 instead, the δ does not change - even when x_0 is set to 0.1 instead of 1, we see that $g(x)$ will stay 0.1-close to $g(x_0)$ whenever x is 0.05-close to x_0. Indeed, the same δ works for every x_0. When this happens, we say that the function g is *uniformly continuous*. More precisely:

- **Definition** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x, x_0 \in X$ are two points in X which are δ-close.

- In contrast, a function f is merely *continuous* if for every $\varepsilon > 0$, and every $x_0 \in X$, there is a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x \in X$ is δ-close to x_0. The difference between uniform continuity and continuity is that in uniform continuity one can take a single δ which works for all $x_0 \in X$; for ordinary continuity, each $x_0 \in X$ might use a different δ. Thus every uniformly continuous function is continuous, but not conversely.

- **Example.** The function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$ is continuous on $(0, 2)$, but not uniformly continuous, because the continuity (or more precisely, the dependence of δ on ε) becomes worse and worse as $x \to 0$. (We will make this more precise later on, and give a rigorous proof that this function is not uniformly continuous).

- Recall that the notions of adherent point and of continuous function had several equivalent formulations; both had "epsilon-delta" type formulations (involving the notion of ε-closeness), and both had "sequential" formulations (involving the convergence of sequences). See Lemma 5 and Proposition 6 from last week's notes. The concept of uniform convergence can similarly be phrased in a sequential formulation, this time using the concept of *equivalent sequences*. We have not had to deal much with this concept since Week 2, so let us review it again (this time using sequences of real numbers instead of rationals):

- **Definition.** Let $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ be two sequences of real numbers, m be an integer, and let $\varepsilon > 0$. We say that $(a_n)_{n=m}^\infty$ is *ε-close* to $(b_n)_{n=m}^\infty$ iff a_n is ε-close to b_n for each $n \geq m$. We say that $(a_n)_{n=m}^\infty$ is *eventually ε-close* to $(b_n)_{n=0}^\infty$ iff there exists an $N \geq m$ such that the sequences $(a_n)_{n=N}^\infty$ and $(b_n)_{n=N}^\infty$ are ε-close. Two sequences $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are *equivalent* iff for each $\varepsilon > 0$, the sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ are eventually ε-close.

- Note that one could quibble about whether ε should be assumed to be rational or real, but as discussed in Week 3/4 notes (cf. Proposition

15 from those notes) this does not make any difference to the above definitions.

- The notion of equivalence can now be phrased more succinctly using our language of limits:

- **Lemma 4.** Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences of real numbers (not necessarily bounded or convergent). Then $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are equivalent if and only if $\lim_{n\to\infty}(a_n - b_n) = 0$.

- **Proof.** See Week 7 homework. $\square$

- Meanwhile, the notion of uniform continuity can be phrased using equivalent sequences:

- **Proposition 5.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Then the following two statements are equivalent (i.e. (a) is true if and only if (b) is true):

- (a) f is uniformly continuous on X.

- (b) For every two *equivalent* sequences $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ consisting of elements of X, the sequences $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ are also equivalent sequences.

- **Proof.** See Week 7 homework. $\square$

- The reader should compare this with Proposition 6 from last week's notes. That Proposition asserted that if f was continuous, then f maps convergent sequences to convergent sequences. In contrast, Proposition 5 here asserts that if f is *uniformly* continuous, then f maps *equivalent* pairs of sequences to equivalent pairs of sequences. (To see how the two Propositions are connected, observe that $(x_n)_{n=0}^\infty$ will converge to x_* if and only if the sequences $(x_n)_{n=0}^\infty$ and $(x_*)_{n=0}^\infty$ are equivalent.)

- **Example.** Consider the function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$ considered earlier. The sequence $(1/n)_{n=1}^\infty$ and $(1/2n)_{n=1}^\infty$ are equivalent sequences in $(0, 2)$ (why? Use Lemma 4). However, the sequences $(f(1/n))_{n=1}^\infty$ and $(f(1/2n))_{n=1}^\infty$ are not equivalent (why? Use Lemma 4 again). So by Proposition 5, f is not uniformly continuous. (These sequences start at 1 instead of 0, but the reader can easily see that this makes no difference to the above discussion).

- **Example.** Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$. This is a continuous function on $\mathbf{R}$, but it turns out to not be uniformly continuous; in some sense the continuity gets "worse and worse" as one approaches infinity. One way to quantify this is via Proposition 5. Consider the sequences $(n)_{n=1}^\infty$ and $(n + \frac{1}{n})_{n=1}^\infty$. By Lemma 4, these sequences are equivalent. But the sequences $(f(n))_{n=1}^\infty$ and $(f(n + \frac{1}{n}))_{n=1}^\infty$ are not equivalent, since $f(n+\frac{1}{n}) = n^2 + 2 + \frac{1}{n^2} = f(n) + 2 + \frac{1}{n^2}$ does not become eventually 2-close to $f(n)$. By Proposition 5 we can thus conclude that f is not uniformly continuous.

