Assignment 1 (Due January 16). Covers: Week 1 notes

- Q1. Prove Proposition 4 from the Week 1 notes. (Hint: fix two of the variables and induct on the third).
- Q2. Prove Proposition 8 from the Week 1 notes. (Hint: you will need many of the preceding propositions, corollaries, and lemmas. For the last part of the Proposition, it may be helpful to prove the following auxiliary result: every natural number is either equal to zero, or is the successor of a natural number).
- Q3. Justify the three statements marked (why?) in the proof of Proposition 9.
- Q4. Prove the statements at the bottom of page 23 of the Week 1 notes, i.e. that $n \times 0 = 0$, $n \times (m++) = n \times m + n$, $n \times m = m \times n$, and $(a \times b) \times c = a \times (b \times c)$ for all natural numbers n, m, a, b, c.
- Q5. Prove the statements at the top of page 24 of the Week 1 notes, i.e. that if a and b are positive, then ab is positive, and that if $ab = 0$, then either $a = 0$ or $b = 0$ or both.
- Q6. Prove Proposition 13 of the Week 1 notes. (Hint: Fix q and induct on n).
- Q7. Prove Proposition 17 of the Week 1 notes. (Note that this is more general than the corresponding statement in Q5, because we are now considering integers rather than natural numbers. Of course, you may use the results of Q5 in order to prove Q7).
- Q8. Prove Lemma 18 of the Week 1 notes. (You will have to use many of the preceding Propositions and Lemmas).
- Q9. Prove Lemma 21 of the Week 1 notes. (Note that, as in Proposition 9, you have to prove two different things: firstly, that *at least* one of (a), (b), (c) is true; and secondly, that *at most* one of (a), (b), (c) is true).
- Q10. Prove Proposition 22 of the Week 1 notes.

Some general notes on proofs: One purpose of this course is to help you write proofs in a correct and professional manner. To this end, you are encouraged to make your proofs as detailed as possible. You are certainly encouraged to write several English sentences in your proofs, and to use logical connectives ("since", "if... then", "because", "we have", "let", "therefore", "thus", "by hypothesis", etc.) to clarify the logical structure of your proofs; merely laying out a long sequence of mathematical equations without supplying any explanation may (barely) be acceptable, but don't count on it. Also, the following types of proofs may result in partial credit at best:

- **Circular reasoning.** This can occur if you use a result from later in the notes to prove a result from earlier in the notes, or if you use more advanced theory to prove simpler concepts. Of course, using results from earlier in the notes to prove results later in the notes is OK; so is using earlier HW questions to prove later HW questions. Because of the need to avoid circular reasoning, some of the most elementary results may, paradoxically, be the hardest for you to prove (because so many results that you know are not permitted to be used).

- **Proof by example.** This occurs when one is asked to prove a *universal* statement (e.g. "Property P is true for all integers n") and instead one just supplies a single example ("Property P is true for a single integer n"). However, proofs by example are valid for *disproving* a universal statement (i.e. finding a counterexample), or in proving *existence* questions (e.g. "Show that there exists an integer n which satisfies property P").

- **Proof by appeal to intuition.** You have plenty of intuition already about the natural numbers and related number systems, and this will be very helpful for you in constructing proofs. However, even when claiming an intuitively obvious statement in a proof, you still have to supply rigorous justification (though you are certainly encouraged to also *add* your own intuitive remarks; just don't rely on them by themselves!). Also, your intuition may use more advanced concepts than what is needed for the question at hand, and so by relying on intuition too much you may fall into circular reasoning (see above). One of the purposes of this course is to allow you to analyze your own intuition and see what assumptions it is really based on.