Mathematics 131AH
Terence Tao
Final, Mar 19, 2003

Instructions: Do nine out of the 12 problems; they are all of equal value. There is plenty of working space, and a blank page at the end. On the first page you will be supplied a list of standard definitions for easy reference.

You may use any result from the textbook or notes (or from any other mathematics book); you do not need to give precise theorem numbers or page numbers (e.g. saying "by a theorem from the notes" will suffice). You are encouraged to be verbose in your proofs and explanations; a chain of equations with no explanation given may be insufficient for full credit.

You may enter in a nickname if you want your final score posted.

Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points). ________

Problem 2 (10 points). ________

Problem 3 (10 points). ________

Problem 4 (10 points). ________

Problem 5 (10 points). ________

Problem 6 (10 points). ________

Problem 7 (10 points). ________

Problem 8 (10 points). ________

Problem 9 (10 points). ________

Problem 10 (10 points). ________

Problem 11 (10 points). ________

Problem 12 (10 points). ________

Best 9 of 12 (90 points): ________________

Definitions

- **Absolutely convergent series.** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is *absolutely convergent* if the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

- **Adherent points.** Let X be a subset of $\mathbf{R}$, let $\varepsilon > 0$, and let $x \in \mathbf{R}$. We say that x is *ε-adherent to X* iff there exists a $y \in E$ which is ε-close to x. We say that x is *adherent to X* iff it is ε-adherent to X for every $\varepsilon > 0$.

- **Bounded functions.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *bounded* if there exists a number M such that $|f(x)| \leq M$ for all $x \in X$.

- **Bounded sequences.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. We say this sequence is *bounded* if there exists a number M such that $|a_n| \leq M$ for all $n \in \mathbf{N}$.

- **Closure.** Let X be a subset of $\mathbf{R}$. The *closure* of X, sometimes denoted $\overline{X}$ is defined to be the set of all the adherent points of X.

- **Connected sets.** Let X be a subset of $\mathbf{R}$. We say that X is *connected* iff the following property is true: whenever x, y are elements in X such that $x < y$, the interval $[x, y]$ is a subset of X (i.e. every number between x and y is also in X).

- **Continuity.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is *continuous at x_0* iff we have

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0).$$

  We say that f is *continuous on X* iff f is continuous at x_0 for every $x_0 \in X$.

- **Convergent series.** Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the *N^{th} partial sum S_N* of this series to be $S_N := \sum_{n=m}^{N} a_n$. If the sequence $(S_N)_{n=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *convergent*, and *converges to L*; we also write $L = \sum_{n=m}^{\infty} a_n$.

- **Differentiable functions.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Let $x_0 \in X$ be an element of X which is also a limit point of X. If the limit

$$\lim_{x \to x_0; x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

  converges to some real number L, then we say that f is *differentiable at x_0 on X with derivative L*, and write $f'(x_0) := L$. If f is differentiable at every element of X, we say that f is *differentiable on X*.

- **Generalized interval.** A *generalized interval* is a subset I of $\mathbf{R}$ which is either an interval (i.e. a set of the form $[a, b]$, (a, b), $[a, b)$, or $(a, b]$); a point $\{a\}$; or the empty set $\emptyset$.

- **Length.** If I is a generalized interval, we define the *length* of I, denoted $|I|$ as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $|I| := b - a$. Otherwise, if I is a point or the empty set, we define $|I| = 0$.

- **α-Length.** If I is a generalized interval and α is a monotone increasing function on a domain containing I, we define the *α-length* of I, denoted $\alpha[I]$ as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $\alpha[I] := \alpha(b) - \alpha(a)$. Otherwise, if I is a point or the empty set, we define $|I| = 0$.

- **Limiting values of functions.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a number. We say that f *converges to L at x_0 in E*, and write $\lim_{x \to x_0; x \in E} f(x) = L$, iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$.

- **Majorizing/Minorizing.** Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. We say that f *majorizes* g if $f(x) \geq g(x)$ for all $x \in X$, and f *minorizes* g if $f(x) \leq g(x)$ for all $x \in X$.

- **Monotone increasing.** Let $f : X \to \mathbf{R}$ be a function. We say that f is *monotone increasing* iff we have $f(y) \geq f(x)$ whenever $x, y \in X$ are such that $y > x$.

- **Partitions.** Let I be a generalized interval. A *partition* of I is a finite set $\mathbf{P}$ of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in $\mathbf{P}$.