- Another property of uniformly continuous functions is that they map Cauchy sequences to Cauchy sequences.

- **Proposition 6.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a uniformly continuous function. Let $(x_n)_{n=0}^\infty$ be a Cauchy sequence consisting entirely of elements in X. Then $(f(x_n))_{n=0}^\infty$ is also a Cauchy sequence.

- **Proof.** See Week 7 homework. □

- **Example.** Once again, we demonstrate that the function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$ is not uniformly continuous. The sequence $(1/n)_{n=1}^\infty$ is a Cauchy sequence in $(0, 2)$, but the sequence $(f(1/n))_{n=1}^\infty$ is not a Cauchy sequence (why?). Thus by Lemma 6, f is not uniformly continuous.

- **Corollary 7.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a uniformly continuous function, and let x_0 be an adherent point of X. Then the limit $\lim_{x \to x_0 : x \in X} f(x)$ exists (in particular, it is a real number).

- **Proof.** See Week 7 homework. □

- Again, we could use this Corollary (or the next Proposition) to show once again that the function $f : (0, 2) \to \mathbf{R}$ defined by $f(x) := 1/x$ is not uniformly continuous; this time we leave it to the reader.

- We now show that uniformly continuous functions map bounded sets to bounded sets.

- **Proposition 8.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a uniformly continuous function. Suppose that E is a bounded subset of X. Then $f(E)$ is also bounded.

- **Proof.** See Week 7 homework. $\square$

- As we have just seen repeatedly, not all continuous functions are uniformly continuous. However, if the domain of the function is a closed interval, then continuous functions are in fact uniformly continuous:

- **Theorem 9.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is continuous on $[a, b]$. Then f is also uniformly continuous.

- **Proof. (Optional)** Suppose for contradiction that f is not uniformly continuous. By Proposition 5, there must therefore exist two equivalent sequences $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ in $[a, b]$ such that the sequences $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ are not equivalent. In particular, we can find an $\varepsilon > 0$ such that $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ are not eventually ε-close.

- Fix this value of ε, and let E be the set

$$E := \{n \in N : f(x_n) \text{ and } f(y_n) \text{ are not } \varepsilon\text{-close}\}.$$

We must have E infinite, since if E were finite then $(f(x_n))_{n=0}^\infty$ and $(f(y_n))_{n=0}^\infty$ would be eventually ε-close (why?). By Proposition 2 of Week 3/4 notes, E is countable; in fact from the proof of that proposition we see that we can find an infinite sequence

$$n_0 < n_1 < n_2 < \ldots$$

consisting entirely of elements in E. In particular, we have

$$|f(x_{n_j}) - f(y_{n_j})| > \varepsilon \text{ for all } j \in \mathbf{N}. \qquad (1)$$

On the other hand, the sequence $(x_{n_j})_{j=0}^\infty$ is a sequence in $[a, b]$, and so by the Bolzano-Weierstrass theorem (cf. the proof of Lemma 1) there must be a subsequence $(x_{n_{j_k}})_{k=0}^{\infty}$ which converges to some limit L in $[a,b]$. In particular, f is continuous at L, and so by Proposition 4 of last week's notes,

$$\lim_{k\to\infty} f(x_{n_{j_k}}) = f(L). \qquad (2)$$

Note that $(x_{n_{j_k}})_{k=0}^{\infty}$ is a subsequence of $(x_n)_{n=0}^{\infty}$, and $(y_{n_{j_k}})_{k=0}^{\infty}$ is a subsequence of $(y_n)_{n=0}^{\infty}$, by Lemma 1 of Week 6 notes. On the other hand, from Lemma 4 we have

$$\lim_{n\to\infty} x_n - y_n = 0.$$

By Proposition 2 from Week 6 notes, we thus have

$$\lim_{k\to\infty} x_{n_{j_k}} - y_{n_{j_k}} = 0.$$

Since $x_{n_{j_k}}$ converges to L as $k \to \infty$, we thus have by limit laws

$$\lim_{k\to\infty} y_{n_{j_k}} = L$$

and hence by continuity of f at L

$$\lim_{k\to\infty} f(y_{n_{j_k}}) = f(L).$$

Subtracting this (2) using limit laws, we obtain

$$\lim_{k\to\infty} f(x_{n_{j_k}}) - f(y_{n_{j_k}}) = 0.$$

But this contradicts (1) (why?). From this contradiction we conclude that f was in fact uniformly continuous. □

- The reader should compare Lemma 1, Proposition 8, and Theorem 9 with each other. No two of these results imply the third, but they are all consistent with each other.

