Week 1.

(1) Prove that the isotopy class of the positive trefoil $T_{2,3}$ is unchanged under orientation reversal.

(2) Let $K\colon S^1 \to \mathbb{R}^3$ be a smooth knot. Prove that there exists a continuous map $H\colon [0,1] \times S^1 \to \mathbb{R}^3$ such that $K_t := H|_{t\times S^1}$ is a smooth knot for all t, and $K_0 = K$ and K_1 is the unknot. (You need to prove continuity of the map that you construct.)

(3) Prove that the Seifert form is well-defined, that is, independent of the choice of curves representing the homology elements.

(4) Prove that $\sigma(K)$, $\det(K)$, $\mathrm{Det}(K)$, and $\Delta_K(t)$ are independent of the choice of Seifert surface for K.

(5) Draw two non-isotopic diagrams of the positive trefoil $T_{2,3}$. (Non-isotopic diagrams means they are not related by an isotopy of $\mathbb{R}^2$.)

(6) Prove that two Seifert surfaces obtained by applying Seifert's algorithm to above two diagrams are isotopic (in $\mathbb{R}^3$).

(7) Compute the Seifert matrix of $T_{2,3}$ using the above Siefert surface. Compute $\sigma(T_{2,3})$, $\det(T_{2,3})$, $\mathrm{Det}(T_{2,3})$, and $\Delta_{T_{2,3}}(t)$.

(8) We proved in class that for a knot K, $\Delta_K(1) = 1$. What is $\Delta_L(1)$ for a link L?

(9) Complete the proof from class of the skein relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})\Delta_{L_0}(t).$$

(10) We proved in class that for a knot K, $\Delta_K(1/t) = \Delta_K(t)$. What is the corresponding statement for $\Delta_L(1/t)$ for a link L?

(11) Prove that the skein relation uniquely determines the Alexander polynomial. That is, if $f_L(t)$ is a polynomial link invariant with $f_U(t) = 1$ and it satisfies

$$f_{L_+}(t) - f_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})f_{L_0}(t)$$

then prove $f_L(t) = \Delta_L(t)$.

Week 2.

(12) If L_0 is obtained from L by resolving a crossing, prove that there exists a saddle cobordism connecting them.

(13) Prove that connected sum is a well-defined operation on oriented knots.

(14) If $-K$ denotes the reverse of the mirror of K, prove that $K\# - K$ is slice for all knots K.

(15) Prove that the torus links $T_{p,q}$ and $T_{q,p}$ are isotopic.

(16) Prove $\sigma(L_+) \leq \sigma(L_-) \leq \sigma(L_+) + 2$.

(17) Prove that any link cobordism in $I \times S^3$ can be isotoped so that all the births happen at the beginning and all the deaths happen at the end. (Hint: Try to prove it by viewing the cobordism as a movie.)

(18) What is the genus of the surface produced by Seifert's algorithm on the standard diagram of a (p,q) torus link? (p,q need not be relatively prime.)

(19) If a knot K bounds an immersed surface of genus g with k ribbon singularities and no other singularities, prove that K also bounds a Seifert surface of genus $g+k$.

(20) Prove that the homology of a chain complex over a graded ring R is a graded module over R.

(21) Prove that the homotopy category $K(R)$ is a well-defined category; that is, prove chain homotopy equivalence is an equivalence relation and it respects composition.

(22) Find a ring R and finitely and freely generated chain complexes C, D over R which are not isomorphic in the homotopy category $K(R)$, but become isomorphic in the derived category $D(R)$.

Week 3.

(23) Write down all the variants for grid commutation and grid stabilization, and prove that they all keep the link type unchanged.

(24) Prove that each of the three Reidemeister moves can be realized by grid moves.

(25) Prove that the (p, q) torus link $T_{p,q}$ is indeed represented by the index $(p+q)$ grid diagram that I drew in class. Use that to show $T_{p,q} = T_{q,p}$ and $T_{p,q} = r(T_{p,q})$. What is the grid diagram that represents its mirror?

(26) Complete the check that the differential ∂ in the fully blocked grid complex satisfies $\partial^2 = 0$.

(27) Complete the proof that the Maslov grading $M(x) = 1 + \mathcal{I}(x - O, x - O)$ is well-defined. Show that it is an integer.

(28) Compute the bigraded fully blocked chain complex for an index-3 picture of an unknot.

(29) Compute the bigraded fully blocked chain complex of the standard index-4 diagram for the Hopf link $T_{2,2}$.

Week 4.

(30) Compute $\widehat{GH}$ for the index-1 and index-2 grid diagrams for the unknot.

(31) If $f, g\colon C \to D$ are chain homotopic chain maps, prove that their cones are chain homotopy equivalent chain complexes.

(32) Assume C is a freely generated chain complex over $R[U]$. Prove that the chain complex $C/U = C \otimes_{R[U]} R$ is chain homotopy equivalent to the cone of U, over R.

(33) Use the above two to prove that $\widetilde{CG}$ is chain homotopy equivalent to $2^{n-\ell}$ copies of $\widehat{CG}$.

Week 5–6.

(34) For commutation $G \to G'$, prove that the pentagon map $CG^-(G) \to CG^-(G')$ is a chain map. Prove that the hexagon map $CG^-(G) \to CG^-(G)$ is a chain homotopy from the composition of two pentagon maps to the identity.

(35) For stabilization $G \to G'$, construct the chain maps and chain homotopies between $CG^-(G')$ and the mapping cone $CG^-(G)[W_{n+1}] \xrightarrow{W_1 - W_{n+1}} CG^-(G)[W_{n+1}]$.

(36) Prove that the above mapping cone is chain homotopy equivalent to $CG^-(G)$ over $\mathbb{F}[W_1, \dots, W_n]$ (but not over the bigger ring $\mathbb{F}[W_1, \dots, W_{n+1}]$).

Week 7.

Week 8.

Week 9.

Week 10.