- **Piecewise constant functions.** Let I be a generalized interval, let $f : I \to \mathbf{R}$ be a function, and let $\mathbf{P}$ be a partition of I. We say that f is *piecewise constant with respect to* $\mathbf{P}$ if for every $J \in \mathbf{P}$, f is constant on J. We say that f is *piecewise constant on* I if it is piecewise constant with respect to some partition $\mathbf{P}$ of I.

- **Piecewise constant integrals.** Let I be a generalized interval, and let $f : I \to \mathbf{R}$ be a function which is piecewise constant with respect to some partition $\mathbf{P}$ of I. Then we define the *piecewise constant integral p.c.* $\int_I f$ of f by the formula

  $$p.c. \int_I f := \sum_{J \in \mathbf{P}: J \neq \emptyset} c_J |J|,$$

  where for each J we let c_J be the constant value of f on J. More generally, if α is a monotone increasing function on a domain containing I, we define

  $$p.c. \int_I f \, d\alpha := \sum_{J \in \mathbf{P}: J \neq \emptyset} c_J \alpha[J].$$

- **Riemann integral.** Let $f : I \to \mathbf{R}$ be a bounded function defined on a generalized interval I. We define the *upper Riemann integral* $\overline{\int}_I f$ by the formula

  $$\overline{\int}_I f := \inf\{p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which majorizes } f\}$$

and the *lower Riemann integral* $\underline{\int}_I f$ by the formula

$$\underline{\int}_I f := \sup\{p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which minorizes } f\}.$$

If $\overline{\int}_I f = \underline{\int}_I f$, we say that f is *Riemann integrable* and write $\int_I f = \overline{\int}_I f = \underline{\int}_I f$.

- **Riemann-Stieltjes integral.** Let $f : I \to \mathbf{R}$ be a bounded function defined on a generalized interval I. We define the *upper Riemann-Stieltjes integral* $\overline{\int}_I f \, d\alpha$ by the formula

$$\overline{\int}_I f \, d\alpha := \inf\{p.c. \int_I g \, d\alpha : g \text{ is a piecewise constant function on } I \text{ which majorizes } f\}$$

and the *lower Riemann-Stieltjes integral* $\underline{\int}_I f \, d\alpha$ by the formula

$$\underline{\int}_I f \, d\alpha := \sup\{p.c. \int_I g \, d\alpha : \text{ is a piecewise constant function on } I \text{ which minorizes } f\}.$$

If $\overline{\int}_I f \, d\alpha = \underline{\int}_I f \, d\alpha$, we say that f is *Riemann-Stieltjes integrable* and write $\int_I f \, d\alpha = \overline{\int}_I f \, d\alpha = \underline{\int}_I f \, d\alpha$.

- **Uniformly continuous functions.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x, x_0 \in X$ are two points in x are δ-close.

Problem 1. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function whose derivative $f' : \mathbf{R} \to \mathbf{R}$ is a bounded function. Show that f is uniformly continuous. (Hint: use the mean-value theorem to get some sort of upper bound on $|f(x) - f(y)|$ for $x, y \in \mathbf{R}$).

Since f' is bounded, there exists an $M > 0$ such that $|f'(z)| \leq M$ for all real numbers z.

Lemma. For any real numbers x and y, we have $|f(x) - f(y)| \leq M|x - y|$.

Proof. The claim is obvious for $x = y$. The remaining cases are $x > y$ and $x < y$; without loss of generality we may take $x > y$. By the mean value theorem, there exists a $z \in [x, y]$ such that $\frac{f(x)-f(y)}{x-y} = f'(z)$; taking absolute values, we obtain $\frac{|f(x)-f(y)|}{|x-y|} \leq M$, and the claim follows. (Note: one can also prove this Lemma using the fundamental theorem of calculus, although this may require the additional assumption that f' is Riemann integrable). $\square$

Now let $\varepsilon > 0$. We need to find a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. But if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq M\delta$ by the above lemma. Thus if we choose δ to equal ε/M, we see that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$, as desired.

Note: one can also proceed using the equivalent sequences formulation of uniform continuity.

Problem 2. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers such that $\sum_{n=0}^{\infty} |a_n| = 0$. Show that $a_n = 0$ for every natural number n.