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Limits at infinity

- Until now, we have discussed what it means for a function $f : X \to \mathbf{R}$ to have a limit as $x \to x_0$... as long as x_0 is a *real* number. We now briefly discuss what it would mean to take limits when x_0 is equal to $+\infty$ or $-\infty$. (We will not use those notions much in this course, but we include this for completeness, since they are used elsewhere in mathematics. See pages 97-98 of the textbook for more information).

- First, we need a notion of what it means for $+\infty$ or $-\infty$ to be adherent to a set.

- **Definition** Let X be a subset of $\mathbf{R}$. We say that $+\infty$ is *adherent* to X iff for every $M \in \mathbf{R}$ there exists an $x \in X$ such that $x > M$; we say that $-\infty$ is *adherent* to X iff for every $M \in \mathbf{R}$ there exists an $x \in X$ such that $x < M$.

- In other words, $+\infty$ is adherent to X iff X has no upper bound, or equivalently iff $\sup(X) = +\infty$. Similarly $-\infty$ is adherent to X iff X has no lower bound, or iff $\inf(X) = -\infty$. Thus a set is bounded if and only if $+\infty$ and $-\infty$ are not adherent points.

- The reader may compare this definition with the notion of a real number x_0 being an adherent point. The two definitions may seem dissimilar, but they can in fact be viewed as different special cases of a unified definition; you will see this in more detail if you take Math 121, Introduction to Topology.

- **Definition** Let X be a subset of $\mathbf{R}$ with $+\infty$ as an adherent point, and let $f : X \to \mathbf{R}$ be a function. We say that $f(x)$ *converges to* L as $x \to +\infty$ in X, and write $\lim_{x \to +\infty; x \in X} f(x) = L$, iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (M, +\infty)$ (i.e. $|f(x) - L| \leq \varepsilon$ for all $x \in X$ such that $x > M$). Similarly we say that $f(x)$ *converges to* L as $x \to -\infty$ iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (-\infty, M)$.

- **Example.** Let $f : (0, \infty) \to \mathbf{R}$ be the function $f(x) := 1/x$. Then we have $\lim_{x \to +\infty; x \in (0,\infty)} 1/x = 0$ (can you see why, from the definition?).

- One can do many of the same things with these limits at infinity as we have been doing with limits at other points x_0; for instance, it turns out that all of the limit laws continue to hold. However, as we will not be using these limits in this course, we will not devote much attention to these matters.

- We will note one thing though: if $(a_n)_{n=0}^\infty$ is a sequence of real numbers, then a_n can also be thought of as a function from $\mathbf{N}$ to $\mathbf{R}$, which takes each natural number n to a real number a_n. Thus we can use the above definition to define the expression $\lim_{n\to+\infty; n\in\mathbf{N}} a_n$. But it turns out that this expression is exactly the same as the ordinary limit $\lim_{n\to\infty} a_n$ defined back in Week 5 notes; i.e. if one limit exists then so does the other, and they have the same value. This is an easy matter of inspecting the definition of both types of limits.

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Derivatives

- We are almost ready now to define a notion of derivative. But first we must modify the notion of adherent point mentioned earlier, to that of limit point.

- **Definition** Let X be a subset of $\mathbf{R}$, and let x be a real number. We say that x is a *limit point* (or *cluster point*) of X iff it is an adherent point of $X - \{x\}$.

- Equivalently, x is a limit point of X iff for every $\varepsilon > 0$ there exists a point $y \in X$ which is ε-close to x, but which is not equal to x. From Lemma 5 of last week's notes, we immediately have

- **Lemma 10.** Let X be a subset of $\mathbf{R}$, and let x be a real number. Then x is a limit point of X iff there exists a sequence $(x_n)_{n=m}^\infty$ of elements in $X - \{x\}$ which converge to x.

- **Example.** Let X be the set $X = (1,2) \cup \{3\}$. Then 3 is an adherent point of X, but it is not a limit point of X, since 3 is not adherent to $X - \{3\} = (1,2)$. On the other hand, 2 is still a limit point of X, since 2 is adherent to $X - \{2\} = X$.

- **Lemma 11.** Let I be a (possibly infinite) interval, i.e. I is a set of the form (a,b), $(a,b]$, $[a,b)$, $[a,b]$, $(a,+\infty)$, $[a,+\infty)$, $(-\infty,a)$, or $(-\infty,a]$. Then every element of I is a limit point of I.

- **Proof.** We show this for the case $I = [a, b]$; the other cases are similar and are left to the reader.

