Problem 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers which converge to 0, i.e. $\lim_{n\to\infty} a_n = 0$. Show that the series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$ is conditionally convergent, and converges to a_0. (**Hint:** First work out what the partial sums $\sum_{n=0}^{N}(a_n - a_{n+1})$ should be, and prove your assertion using induction).

This question is somewhat similar to the proof of the Alternating Series Test (Proposition 16 of Week 5 notes).

We first claim that

$$\sum_{n=0}^{N}(a_n - a_{n+1}) = a_0 - a_{N+1}$$

for all natural numbers N. To see this we use induction. We first check the base case $N = 0$:

$$\sum_{n=0}^{0}(a_n - a_{n+1}) = a_0 - a_1 = a_0 - a_{N+1}.$$

Now suppose inductively that $N \geq 0$, and we have already proven

$$\sum_{n=0}^{N}(a_n - a_{n+1}) = a_0 - a_{N+1}.$$

Then by the laws of finite summation, followed by the induction hypothesis, followed by some algebra, we have

$$\begin{aligned}\sum_{n=0}^{N++}(a_n - a_{n+1}) &= \sum_{n=0}^{N}(a_n - a_{n+1}) + (a_{N+1} - a_{N+2}) \\ &= (a_0 - a_{N+1}) + (a_{N+1} - a_{N+2}) \\ &= a_0 - a_{N++\,+1},\end{aligned}$$

closing the induction. This proves the claim.

Thus the partial sums $S_N := \sum_{n=0}^{N}(a_n - a_{n+1})$ are given by the formula $a_0 - a_{N+1}$. Now observe from the definition of limit that since a_N converges to 0 as $N \to \infty$, a_{N+1} also converges to 0 as $N \to \infty$ (why?). Thus by limit laws

$$\lim_{N\to\infty} S_N = a_0 - \lim_{N\to\infty} a_{N+1} = a_0 - 0 = a_0$$

and so the series is conditionally convergent to a_0.

A linguistic remark: Note that when we assert a series to be conditionally convergent, this does not automatically mean that the series has to be absolutely divergent; we consider the class absolutely convergent series to be a subclass of the class of conditionally convergent series (cf. Proposition 15 of Week 5 notes). In mathematics, a statement only means what it says explicitly, not what it hints at or otherwise implies; the reason for this is that hints or inferences are hard to quantify objectively and precisely (different readers may take different implications from the same sentence, or may simply "not get the hint"). In everyday English, if one makes a statement such as "My friend X has a college degree, and my friend Y has a high school diploma", then the implication is that Friend Y does not have a college degree, even though this is not explicitly stated. However, in mathematics, if one makes a statement such as "The series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, and the series $\sum_{n=0}^{\infty} b_n$ is conditionally convergent", this does NOT imply that $\sum_{n=0}^{\infty} b_n$ is absolutely divergent. (If one DID want to make this point, one would have said something like "..., but $\sum_{n=0}^{\infty} b_n$ is only conditionally convergent" or "..., while $\sum_{n=0}^{\infty} b_n$ converges conditionally but not absolutely" instead). Another example: in everyday English, if one asserts "My friend X is at work from 9am to 5pm", the implication is that X is not at work after 5pm or before 9am, although this is not explicitly stated. In mathematics, however, if one asserts a statement such as "If $x > 3$, then $f(x)$ is positive", this does not imply that $f(x)$ ceases to be positive when $x \leq 3$; if we wanted to say this also, we would have said "$f(x)$ is positive if and only if $x > 3$" instead. (Cf. the logic handout, especially the distinction between a statement being *true*, versus being *useful*, or being *efficient*, or being *informative*. In everyday English one wants sentences to be all of these, but in mathematics what matters most is that statements are true.).

Note: It is also tempting to proceed by splitting the series as

$$\sum_{n=0}^{\infty}(a_n - a_{n+1}) = \sum_{n=0}^{\infty} a_n - \sum_{n=0}^{\infty} a_{n+1}.$$

However, this is not rigorous because we are not assuming $\sum_{n=0}^{\infty} a_n$ to be convergent (note that the zero test has nothing to say on this matter).