We prove by contradiction. Suppose that there existed a natural number n such that $a_n \neq 0$. Then $|a_n| > 0$. Then for any $N > n$, we have

$$\sum_{m=0}^{N} |a_m| = \sum_{m=0}^{n-1} |a_m| + |a_n| + \sum_{m=n+1}^{N} |a_m| \geq |a_n|;$$

taking limits as $N \to \infty$, we obtain

$$\sum_{m=0}^{\infty} |a_m| \geq |a_n|.$$

But this contradicts the assumption that $\sum_{n=0}^{\infty} |a_n| = 0$.

Problem 3. Let $f : [0, \infty) \to \mathbf{R}$ be a monotone decreasing function which is non-negative (i.e. $f(x) \geq 0$ for all $x \geq 0$). Suppose that there exists a number $M > 0$ such that $\int_{[0,N]} f \leq M$ for all natural numbers N. Show that the sum $\sum_{n=1}^{\infty} f(n)$ is convergent. (Note: you may only use the integral test for this problem if you provide an explanation as to why the integral test works). Hint: what is the relationship between the sum $\sum_{n=1}^{N} f(n)$ and the integral $\int_{[0,N]} f$?

Lemma. For any natural number $N \geq 1$, we have $\sum_{n=1}^{N} f(n) \leq \int_{[0,N]} f$.

Proof. First observe that $\int_{[0,N]} f = \int_{[0,N)} f$, because $\int_{\{N\}} f = 0$ (why?). Observe that the set

$$\mathbf{P} := \{[j-1, j) : 1 \leq j \leq N; j \text{ is a natural number}\}$$

is a partition of $[0, N)$ (why?). Now define the function $g : [0, N) \to \mathbf{R}$ by setting $g(x) := f(j)$ for all $x \in [j-1, j)$ and all natural numbers $1 \leq j \leq N$. By construction g is piecewise constant with respect to $\mathbf{P}$, and minorizes f since f is monotone decreasing, so

$$\int_{[0,N)} f \geq p.c. \int_{[0,N)} g = \sum_{j=1}^{N} f(j)|[j-1, j)| = \sum_{j=1}^{N} f(j)$$

and the claim follows. □

From the Lemma we see that

$$\sum_{n=1}^{N} f(n) \leq M$$

for all N; in other words, the partial sums of $f(n)$ are bounded. Since $f(n)$ is non-negative, we thus see that $\sum_{n=1}^{N} f(n)$ is absolutely convergent.

Problem 4. Let X be a finite subset of $\mathbf{R}$. Show that $\overline{X} = X$, i.e. the closure of X is the same set as X itself.

We have to show that every element of X is also an element of $\overline{X}$, and vice versa. Clearly every element of X is adherent to X (since it is 0-close to X), and so lies in $\overline{X}$. Now suppose, conversely, that x is an element of $\overline{X}$. We have to show that $x \in X$.

Suppose for contradiction that $x \notin X$. Since X is finite, it takes the form $\{x_1, x_2, \ldots, x_n\}$ for some real numbers $x_1, \ldots, x_n$. The numbers $x - x_j$ are non-zero for all $j = 1, \ldots, n$, so in particular $|x - x_j| > 0$ for all $j = 1, \ldots, n$. Let $|x - x_k|$ be the minimum of all the $|x - x_j|$, then $|x - x_k|$ is also positive. If we set $\varepsilon = |x - x_k|/2$, ten x is not ε-close to x_k or to any other x_j, and so x is not adherent to X, a contradiction.

An alternate way to proceed is to use induction on the cardinality of X. One way to do this is to show that if x is adherent to $X \cup \{y\}$, then either x is adherent to X or x is equal to y.

Problem 5. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a Riemann integrable function. Let $g : [-b, -a] \to \mathbf{R}$ be defined by $g(x) := f(-x)$. Show that g is also Riemann integrable, and $\int_{[-b,-a]} g = \int_{[a,b]} f$.

Lemma. Let I be a generalized interval, and let $-I$ be the set $-I := \{-x : x \in I\}$. Then $-I$ is also a generalized interval, and $|-I| = |I|$.

Proof. If I is a point or the empty set then this is easy to check. If I is an interval such as $[a, b]$, (a, b), $[a, b)$, or $(a, b]$, then $-I$ is an interval of the form $[-b, -a]$, $(-b, -a)$, $(-b, -a]$, or $[-b, -a)$, and has length $-a - (-b) = b - a = |I|$. □

Lemma. Let $f : [a, b] \to \mathbf{R}$ be a piecewise constant function, and let $g : [-b, -a] \to \mathbf{R}$ be the function $g(x) := f(-x)$. Then $p.c. \int_{[-b,-a]} g = p.c. \int_{[a,b]} f$.