- Let $x \in I$; we have to show that x is a limit point of I. There are three cases: $x = a$, $a < x < b$, $x = b$. If $x = a$, then consider the sequence $(x + \frac{1}{n})_{n=N}^{\infty}$. This sequence converges to x, and will lie inside $I - \{a\} = (a, b]$ if N is chosen large enough (why?). Thus by Lemma 10 we see that $x = a$ is a limit point of $[a, b]$. A similar argument works when $a < x < b$. When $x = b$ one has to use the sequence $(x - \frac{1}{n})_{n=N}^{\infty}$ instead (why?) but the argument is otherwise the same. $\square$

- We can now define derivatives analytically, using limits (this is opposed to the geometric definition of derivatives, which uses tangents. The advantage of working analytically is that (a) we do not need to know the axioms of geometry, and (b) these definitions can be modified to handle functions of several variables, or functions whose values are vectors instead of scalars, whereas one's geometric intuition becomes difficult to rely on once one has more than three dimensions in play).

- **Definition** Let X be a subset of $\mathbf{R}$, and let $x_0 \in X$ be an element of X which is also a limit point of X. Let $f : X \to \mathbf{R}$ be a function. If the limit
$$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$
converges to some real number L, then we say that f is *differentiable at x_0 on X with derivative L*, and write $f'(x_0) := L$. If the limit does not exist, or if x_0 is not an element of X or not a limit point of X, we leave $f'(x_0)$ undefined, and say that f is *not differentiable at x_0 on X*.

- Note that we need x_0 to be a limit point in order for x_0 to be adherent to $X - \{x_0\}$, otherwise the limit $\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$ would automatically be undefined. In practice, the domain X will almost always be one of the sets in Lemma 11, and so all elements x_0 of X will automatically be limit points and we will not have to care much about these issues.

- **Example.** Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := x^2$, and let x_0 be any real number. To see whether f is differentiable at x_0 on $\mathbf{R}$, we compute the limit

$$\lim_{x \to x_0; x \in \mathbf{R} - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0; x \in \mathbf{R} - \{x_0\}} \frac{x^2 - x_0^2}{x - x_0}.$$

We can factor the numerator as $(x^2 - x_0^2) = (x - x_0)(x + x_0)$. Since $x \in \mathbf{R} - \{x_0\}$, we may legitimately cancel the factors of $x - x_0$ and write the above limit as

$$\lim_{x \to x_0; x \in \mathbf{R} - \{x_0\}} x + x_0$$

which by limit laws is equal to $2x_0$. Thus the function $f(x)$ is differentiable at x_0 and its derivative there is $2x_0$.

- This point is trivial, but it is worth mentioning: if $f : X \to \mathbf{R}$ is differentiable at x_0, and $g : X \to \mathbf{R}$ is equal to f (i.e. $g(x) = f(x)$ for all $x \in X$), then g is also differentiable at x_0 and $g'(x_0) = f'(x_0)$. (Why?). However, if two functions f and g merely have the same *value* at x_0, i.e. $g(x_0) = f(x_0)$, this does not imply that $g'(x_0) = f'(x_0)$ (can you see a counterexample?). Thus there is a big difference between two functions being equal on their whole domain, and merely being equal at one point.

- One sometimes writes $\frac{df}{dx}$ instead of f'. This notation is of course very familiar and convenient, but one has to be a little careful, because it is only safe to use as long as x is the only variable used to represent the input for f; otherwise one can get into all sorts of trouble. For instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$ has derivative $\frac{df}{dx} = 2x$, but the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(y) := y^2$ would seem to have derivative $\frac{dg}{dx} = 0$ if y and x are independent variables, despite the fact that g and f are exactly the same function. Because of this possible source of confusion, we will refrain from using the notation $\frac{df}{dx}$ whenever it could possibly lead to confusion. (This confusion becomes even worse in several variable calculus, and the standard notation of $\frac{\partial f}{\partial x}$ can lead to some serious ambiguities. There are ways to resolve these ambiguities, most notably by introducing the notion of differentiation along vector fields, but this is beyond the scope of this course).

- **Example.** Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := |x|$, and let $x_0 = 0$. To see whether f is differentiable at 0 on $\mathbf{R}$, we compute the limit

$$\lim_{x\to 0: x\in \mathbf{R}-\{0\}} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0: x\in \mathbf{R}-\{0\}} \frac{|x|}{x}.$$

Now we take left limits and right limits. The right limit is

$$\lim_{x\to 0: x\in (0,\infty)} \frac{|x|}{x} = \lim_{x\to 0: x\in (0,\infty)} \frac{x}{x} = \lim_{x\to 0: x\in (0,\infty)} 1 = 1,$$

while the left limit is

$$\lim_{x\to 0: x\in (-\infty,0)} \frac{|x|}{x} = \lim_{x\to 0: x\in (0,\infty)} \frac{-x}{x} = \lim_{x\to 0: x\in (0,\infty)} -1 = -1,$$

and these limits do not match. Thus $\lim_{x\to 0: x\in \mathbf{R}-\{0\}} \frac{|x|}{x}$ does not exist, and f is not differentiable at 0 on $\mathbf{R}$. However, if one restricts f to $[0,\infty)$, then the restricted function $f|_{[0,\infty)}$ *is* differentiable at 0 on $[0,\infty)$, with derivative 1:

$$\lim_{x\to 0: x\in [0,\infty)-\{0\}} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0: x\in (0,\infty)} \frac{|x|}{x} = 1.$$

Similarly, when one restricts f to $(-\infty,0]$, the restricted function $f|_{(-\infty,0]}$ is differentiable at 0 on $(-\infty,0]$, with derivative -1. Thus even when a function is not differentiable, it is sometimes possible to restore the differentiability by restricting the domain of the function.

