Assignment 2 (Due January 23). Covers: Week 2 notes

Note: For this assignment you may freely use any laws of algebra or order from the Week 1 notes, or any material in the Week 2 notes preceding the result referenced in the homework question (there is no need to give an exact reference). You can use any other laws of algebra that you know about, provided you give a short explanation of why that law follows from the material in the Week 1 or Week 2 notes.

- Q1. Prove Proposition 1 from the Week 2 notes.
- Q2. Prove Proposition 2 from the Week 2 notes. (You do not need to prove 2(h) since this is already done in the notes).
- Q3. Prove Proposition 5 from the Week 2 notes. (Hint: use the Euclidean algorithm - Proposition 13 from Week 1 notes).
- Q4. A definition: a sequence $a_0, a_1, a_2, \ldots$ of numbers (natural numbers, integers, rationals, or reals) is said to be in *infinite descent* if we have $a_n > a_{n++}$ for all natural numbers n (i.e. $a_0 > a_1 > a_2 > \ldots$).
- Q4(a). Prove the *principle of infinite descent*: that it is not possible to have a sequence of *natural numbers* which is in infinite descent. (Hint: Assume for contradiction that you can find a sequence of natural numbers which is in infinite descent. Since all the a_n are natural numbers, you know that $a_n \geq 0$ for all n. Now use induction to show in fact that $a_n \geq k$ for all $k \in \mathbf{N}$ and all $n \in \mathbf{N}$, and obtain a contradiction).
- Q4(b) Does the principle of infinite descent work if the sequence $a_1, a_2, a_3, \ldots$ is allowed to take integer values instead of natural number values? What about if it is allowed to take positive rational values instead of natural numbers? Explain.
- Q4(c). Fill in the gaps marked (why?) in the proof of Proposition 6 of Week 2 notes.
- Q5. Prove Lemma 9 of the Week 2 notes. (Hint: Use the fact that a_n is eventually 1-steady, and thus can be split into a finite sequence and a 1-steady sequence. Then use Lemma 8 for the finite part.).
- Q6. Prove Proposition 10 of the Week 2 notes.

- Q7. Prove Proposition 13 of the Week 2 notes. (Hint: You will need to use Proposition 2(g) from Week 2 notes).
- Q8. Prove Proposition 18 of the Week 2 notes. (Hint: if x is not zero, and x is the formal limit of some sequence $(a_n)_{n=1}^{\infty}$, then this sequence cannot be eventually ε-close to the zero sequence $(0)_{n=1}^{\infty}$ for every single $\varepsilon > 0$. Use this to show that the sequence $(a_n)_{n=1}^{\infty}$ is eventually either positively bounded away from zero or negatively bounded away from zero).
- Q9*. Prove Proposition 25 of the Week 2 notes. (Hint: you may first need to find a natural number N such that $1/N < y - x$. You may also need to argue by contradiction.).
- Q10. Prove Proposition 29 of the Week 2 notes.