Proof. Let's say that f is piecewise constant with respect to some partition $\mathbf{P}$ of $[a, b]$, and let's say that f has constant value c_J on each interval J on $\mathbf{P}$. Then $p.c. \int_{[a,b]} f = \sum_{J \in \mathbf{P}} c_J |J|$ (we may assume that $\mathbf{P}$ does not contain the empty set interval). Since f is constant on J, g is constant on $-J$ with the same constant value c_J. Since the intervals $\{-J : J \in \mathbf{P}\}$ partition $-[a, b] = [-b, -a]$ (why?), we thus have (by the above Lemma)

$$p.c. \int_{[a,b]} g = \sum_{J \in \mathbf{P}} c_J |-J| = \sum_{J \in \mathbf{P}} c_J |J| = p.c. \int_{[a,b]} f$$

as desired. □

Now let $f : [a, b] \to \mathbf{R}$ be a Riemann integrable function. Let $\varepsilon > 0$, then we can a piecewise constant function $\overline{f}$ majorizing f such that

$$p.c. \int_{[a,b]} \overline{f} \leq \int_{[a,b]} f + \varepsilon.$$

By the above Lemma we thus have

$$p.c. \int_{[-b,-a]} \overline{g} \leq \int_{[a,b]} f + \varepsilon$$

where $\overline{g} : [-b, -a] \to \mathbf{R}$ is the function $\overline{g}(x) := \overline{f}(-x)$. Since $\overline{f}$ majorizes f, $\overline{g}$ majorizes g, and so

$$\overline{\int}_{[-b,-a]} g \leq \int_{[a,b]} f + \varepsilon$$

A similar argument shows that

$$\underline{\int}_{[-b,-a]} g \geq \int_{[a,b]} f - \varepsilon.$$

Since the upper Riemann integral is always greater than or equal to the lower Riemann integral, we thus see that both the upper and lower Riemann integrals of g are ε-close to

$\int_{[a,b]} f$. Since ε is arbitrary, this means that $\overline{\int}_{[-b,-a]} g = \underline{\int}_{[-b,-a]} g = \int_{[a,b]} f$, and the claim follows.

(It is also possible to proceed by taking an antiderivative F of f, defining $G(x) := F(-x)$, and then showing that $G' = g$ and applying the fundamental theorem of calculus).

Problem 6. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a continuous, non-negative function (so $f(x) \geq 0$ for all $x \in [a, b]$). Suppose that $\int_{[a,b]} f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$. (Hint: argue by contradiction).

Suppose for contradiction that there exists an $x \in [a, b]$ such that $f(x) \neq 0$; since f is non-negative, this means that $f(x) > 0$. Let $\varepsilon := f(x)/2$; by continuity, we know that there exists a $\delta > 0$ such that $|f(y) - f(x)| \leq \varepsilon$ whenever $y \in [a, b]$ is such that $|y - x| < \delta$. In particular, we see that $f(y) \geq f(x)/2$ for all y in the interval $I = [x - \delta, x + \delta] \cap [a, b]$. Thus if we let $g : [a, b] \to \mathbf{R}$ be the function such that $g(y) := f(x)/2$ for $y \in I$, and $g(y) = 0$ otherwise, then g is piecewise constant minorizes f, thus $\int_I f \geq p.c. \int_I g = |I| f(x)/2 > 0$, a contradiction.

Problem 7. Let sgn : $\mathbf{R} \to \mathbf{R}$ be the function

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when} x < 0. \end{cases}$$

Let $f : [-1, 1] \to \mathbf{R}$ be a continuous function. Show that f is Riemann-Stieltjes integrable with respect to sgn, and $\int_{[-1,1]} f \, d\text{sgn} = 2f(0)$. (Hint: for every $\varepsilon > 0$, find piecewise constant functions majorizing and minorizing f whose Riemann-Stieltjes integral is ε-close to $2f(0)$).

Since f is continuous, it is bounded on $[-1, 1]$, and so there exists an M such that $-M \leq f(x) \leq M$ for all $x \in [-1, 1]$.