- An element of X which is not a limit point of X is known as an *isolated point*; for instance, 3 is an isolated point of the set $X := (1,2) \cup \{3\}$. Given the above definition, it is not possible for a function $f : X \to \mathbf{R}$ to be differentiable at an isolated point of X; for instance the function $f : (1,2) \cup \{3\} \to \mathbf{R}$ defined by $f(x) = x^2$ is not differentiable at 3 on $(1,2) \cup \{3\}$. This is despite f being the restriction of the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) := x^2$, which *is* differentiable at 3. Thus it is possible for a function which is differentiable at x_0 to cease being differentiable if the domain is restricted so that x_0 becomes an isolated point. (However, if $f : X \to \mathbf{R}$ is differentiable at x_0, and $Y \subset X$ is such that x_0 is *still* a limit point of Y, then the restricted $f|_Y : Y \to \mathbf{R}$ is also differentiable at x_0, and with the same derivative. Why?)

- If a function is differentiable at x_0, then it is approximately linear near x_0:

- **Proposition 12 (Newton's approximation).** Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, let $f : X \to \mathbf{R}$ be a function, and let L be a real number. Then the following two statements are equivalent.

- (a) f is differentiable at x_0 on X with derivative L.

- (b) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ is $\varepsilon|x-x_0|$-close to $f(x_0) + L(x - x_0)$ whenever $x \in X$ is δ-close to x_0, i.e.

  $$|f(x)-(f(x_0)+L(x-x_0))| \leq \varepsilon|x-x_0| \text{ whenever } x \in X \text{ and } |x-x_0| \leq \delta.$$

- **Proof.** See Week 8 homework. $\square$

- To phrase Proposition 12 in a more informal way: if f is differentiable at x_0, then one has the approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, and conversely.

- As the example of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := |x|$ shows, a function can be continuous at a point without being differentiable at that point. However, in the converse direction, differentiability implies continuity:

- **Proposition 13.** Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbf{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.

- **Proof.** See Week 8 homework. $\square$

- **Definition** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *differentiable on* X if, for every $x_0 \in X$, the function f is differentiable at x_0 on X.

- From Proposition 13 and the above definition we have an immediate corollary:

- **Corollary 14.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function which is differentiable on X. Then f is also continuous on X.

- Now we state the basic properties of derivatives which you are all familiar with.

- **Theorem 15.** Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions.

- (a) If f is a constant function, i.e. there exists a real number c such that $f(x) = c$ for all $x \in \mathbf{R}$, then f is differentiable at x_0 and $f'(x_0) = 0$.

- (b) If f is the identity function, i.e. $f(x) = x$ for all $x \in \mathbf{R}$, then f is differentiable at x_0 and $f'(x_0) = 1$.

- (c) (Sum rule) If f and g are differentiable at x_0, then $f + g$ is also differentiable at x_0, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

- (d) (Product rule) If f and g are differentiable at x_0, then fg is also differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

- (e) If f is differentiable at x_0 and c is a real number, then cf is also differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.

- (f) (Difference rule) If f and g are differentiable at x_0, then $f - g$ is also differentiable at x_0, and $(f - g)'(x_0) = f'(x_0) - g'(x_0)$.

- (g) If g is differentiable at x_0, and g is non-zero on X (i.e. $g(x) \neq 0$ for all $x \in X$), then $1/g$ is also differentiable at x_0, and $(\frac{1}{g})'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$.

- (h) (Quotient rule) If f and g are differentiable at x_0, and g is non-zero on X, then f/g is also differentiable at x_0, and

$$(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

- **Proof.** See Week 8 homework. □

- As you are well aware, the above rules allow one to compute many derivatives easily. For instance, if $f : \mathbf{R} - \{1\} \to \mathbf{R}$ is the function $f(x) := \frac{x-2}{x-1}$, then it is easy to use the above rules to show that $f'(x_0) = \frac{1}{(x_0-1)^2}$ for all $x_0 \in \mathbf{R} - \{1\}$. (Why? Note that every point x_0 in $\mathbf{R} - \{1\}$ is a limit point of $\mathbf{R} - \{1\}$).

