Assignment 3 (Due January 30). Covers: Week 3/4 notes

Note: For this assignment you may freely use any laws of algebra or order concerning the natural numbers, integers, rationals, or reals, and anything from Weeks 1-2. For Questions 8-10, you can use everything from Weeks 1-3 (there is no danger of circularity since the results of those questions are not required in those week's notes).

- Q1. Prove Proposition 1 from the Weeks 3/4 notes.
- Q2 (a). Prove the Well-ordering principle from the Weeks 3/4 notes. (Hint: You can either use induction in a way similar to Q4(a) from Assignment 2, or use the least upper bound (or greatest lower bound) principle).
- Q2 (b). Does the well-ordering principle work if we replace the natural numbers by the integers? What if we replace the natural numbers by the positive rationals? Explain.
- Q3. Fill in the gaps marked (?) in the proof of Proposition 2 from the Weeks 3/4 notes. (Some hints. To prove that the sets A_n are infinite, one needs the following lemma: if X is infinite, and $x \in X$, then $X - \{x\}$ is also infinite; but this follows quickly from Proposition 1(a). To show that the sets A_n and the numbers a_n are well-defined, one may need to use induction on a fairly complicated property, such as

  $$P(n) = \text{“}A_n \text{ is well-defined and infinite, and } a_n \text{ is well-defined”}.$$

  For the rest of the proof, expect to use induction. A lot.)
- Q4. Prove Proposition 4 from the Weeks 3/4 notes. (Hint: The basic problem here is that f is not assumed to be one-to-one. Define A to be the set

  $$A := \{n \in \mathbf{N} : f(m) \neq f(n) \text{ for all } 0 \leq m < n\};$$

  informally speaking, A is the set of natural numbers n for which $f(n)$ does not appear any earlier in the sequence $f(0), f(1), f(2), \ldots$ than in the n^{th} position. Prove that when f is restricted to A, it becomes a bijection from A to $f(\mathbf{N})$. Then use Proposition 2.)

- Q5. Prove Corollary 5 from the Weeks 3/4 notes. (Hint: use Proposition 4).

- Q6. Prove Proposition 6 from the Weeks 3/4 notes. (Hint: By hypothesis, we have a bijection $f : \mathbf{N} \to X$ from $\mathbf{N}$ to X, and a bijection $g : \mathbf{N} \to Y$ from $\mathbf{N}$ to Y. Now define $h : \mathbf{N} \to X \cup Y$ by defining $h(2n) := f(n)$ and $h(2n+1) := g(n)$ for every natural number n. Then show that $h(\mathbf{N}) = X \cup Y$. Then use Corollary 5, and show that $X \cup Y$ cannot possibly be finite.)

- Q7. Prove Corollary 10 from the Weeks 3/4 notes. (You should not need anything other than Corollary 9 to prove this).

- Q8. Let X be a finite set with n elements, and let Y be a finite set with m elements. Prove that $X \times Y$ is a finite set with nm elements.

- Q9. Let X and Y be non-empty sets. Show that $X \times Y$ is uncountable if and only if at least one of X, Y is uncountable.

- Q10. Show that the set $\mathbf{R} - \mathbf{Q} = \{x \in \mathbf{R} : x \notin \mathbf{Q}\}$ of irrational numbers is an uncountable set. (Hint: Prove by contradiction).