Problem 2. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers. Let $f : \mathbf{N} \to \mathbf{N}$ be an increasing function (i.e. $f(n++) > f(n)$ for all $n \in \mathbf{N}$). Show that $\sum_{n=0}^{\infty} a_{f(n)}$ is also an absolutely convergent series. (**Hint:** try to compare each partial sum of $\sum_{n=0}^{\infty} a_{f(n)}$ with a (slightly different) partial sum of $\sum_{n=0}^{\infty} a_n$).

This is similar to Proposition 23 from Week 5 notes, although it does not follow directly from that propssition (since f is not assumed to be a bijection).

We want to show that $\sum_{n=0}^{\infty} |a_{f(n)}|$ is convergent. What we will in fact show is that there is a number M such that the partial sums $\sum_{n=0}^{N} |a_{f(n)}|$ are bounded by M for all N; this, combined with Proposition 18 of Week 5 notes, will imply the convergence of $\sum_{n=0}^{\infty} |a_{f(n)}|$.

The function $f(n)$ is increasing (by Q3 of Midterm 1): $f(n) > f(m)$ whenever $n > m$. In particular, the function $f(n)$ is one-to-one. Thus, when restricted to the set $\{0, 1, \ldots, N\} = \{n \in \mathbf{N} : 0 \leq n \leq N\}$, the function f is a bijection from $\{0, 1, 2, \ldots, N\}$ to $\{f(0), f(1), \ldots, f(N)\} = \{f(n) : 0 \leq n \leq N\}$. Thus we can write $\sum_{n=0}^{N} |a_{f(n)}| = \sum_{m \in X} |a_m|$, by the definition of summation over finite sets, where X is the set $X := \{f(n) : 0 \leq n \leq N\}$.

Since f is increasing, the set X is a subset of the set $\{0, 1, 2, \ldots, f(n)\}$. Thus, by the laws of summation over finite sets,

$$\sum_{m \in X} |a_m| + \sum_{m \in \{0,1,2,\ldots,f(n)\} - X} |a_m| = \sum_{m=0}^{f(n)} |a_m|.$$

Since $|a_m|$ is non-negative for every m, the sum $\sum_{m \in \{0,1,2,\ldots,f(n)\} - X} |a_m|$ is also non-negative (why?), and thus

$$\sum_{m \in X} |a_m| \leq \sum_{m=0}^{f(n)} |a_m|.$$

Thus

$$\sum_{n=0}^{N} |a_{f(n)}| \leq \sum_{m=0}^{f(n)} |a_m| \leq \sum_{m=0}^{\infty} |a_m|,$$

and in particular all the partial sums of $\sum_{n=0}^{\infty} |a_{f(n)}|$ are bounded by the number $\sum_{m=0}^{\infty} |a_m|$ (which is independent of N, and is also a real number, since $\sum_{n=0}^{\infty} a_n$ was assumed to be absolutely convergent). By Proposition 18 of Week 5 notes we thus see that $\sum_{n=0}^{\infty} |a_{f(n)}|$ is convergent as desired.

(Note: it is also possible to proceed using Proposition 13 of Week 5 notes by similar arguments).

Problem 3. Let E be a bounded subset of $\mathbf{R}$, and let $S := \sup(E)$ be the least upper bound of E. (Note from the least upper bound principle that S is a real number). Show that S is an adherent point of E, and is also an adherent point of $\mathbf{R}\backslash E$.

This question is somewhat similar to Proposition 27(e) from Weeks 3/4 notes.

We first show that S is an adherent point of E. Let $\varepsilon > 0$ be any real number. We have to show that S is ε-adherent to E. Since S is the least upper bound of E, and $S - \varepsilon$ is less than S, $S - \varepsilon$ is not an upper bound for E, thus there exists an $x \in E$ such that $S - \varepsilon < x$. On the other hand, since S is an upper bound for E we have $x \leq S$. In particular we see that x must be ε-close to S, and so S is ε-adherent to E as desired.