- Another fundamental property of differentiable functions is the following:

- **Theorem 16 (Chain rule).** Let X, Y be subsets of $\mathbf{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f : X \to Y$ be a function such that $f(x_0) = y_0$ and f is differentiable at x_0. Suppose that $g : Y \to \mathbf{R}$ is a function which is differentiable at y_0. Then the function $g \circ f : X \to \mathbf{R}$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(y_0) f'(x_0).$$

- **Proof.** See Week 8 homework. □

- For instance, if $f : \mathbf{R} - \{1\} \to \mathbf{R}$ is the function $f(x) := \frac{x-2}{x-1}$, and $g : \mathbf{R} \to \mathbf{R}$ is the function $g(y) := y^2$, then $g \circ f(x) = (\frac{x-2}{x-1})^2$, and the chain rule gives

$$(g \circ f)'(x_0) = 2(\frac{x_0 - 2}{x_0 - 1}) \frac{1}{(x_0 - 1)^2}.$$

- If one writes y for $f(x)$, and z for $g(y)$, then the chain rule can be written in the more visually appealing manner $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$. However, this notation can be misleading (for instance it blurs the distinction between dependent variable and independent variable, especially for y), and leads one to believe that the quantities dz, dy, dx can be manipulated like real numbers. However, these quantities are not real numbers (in fact, we have not assigned any meaning to them at all), and treating them as such can lead to problems in the future. For instance, if f depends on x_1 and x_2, which depend on t, then the several variable calculus chain rule asserts that $\frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt}$, but this rule might seem suspect if one treated df, dt, etc. as real numbers. It is possible to think of dy, dx, etc. as "infinitesimal real numbers" if one knows what one is doing, but at this stage I would not recommend it, especially if one wishes to work rigorously. (There is a way to make all of this rigorous, even for several variable calculus, but it requires the notion of a tangent vector, and the derivative map, both of which are beyond the scope of this course).

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Local maxima, local minima, and derivatives

- As you learnt in lower-division calculus, one very common application of using derivatives is to locate maxima and minima. We now present this material again, but more rigorously than in lower division.

- The notion of a function $f : X \to \mathbf{R}$ attaining a maximum or minimum at a point $x_0 \in X$ was defined in last week's notes. We now localize this definition:

- **Definition** Let $f : X \to \mathbf{R}$ be a function, and let $x \in X$. We say that f attains a *local maximum* at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0 - \delta, x_0 + \delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a maximum at x_0. We say that f attains a *local minimum* at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0 - \delta, x_0 + \delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a minimum at x_0.

- If f attains a maximum at x_0, we sometimes say that f attains a *global* maximum at x_0, in order to distinguish it from the local maxima defined here. Note that if f attains a global maximum at x_0, then it certainly also attains a local maximum at this x_0, and similarly for minima.

- **Example.** Let $f : \mathbf{R} \to \mathbf{R}$ denote the function $f(x) := x^2 - x^4$. This function does not attain a global minimum at 0, since for example $f(2) = -12 < 0 = f(0)$, however it does attain a local minimum, for if we choose $\delta := 1$ and restrict f to the interval $(-1, 1)$, then for all $x \in (-1, 1)$ we have $x^4 \leq x^2$ and thus $f(x) = x^2 - x^4 \geq 0 = f(0)$, and so $f|_{(-1,1)}$ has a local minimum at 0.

- **Example.** Let $f : \mathbf{Z} \to \mathbf{R}$ be the function $f(x) = x$, defined on the integers only. Then f has no global maximum or global minimum (why?), but attains both a local maximum and local minimum at every integer n (why?).

- Note that if $f : X \to \mathbf{R}$ attains a local maximum at a point x_0 in X, and $Y \subset X$ is a subset of X which contains x_0, then the restriction $f|_Y : Y \to \mathbf{R}$ also attains a local maximum at x_0 (why?). Similarly for minima.

- The connection between local maxima, minima and derivatives is the following.

- **Proposition 17.** Let $a < b$ be real numbers, and let $f : (a, b) \to \mathbf{R}$ be a function. If $x_0 \in (a, b)$, f is differentiable at x_0, and f attains either a local maximum or local minimum at x_0, then $f'(x_0) = 0$.

- **Proof.** See Week 8 homework. $\square$

- Note that f must be differentiable for this to work; see Week 8 homework. Also, this Proposition also does not work if the open interval (a, b) is replaced by a closed interval $[a, b]$. For instance, the function $f : [1, 2] \to \mathbf{R}$ defined by $f(x) := x$ has a local maximum at $x_0 = 2$ and a local minimum $x_0 = 1$ (in fact, these local extrema are global extrema), but at both points the derivative is $f'(x_0) = 1$, not $f'(x_0) = 0$. Thus the endpoints of an interval can be local maxima or minima even if the derivative is not zero there.

