Assignment 7 (Due Feb 28). Covers: Week 7/8 notes

Many of these questions require you to provide examples of objects (sets, functions, etc.) which obey various properties. You should not only describe these objects, but also provide a brief explanation of why your objects obey the required properties. This explanation does not have to be as rigorous as for some of the proof-type questions, but should give some details beyond just vague words or pictures.

- Q1. Give examples of
- (a) A function $f : (1, 2) \to \mathbf{R}$ which is continuous and bounded, attains its minimum somewhere, but does not attain its maximum anywhere;
- (b) A function $f : [0, \infty) \to \mathbf{R}$ which is continuous, bounded, attains its maximum somewhere, but does not attain its minimum anywhere;
- (c) A function $f : [-1, 1] \to \mathbf{R}$ which is bounded but does not attain its minimum anywhere or its maximum anywhere.
- (d) A function $f : [-1, 1] \to \mathbf{R}$ which has no upper bound and no lower bound.
- Explain why none of the examples you construct violate the Maximum principle. (Note: read the assumptions *carefully*!)
- Q2 (a). Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a continuous function. If Y is a subset of X, show that the restriction $f|_Y : Y \to \mathbf{R}$ of f to Y is also a continuous function. (This is a simple result, but it requires you to follow the definitions carefully).
- Q2 (b). Prove Corollary 2 of Week 7/8 notes. (Hint: Use the intermediate value theorem and Q2(a)).
- Q3 (a). Explain why the Maximum principle remains true if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone. (You can use the same explanation for both cases).

- Q3 (b). Given an example to show that the intermediate value theorem becomes false if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone. (You can use the same counterexample for both cases).

- Q4*. In this question we give an example of a function which has a discontinuity at every rational point. Since the rationals are countable, we can write them as $\mathbf{Q} = \{q(0), q(1), q(2), \ldots\}$, where $q : \mathbf{N} \to \mathbf{Q}$ is a bijection from $\mathbf{N}$ to $\mathbf{Q}$. Now define a function $g : \mathbf{Q} \to \mathbf{R}$ by setting $g(q(n)) := 2^{-n}$ for each natural number n; thus g maps $q(0)$ to 1, $q(1)$ to 2^{-1}, etc. Since $\sum_{n=0}^{\infty} 2^{-n}$ is absolutely convergent, we see that $\sum_{r \in \mathbf{Q}} g(r)$ is also absolutely convergent. Now define the function $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) := \sum_{r \in \mathbf{Q} : r < x} g(r).$$

  Since $\sum_{r \in \mathbf{Q}} g(r)$ is absolutely convergent, we know that $f(x)$ is well-defined for every real number x.

- (a) Show that f is strictly monotone increasing. (Hint: You will need Proposition 25 from Week 2 notes).

- (b) Show that for every rational number r, f is discontinuous at r. (Hint: Since r is rational, $r = q(n)$ for some natural number n. Show that $f(x) \geq f(r) + 2^{-n}$ for all $x > r$.)

- Q5. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbf{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (Hint: divide into the three cases $f(a) < f(b)$, $f(a) = f(b)$, $f(a) > f(b)$. The second case leads directly to a contradiction. In the first case, use contradiction and the intermediate value theorem to show that f is strictly monotone increasing; in the third case, argue similarly to show f is strictly monotone decreasing).

- Q6*. Prove Proposition 3 from Week 7/8 notes. (Hint: to show that f^{-1} is continuous, it is easiest to use the "epsilon-delta" definition of continuity (Proposition 10(c) from Week 6 notes)).

- Q7. Prove Lemma 4 from Week 7/8 notes.

- Q8. Prove Proposition 5 from Week 7/8 notes. (Hint: You should avoid Lemma 4, and instead go back to the definition of equivalent sequences).
- Q9. (a) Prove Proposition 6 from Week 7/8 notes. (Hint: you should use the definition of uniform continuity directly).
- Q9. (b) Use Proposition 6 to prove Corollary 7 from Week 7/8 notes.
- Q10. Prove Proposition 8 from Week 7/8 notes. (Hint: Mimic the proof of Lemma 1. At some point you will need either Proposition 6 or Corollary 7).