Let $\varepsilon > 0$. There exists $0 < \delta < 1$ such that $f(x)$ is ε-close to $f(0)$ for all $x \in [-\delta, \delta]$. Thus if we define $\overline{f} : [-1, 1] \to \mathbf{R}$ by setting $\overline{f}(x) := f(0) + \varepsilon$ for $x \in [-\delta, \delta]$, an $\overline{f}(x) := M$ on the remaining intervals $[-1, -\delta)$ and $(\delta, 1]$, then $\overline{f}$ is piecewise constant and majorizes f. Since $[-1, -\delta)$ and $(\delta, 1]$ has sgn-length 0, and $[-\delta, \delta]$ has sgn-length 2, we have

$$p.c. \int_{[-1,1]} \overline{f} \, d\text{sgn} = 0 \times M + 2 \times (f(0) + \varepsilon) + 0 \times M = 2f(0) + 2\varepsilon.$$

Thus we have

$$\overline{\int}_{[-1,1]} f \, d\text{sgn} \leq 2f(0) + 2\varepsilon.$$

A similar argument gives that

$$\underline{\int}_{[-1,1]} f \, d\text{sgn} \geq 2f(0) - 2\varepsilon.$$

By arguing as in Question 5 we thus see that

$$\underline{\int}_{[-1,1]} f \, d\text{sgn} = 2f(0)$$

as desired.

Problem 8. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a monotone increasing function. Let $F : [a, b] \to \mathbf{R}$ be the function $F(x) := \int_{[a,x]} f$. Let x_0 be an element of (a, b). Show that F is differentiable at x_0 if and only of f is continuous at x_0. (Hint: One direction is taken care of by one of the fundamental theorems of calculus. For the other, consider left and right limits of f and argue by contradiction).

Suppose for contradiction that F' is differentiable at x_0, but f is not continuous at x_0. Let $A := \sup\{f(x) : x \in [a, x_0)\}$, and $B := \inf\{f(x) : x \in (x_0, b]\}$. Since f is monotone increasing, then $A \leq f(x_0) \leq B$.

We claim that in fact $A < B$. To see this, suppose for contradiction that $A = B$, which implies $A = B = f(x_0)$. Then for any ε, there exists an $x_- \in [a, x_0)$ such that $f(x_-) > A - \varepsilon = f(x_0) - \varepsilon$, while similarly there exists $x_+ \in (x_0, b]$ such that $f(x_+) < B + \varepsilon = f(x_0) + \varepsilon$. Since f is monotone increasing, this implies that f is ε-close to $f(x_0)$ on $[x_-, x_+]$. In particular, if we set $\delta := \min(|x_0 - x_-|, |x_0 - x_+|) > 0$, then $f(x)$ is ε-close to $f(x_0)$ when x is δ-close to x_0. Since ε was arbitrary, we see that f is continuous at x_0, contradiction.

Now compute left and right limits of F'. If $x > x_0$ then

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{\int_{(x_0,x]} f}{x - x_0} \geq \frac{B(x - x_0)}{x - x_0} = B,$$

and so taking limits we see that $F'(x) \geq B$. Conversely, if $x < x_0$ then

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{-\int_{[x,x_0)} f}{x - x_0} \geq \frac{-A(x_0 - x)}{x - x_0} = A,$$

and so taking limits we see that $F'(x) \leq A$. But these facts contradict the fact that $A < B$, obtained earlier.

Problem 9. Let I be a generalized interval, let $f : I \to \mathbf{R}$ be a Riemann integrable function, and let $\mathbf{P}$ be a partition of I. Show that

$$\int_I f = \sum_{J \in \mathbf{P}} \int_J f.$$

The quickest proof is the following. For each J in $\mathbf{P}$, let $f_J : I \to \mathbf{R}$ be the function defined by setting $f_J(x) := f(x)$ for $x \in J$, and $f_J(x) = 0$ otherwise. Then (Theorem 13(g) of Week 9) we have $\int_J f = \int_I f_J$. Also, we have

$$\sum_{J \in \mathbf{P}} \int_I f_J = \int_I \sum_{J \in \mathbf{P}} f_J$$

(this follows from Theorem 13(a) of Week 9 and induction on the cardinality of $\mathbf{P}$). But for any $x \in I$, the summands in $\sum_{J \in \mathbf{P}} f_J(x)$ are mostly zero, except for the single generalized interval $J \in \mathbf{P}$ which contains x, and on this interval $f_J(x) = f_J(x)$. Thus $\sum_{J \in \mathbf{P}} f_J(x) = f(x)$, and the claim follows.

Problem 10. Let $(a_n)_{n=0}^{\infty}$ be a sequence which is not bounded. Show that there exists a subsequence $(b_n)_{n=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} 1/b_n$ exists and is equal to zero.