- By combining Proposition 17 with the Maximum principle, one can obtain

- **Theorem 18 (Rolle's theorem)** Let $a < b$ be real numbers, and let $g : [a, b] \to \mathbf{R}$ be a continuous function which is differentiable on (a, b). Suppose also that $g(a) = g(b)$. Then there exists an $x \in (a, b)$ such that $g'(x) = 0$.

- **Proof.** See Week 8 homework. $\square$

- Note that we only assume f is differentiable on the open interval (a, b), though of course the theorem also holds if we assume f is differentiable on the closed interval $[a, b]$, since this is larger than (a, b).

- This theorem has an important corollary.

- **Corollary 19 (Mean value theorem)** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists an $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

- **Proof.** See Week 8 homework. $\square$

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Monotone functions and derivatives

- In your lower-division calculus (or perhaps in high-school) you learnt that a positive derivative meant an increasing function, and a negative derivative meant a decreasing function. This statement is not completely accurate, but it is pretty close; we now give the precise version of these statements below.

- **Proposition 20.** Let X be a subset of $\mathbf{R}$, let x_0 be a limit point of X, and let $f : X \to \mathbf{R}$ be a function. If f is monotone increasing *and* f is differentiable at x_0, then $f'(x_0) \geq 0$. If f is monotone decreasing *and* f is differentiable at x_0, then $f'(x_0) \leq 0$.

- **Proof.** See Week 8 homework. □

- Note that we have to assume that f is differentiable at x_0. There exist monotone functions which are not always differentiable (see Week 8 homework), and of course if f is not differentiable at x_0 we cannot possibly conclude that $f'(x_0) \geq 0$ or $f'(x_0) \leq 0$.

- One might naively guess that if f were *strictly* monotone increasing, and f was differentiable at x_0, then the derivative $f'(x_0)$ would be strictly positive instead of merely non-negative. Unfortunately, this is not always the case. For instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^3$ is strictly monotone increasing on $\mathbf{R}$ (why?), but the derivative at 0 is 0.

- On the other hand, we do have a converse result: if the derivative is always strictly positive, then the function is strictly monotone increasing:

- **Proposition 21.** Let $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a, b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant function.

- **Proof.** See Week 8 homework. □

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Inverse functions and derivatives.

- We now ask the following question: if we know that a function $f : X \to Y$ is differentiable, and it has an inverse $f^{-1} : Y \to X$, what can we say about the differentiability of f^{-1}? This will be useful for many applications, for instance if we want to differentiate the function $f(x) := x^{1/n}$.

- We begin with a preliminary result.

- **Lemma 22.** Let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that $y_0 = f(x_0)$ (which also implies that $x_0 = f^{-1}(y_0)$). If f is differentiable at x_0, *and* f^{-1} is differentiable at y_0, then

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

- **Proof.** From the chain rule (Theorem 16) we have

$$(f^{-1} \circ f)'(x_0) = (f^{-1})'(y_0) f'(x_0).$$

But $f^{-1} \circ f$ is the identity function on X, and hence by Theorem 15(b) $(f^{-1} \circ f)'(x_0) = 1$. The claim follows. $\square$

- As a particular corollary of Lemma 22, we see that if f is differentiable at x_0 with $f'(x_0) = 0$, then f^{-1} cannot be differentiable at $y_0 = f(x_0)$, since $1/f'(x_0)$ is undefined in that case. Thus for instance, the function $g : [0, \infty) \to [0, \infty)$ defined by $g(y) := y^{1/3}$ cannot be differentiable at 0, since this function is the inverse $g = f^{-1}$ of the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) := x^3$, and this function has a derivative of 0 at $f^{-1}(0) = 0$.

- If one writes $y = f(x)$, so that $x = f^{-1}(y)$, then one can write the conclusion of Lemma 22 in the more appealing form $dx/dy = 1/(dy/dx)$. However, as mentioned before, this way of writing things, while very convenient and easy to remember, can be misleading and cause errors if applied too carelessly (especially when one begins to work in several variable calculus).

- Lemma 22 seems to answer the question of how to differentiate the inverse of a function, however it has one significant drawback: the lemma only works if one assumes *a priori* that f^{-1} is differentiable. (a priori is Latin for "beforehand" or "before the fact", as opposed to *a posteriori*, "after the fact", and in mathematics refers to a situation in which a certain desirable property is assumed instead of deduced). Thus, if one does not already know that f^{-1} is differentiable, one cannot use Lemma 22 to compute the derivative of f^{-1}.

- However, the following improved version of Lemma 22 will compensate for this fact, by relaxing the requirement on f^{-1} from differentiability to continuity.

- **Theorem 23 (Inverse function theorem)** Let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that $f(x_0) = y_0$. If f is differentiable at x_0, f^{-1} is continuous at y_0, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

- **Proof.** We have to show that

$$\lim_{y \to y_0: y \in Y - \{y_0\}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

By Proposition 6 of Week 6 notes, it suffices to show that

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$$

for any sequence $(y_n)_{n=1}^\infty$ of elements in $Y - \{y_0\}$ which converge to y_0.