We now show that S is an adherent point of $\mathbf{R}\backslash E$. Let $\varepsilon > 0$ be any real number. We have to show that S is ε-adherent to E. The number $S + \varepsilon/2$ is larger than S; since S is an upper bound for E, this implies that $S + \varepsilon/2$ is not in E, and thus lies in $\mathbf{R}\backslash E$. Since $S + \varepsilon/2$ is ε-close to S, we see that S is ε-adherent to $\mathbf{R}\backslash E$ as desired.

Problem 4. Let X, Y, Z be subsets of $\mathbf{R}$. Let $f : X \to Y$ be a function which is uniformly continuous on X, and let $g : Y \to Z$ be a function which is uniformly continuous on Y. Show that the function $g \circ f : X \to Z$ is uniformly continuous on X.

This question is similar to Q10 from Week 6 homework.

First proof: We use Proposition 5 from Weeks 7/8 notes. Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be any two equivalent sequences in X. Since $f : X \to Y$ is uniformly continuous, $(f(a_n))_{n=0}^\infty$ and $(f(b_n))_{n=0}^\infty$ are equivalent sequences in Y. Since $g : Y \to Z$ is uniformly continuous, $(g(f(a_n)))_{n=0}^\infty$ and $(g(f(b_n)))_{n=0}^\infty$ are equivalent sequences in Z, i.e. $(g \circ f(a_n))_{n=0}^\infty$ and $(g \circ f(b_n))_{n=0}^\infty$ are equivalent sequences in Z. But then by Proposition 5 again, this means that $g \circ f$ is uniformly continuous.

Second proof: We use the epsilon-delta definition of uniform continuity. Let $\varepsilon > 0$. We need to find a $\delta > 0$ with the following property: whenever $x, x_0 \in X$ are δ-close, we need $g \circ f(x)$ and $g \circ f(x_0)$ to be ε-close.

Since g is uniformly continuous, we can already find a $\delta' > 0$ with the following property: whenever $y, y_0 \in Y$ are δ'-close, then $g(y)$ and $g(y_0)$ are ε-close.

Choose such a $\delta' > 0$. Since f is uniformly continuous, we can then find a $\delta > 0$ with the following property: whenever $x, x_0 \in X$ are δ-close, then $f(x), f(x_0)$ are δ'-close.

Combining these two properties we see that whenever $x, x_0 \in X$ are δ close, then $f(x), f(x_0) \in Y$ are δ'-close, and hence $g(f(x))$ and $g(f(x_0))$ are ε-close, i.e. $g \circ f(x)$ and $g \circ f(x_0)$ are ε-close, as desired. (Note that one has to choose the epsilons and deltas in this exact order; most other ways of doing it will not work properly).

Problem 5. Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there exists a real number x in $[0,1]$ such that $f(x) = x$. (**Hint:** Apply the intermediate value theorem to the function $f(x) - x$.)

Let $g : [0,1] \to \mathbf{R}$ be the function $g(x) := f(x) - x$. Since f is continuous, and the identity function $x \mapsto x$ is also continuous, we see that g is also continuous on $[0,1]$. Also, $g(0) = f(0) - 0$ is non-negative, since $f(0) \in [0,1]$. Similarly, $g(1) = f(1) - 1$ is non-positive, since $f(1) \in [0,1]$. Thus $g(0) \geq 0 \geq g(1)$, and so by the intermediate value theorem there exists an $x \in [0,1]$ such that $g(x) = 0$, which implies that $f(x) - x = 0$, and hence $f(x) = x$, as desired.

(This result is an example of a *fixed point theorem*; it says that if you take the unit interval $[0,1]$ and distort it continuously, but then put it back on top of $[0,1]$, there must be at least one point which was not affected by this transformation).