Define the sequence $n_0 < n_1 < n_2 < n_3 < \dots$ as follows. Choose $n_0 := 0$. n_1 to be a natural number larger than n_0 such that $|a_{n_1}| \geq 1$; such a number exists since $(a_n)_{n=0}^{\infty}$ is unbounded. Then, choose n_2 to be a natural number larger than n_1 such that $|a_{n_2}| > 2$; again, this exists since $(a_n)_{n=0}^{\infty}$ is unbounded. Proceeding recursively in this manner, we can construct an increasing sequence n_k such that $|a_{n_k}| > k$. If we set $b_k := a_{n_k}$, then $(b_k)_{k=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$ and $|b_k| \geq k$ for all k. Thus $-1/k \leq 1/b_k \leq 1/k$ for all k, and hence $1/b_k$ converges to 0 as desired, by the squeeze test.

Problem 11. Let X be a subset of $\mathbf{R}$, let x_0 be an adherent point of X, and let $f : X \to \mathbf{R}$, $g : X \to \mathbf{R}$, and $h : X \to \mathbf{R}$ be functions on X such that $f(x) \leq g(x) \leq h(x)$ for all $x \in X$. Let L be a real number, and suppose that

$$\lim_{x \to x_0; x \in X} f(x) = \lim_{x \to x_0; x \in X} h(x) = L.$$

Show that $\lim_{x \to x_0; x \in X} g(x) = L$. (Note: You may only use the squeeze test for *functions* if you explain why this test works. On the other hand, the squeeze test for *sequences* is in the notes and thus may be used to help solve this problem).

Let x_n be any sequence in X converging to x_0. We have to show that $g(x_n)$ converges to L as $n \to \infty$. But we already know that $f(x_n)$ converges to L and $h(x_n)$ converges to L. Since $f(x_n) \leq g(x_n) \leq h(x_n)$, the claim then follows from the squeeze test (for sequences).

Problem 12. Let $a < b$ be real numbers, and let $\phi : [a, b] \to \mathbf{R}$ be a continuous, strictly monotone increasing function. Let $\mathbf{P}$ be a partition of $[a, b]$. Let $\mathbf{Q}$ be the set

$$\mathbf{Q} := \{\phi(J) : J \in \mathbf{P}\}$$

where $\phi(J) := \{\phi(x) : x \in J\}$. Show that the sets $\phi(J)$ are all generalized intervals, and show that $\mathbf{Q}$ is a partition of $[\phi(a), \phi(b)]$. (Hint: for the first part, show that $\phi(J)$ is connected).

Since ϕ is continuous and strictly monotone increasing, it is invertible on $[\phi(a), \phi(b)]$ and the inverse is also continuous and strictly monotone increasing (Proposition 3 of Week 7/8 notes).

Since ϕ is continuous on $[a, b]$, it is uniformly continuous. Since J is bounded, $\phi(J)$ is thus also bounded. Now we show that $\phi(J)$ is connected. Let $x < y$ be elements of $\phi(J)$, then $\phi^{-1}(x) < \phi^{-1}(y)$, and $[\phi^{-1}(x), \phi^{-1}(y)]$ is a subset of $\phi(J)$. Since ϕ is continuous and strictly monotone, then $\phi([\phi^{-1}(x), \phi^{-1}(y)])$ is equal to $[\phi(\phi^{-1}(x)), \phi(\phi^{-1}(y))] = [x, y]$. Since $\phi([\phi^{-1}(x), \phi^{-1}(y)])$ is a subset of $\phi(J)$, we thus see that $[x, y]$ is contained inside $\phi(J)$. Thus $\phi(J)$ is connected; since it was also bounded, it is thus a generalized interval.

Now we show that $\mathbf{Q}$ is a partition of $[\phi(a), \phi(b)]$. First observe that ϕ maps $[a, b]$ to $[\phi(a), \phi(b)]$, so all the intervals $\phi(J)$ are indeed contained in $[\phi(a), \phi(b)]$. Now we need to show that every x in $[\phi(a), \phi(b)]$ is contained in exactly one interval $\phi(J)$, where $J \in \mathbf{P}$. But this is the same as saying that $\phi^{-1}(x)$ is contained in exactly one interval J in $\mathbf{P}$. Since $\phi^{-1}(x)$ lies in $[a, b]$, this follows from the assumption that $\mathbf{P}$ is a partition of $[a, b]$.