- To prove this, we set $x_n := f^{-1}(y_n)$. Then $(x_n)_{n=1}^\infty$ is a sequence of elements in $X - \{x_0\}$ (why? Note that f^{-1} is a bijection). Since f^{-1} is continuous by assumption, we know that $x_n = f^{-1}(y_n)$ converges to $f^{-1}(y_0) = x_0$ as $n \to \infty$. Thus, since f is differentiable at x_0, we have (by Proposition 6 of Week 6 notes again)

$$\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0).$$

But since $x_n \neq 0$ and f is a bijection, the fraction $\frac{f(x_n)-f(x_0)}{x_n-x_0}$ is non-zero. Also, by hypothesis $f'(x_0)$ is non-zero. So by limit laws

$$\lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

But since $x_n = f^{-1}(y_0)$ and $x_0 = f^{-1}(y_0)$, we thus have

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$$

as desired. □

- In the homework you will use the inverse function theorem to prove that functions such as $x^{1/n}$ are differentiable, and then compute their derivatives.

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L'Hôpital's rule

- Finally, we present a version of a rule you are all familiar with.

- **Proposition 24 (L'Hôpital's rule, first version)** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions, and let x_0 be a limit point of X. Suppose that $f(x_0) = g(x_0) = 0$, that f and g are both differentiable at x_0, but $g'(x_0) \neq 0$. Then there exists a $\delta > 0$ such that $g(x) \neq 0$ for all $x \in (X \cap (x_0 - \delta, x_0 + \delta)) - \{x_0\}$, and

$$\lim_{x\to x_0; x\in(X\cap(x_0-\delta,x_0+\delta))-\{x_0\}} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

- **Proof.** See Week 8 homework. □

- The presence of the δ may seen somewhat strange, but is needed because $g(x)$ might vanish at some points other than x_0 and so the quotient $\frac{f(x)}{g(x)}$ is not necessarily defined at all points in $X - \{x_0\}$.

- A more sophisticated version of L'Hôpital's rule is the following.

- **Proposition 25 (L'Hôpital's rule, second version)** Let $a < b$ be real numbers, let $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ be a functions which are differentiable on $[a, b]$. Suppose that $f(a) = g(a) = 0$, that g' is non-zero on $[a, b]$ (i.e. $g'(x) \neq 0$ for all $x \in [a, b]$), and $\lim_{x \to a; x \in (a,b]} \frac{f'(x)}{g'(x)}$ exists and equals L. Then $g(x) \neq 0$ for all $x \in (a, b]$, and $\lim_{x \to a; x \in (a,b]} \frac{f(x)}{g(x)}$ exists and equals L.

- This proposition only considers limits to the right of a, but one can easily state and prove a similar proposition for limits to the left of a, or around both sides of a. Speaking very informally, the proposition states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

though one has to ensure all of the conditions of the proposition hold (in particular, that $f(a) = g(a) = 0$, and that the right-hand limit exists), before one can apply L'Hôpital's rule.

- **Proof.** (Optional) We first show that $g(x) \neq 0$ for all $x \in (a, b]$. Suppose for contradiction that $g(x) = 0$ for some $x \in (a, b]$. But since $g(a)$ is also zero, we can apply Rolle's theorem to obtain $g'(y) = 0$ for some $a < y < x$, but this contradicts the hypothesis that g' is non-zero on $[a, b]$.

- Now we show that $\lim_{x \to a; x \in (a,b]} \frac{f(x)}{g(x)} = L$. By Proposition 6 of Week 6 notes, it will suffice to show that

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = L$$

for any sequence $(x_n)_{n=1}^{\infty}$ $(a, b]$ which converges to x.

- Consider a single x_n, and consider the function $h_n : [a, x_n] \to \mathbf{R}$ defined by

$$h_n(x) := f(x)g(x_n) - g(x)f(x_n).$$

Observe that h_n is continuous on $[a, x_n]$ and equals 0 at both a and x_n, and is differentiable on (a, x_n) with derivative $h_n'(x) = f'(x)g(x_n) - g'(x)f(x_n)$. (Note that $f(x_n)$ and $g(x_n)$ are constants with respect to x). By Rolle's theorem, we can thus find $y_n \in (a, x_n)$ such that $h_n'(y_n) = 0$, which implies that

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)}.$$

Since $y_n \in (a, x_n)$ for all n, and x_n converges to a as $n \to \infty$, we see from the squeeze test that y_n also converges to a as $n \to \infty$. Thus $\frac{f'(y_n)}{g'(y_n)}$ converges to L, and thus $\frac{f(x_n)}{g(x_n)}$ also converges to L, as desired. $\square$