Justify all answers.

(1) Let f be a Morse function on closed n-manifold M. Write down a formula for the Euler characteristic $\chi(M)$ in terms of the indices of the critical points.

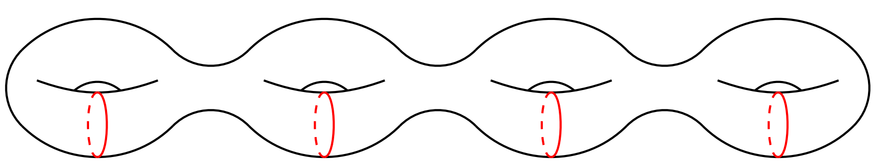
(2) Considering Morse functions f and $-f$, prove closed odd dimensional manifolds have $\chi = 0$.

(3) Let $(Y, \partial Y)$ be a compact n-manifold with boundary. How does χ for the manifold and its boundary change when one attaches an n-dimensional k-handle to its boundary?

(4) For any index-k critical point of a Morse function, prove that a neighborhood of its downward disk and upward disk is a k-handle, with the two disks being identified as the core and the co-core.

(5) Prove that $\mathbb{Z}^n$ is not the π_1 of any closed 3-manfold for $n > 3$.

(6) Let $\alpha_1, \ldots, \alpha_g$ be disjoint embedded circles in Σ_g. Prove that the following are equivalent:
  (a) $\Sigma_g \setminus (\cup_i \alpha_i)$ is connected.
  (b) $[\alpha_i]$'s span a half-dimensional subspace of $H_1(\Sigma_g)$.
  (c) If one attaches g 2-handles to $I \times \Sigma_g$ along $0 \times \alpha_i$, the boundary of the resulting 3-manifold is $1 \times \Sigma_g \amalg S^2$.
  (d) $(\Sigma_g, \alpha_1, \ldots, \alpha_g)$ is diffeomorphic to the standard picture, shown below for $g = 4$:



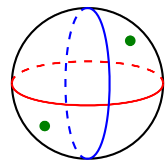
(7) Treat the torus the quotient of $\mathbb{R}^2$ by the standard $\mathbb{Z}^2$ action, and let α, β be the images of the slope 0 and the slope p/q line, with either $p = 1, q = 0$ or $p > 0, q \neq 0, \gcd(p, q) = 1$. Letting $\mathcal{H}_{p,q}$ denote this Heegaard diagram, prove that it is diffeomorphic to the Heegaard diagram $\mathcal{H}_{p,q+kp}$ for any $k \in \mathbb{Z}$.

(8) Let $p > 0, q \neq 0, \gcd(p, q) = 1$. Prove that $\mathbb{Z}/p$ acts by a covering space action on $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$ by $\eta(z, w) = (\eta z, \eta^q w)$ where $\eta = e^{2\pi i/p}$. Prove that it induces covering space actions on the standard Heegaard splitting into $D^2 \times S^1 = \{|z| \leq 1, |w| \geq 1, |z|^2 + |w|^2 = 2\}$ and $S^1 \times D^2 = \{|z| \geq 1, |w| \leq 1, |z|^2 + |w|^2 = 2\}$.

(9) Prove that the Heegaard diagram $\mathcal{H}_{p,q}$ represents the lens space $L(p, q)$, which is the quotient of S^3 by the above $\mathbb{Z}/p$ action.

(10) Let $\alpha_1, \ldots, \alpha_{g+k-1}$ be disjoint embedded circles in Σ_g. Prove that the following are equivalent:
  (a) $\Sigma_g \setminus (\cup_i \alpha_i)$ has exactly k components.
  (b) $[\alpha_i]$'s span a half-dimensional subspace of $H_1(\Sigma_g)$.
  (c) If one attaches $g + k - 1$ 2-handles to $I \times \Sigma_g$ along $0 \times \alpha_i$, the boundary of the resulting 3-manifold is $1 \times \Sigma_g \amalg (\coprod_{i=1}^{k} S^2)$.

(11) What manifold does the following genus 0 Heegaard diagram represent?



(12) If Heegaard diagrams $\mathcal{H}, \mathcal{H}'$ represent 3-manifolds Y, Y', prove that $\mathcal{H}\#\mathcal{H}'$ represents $Y\#Y'$.

(13) Prove that if one changes the attaching sphere of k-handle by an isotopy (preserving its normal bundle trivialization), one gets diffeormophic manifolds after the k-handle attachment.

(14) Prove that given a Morse function with a generic gradient like flow, one can make it self-indexing.

(15) Prove odd dimensional manifolds cannot have almost complex structures.

(16) Let (M^{2n}, ω) be a symplectic manifold and $L \subset M$ be a submanifold with $\omega|_L = 0$. Prove $\dim L \leq n$.

(17) Consider $\mathbb{C}^n$ with complex coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$ with the standard symplectic form $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$. Prove that the set of all compatible almost complex structures is contractible. (Hint: Prove that it is convex.)

(18) Prove that the set of all linear automorphisms of $\mathbb{C}^n$ that preserve the standard symplectic form is $U(n)$. v

(19) Let $\mathbb{R}^n$ be the Lagrangian subspace of $\mathbb{C}^n$ generated by the x_i's. Prove that the subgroup of $U(n)$ that preserves $\mathbb{R}^n$ (setwise) is isomorphic to $O(n)$. Identify this subgroup $O(n)$ inside $U(n)$.

(20) Conclude that the set of all linear Lagrangian subspaces of $\mathbb{C}^n$ is isomorphic to $U(n)/O(n)$.

(21) Let $f\colon M \to \mathbb{R}$ be a Morse function, and x, y be two critical points. Consider paths $u\colon \mathbb{R} \to M$ with $\lim_{t\to-\infty} u(t) = x$ and $\lim_{t\to+\infty} u(t) = y$. Define the energy/action to be $\int_{\mathbb{R}} u^* df$. Prove that this depends only on x, y and is independent of u. If we fix a metric on M, and assume u is a non-constant gradient flow—that is, $u'(t) = \nabla f(u(t))$—prove that the energy is positive.

(22) Let (M, ω, L_0, L_1) as in the setup of Lagrangian Floer homology. Fix $x, y \in L_0 \cap L_1$ and $\phi \in \pi_2(x, y)$. Consider strips $u\colon [0,1] \times \mathbb{R} \to M$ representing ϕ (with $\lim_{t\to-\infty} u(s,t) = x$ and $\lim_{t\to+\infty} u(s,t) = y$). Define the energy action to be $\int_{[0,1]\times\mathbb{R}} u^*\omega$. Prove that this is a well-defined function on $\pi_2(x, y)$, and is additive under the concatenation map $\pi_2(x, y) \times \pi_2(y, z) \to \pi_2(x, z)$. If J is a compatible almost complex structure, and u is a non-constant J-holomorphic map, prove that the energy is positive.

(23) Let $(\mathrm{Sym}^n(\Sigma), T_\alpha, T_\beta)$ be as in the setup of Heegaard Floer homology. For $\phi \in \pi_2(x, y)$, construct its shadow $D(\phi)$ as a well-defined 2-chain on $(\Sigma, \alpha \cup \beta)$. Prove that $\partial(\partial D(\phi) \cap \alpha) = y - x$.