*Differential Geometry of Curves and Surfaces* by Do Carmo.
*Elements of Differential Geometry* by Millman and Parker.
(The notation and conventions are different in the two books. In class, we will follow conventions from Do Carmo and notation from Millman and Parker.)

*Background.*

(1) Derivative. Given differentiable $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$, and a point $a \in \mathbb{R}^m$, the derivative $df_a$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$. If we fix coordinates $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, that fixes a basis, and can represent $df_a$ by a $n \times m$ matrix whose entries are $\partial y_j / \partial x_i$. So $df \colon \mathbb{R}^m \to \mathbb{R}^{mn}$ is the derivative function.
(2) $C^0$ means continuous. If $df$ exists and $df \colon \mathbb{R}^m \to \mathbb{R}^{mn}$ is $C^{k-1}$, then $f$ is $C^k$. Inductive definition. $C^k$ means $k$ times differentiable and $k^{\text{th}}$ derivative is continuous. $C^\infty$ is smooth, infinitely differentiable.
(3) Chain rule. Consider $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^p$, and fix coordinates $x_1 \ldots, x_m, y_1 \ldots, y_n, z_1, \ldots, z_p$, and a starting point $a \in \mathbb{R}^m$. Then $a \mapsto f(a) \mapsto g(f(a))$, and we have linear maps $\mathbb{R}^m \xrightarrow{df_a} \mathbb{R}^n \xrightarrow{dg_{f(a)}} \mathbb{R}^p$. Then as linear maps $d(g \circ f)_a = dg_{f(a)} \circ df_a$. As matrices $d(g \circ f)_a = dg_{f(a)} df_a$. In terms of matrix entries, $\frac{\partial z_k}{\partial x_i} = \sum_{j=1}^n \frac{\partial z_k}{\partial y_j} \frac{\partial y_j}{\partial x_i}$. Double index notation, latter is written $\sum \frac{\partial z_k}{\partial y_j} \frac{\partial y_j}{\partial x_i}$.

*Basic curves in 3D.*

(1) Fix interval $I \subset \mathbb{R}$, open or closed (could be $I = \mathbb{R}$) Curve $\alpha \colon I \to \mathbb{R}^3$, different from its image. Think of particle traveling in space; image is the path traced out.
(2) Velocity $\alpha' = d\alpha/dt \colon I \to \mathbb{R}^3$. We view $\alpha(t)$ as a point in $\mathbb{R}^3$, but $\alpha'(t)$ as a 3D vector, drawn starting at $\alpha(t)$. This is also the tangent vector to the curve. Speed is length of the vector $|\alpha'|$.
(3) To avoid sharp corners, we will assume curves are regular, that is $\alpha'(t) \neq 0 \forall t$. The path  can be the image of a $C^\infty$ (smooth) path, but it has a sharp corner, and doesn't look smooth. If we impose regularity, then well-defined tangent direction at each point, and so no more sharp corners.
(4) For regular curves, unit tangent $T(t) = \alpha'(t)/|\alpha'(t)|$, unit vector in the tangent direction. Velocity $\alpha'$ was extrinsic (depends on how a particle travels a given path), but $T(t)$ is clearly intrinsic (depends only on the path and direction on travel).
(5) Reparametrization. Consider two intervals $I, J$, and $h \colon I \to J$ a bijection, so that both $h$ and $h^{-1}$ are $C^3$. Then if $\alpha \colon I \to \mathbb{R}^3$ and $\beta \colon J \to \mathbb{R}^3$ are related by $\alpha = \beta \circ h$ (equivalently $\beta = \alpha \circ h^{-1}$), then one is a reparametrization of the other. Both of them have the same image, represent the same path in space, but traveled differently (with different speed, etc).
(6) Reparametrization again, $h \colon I \to J$, $g \colon J \to I$, $g \circ h = \mathrm{Id}_I$, $h \circ g = \mathrm{Id}_J$, both $C^k$. If $\alpha, \beta$ related by reparametrization, takes regular curves to regular curves. Clearly $C^k$ to $C^k$. $\frac{\partial \beta}{\partial s}|_a = \frac{\partial \alpha}{\partial t}|_{g(a)} \frac{\partial g}{\partial s}|_a$, so just need to show $\frac{\partial g}{\partial s}|_a \neq 0$. Chain rule again on $h \circ g = \mathrm{Id}_J$.
(7) Alternate description of reparametrization. $h \colon I \to J$ onto, $h \in C^k$, and $h'(t) \neq 0 \forall t$. Injective by Mean Value Theorem; indeed two cases, $h' > 0$ or $h' < 0$ (strictly increasing or strictly decreasing). So if $g = h^{-1}$, why $g \in C^k$?
(8) Review Inverse Function theorem $f \colon \mathbb{R}^n \to \mathbb{R}^n$. If $df_a$ $(n \times n)$ non-singular, then locally has inverse, and inverse is also $C^k$. In case $n = 1$ (like now), has global inverse (as we saw), but for general $n$, not. Example: $\mathbb{C} \to \mathbb{C}, z \mapsto e^z$, so $f \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (e^x \cos y, e^x \sin y)$, compute $|df| = e^x \neq 0$, but no global inverse since not injective.
(9) If two curves $\alpha, \beta$, related by reparametrization $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$, but possibly traveled with different speeds. If $h' > 0$, then same orientation (direction of travel), otherwise opposite.
(10) What about converse? If two regular curves have same image, they are related by reparametrization, since regular curves have a unique canonical reparametrization.

*Arc length parametrization.*

(1) Assume regular curve $\alpha$. Arc length. Consider starting point $\alpha(t_0)$. Arc length is distance traveled from $t_0$ to $t$, integral of speed $s(t) = \int_{t_0}^t |\alpha'(t)| dt$. This is intrinsic, depends only on the starting point, and the direction of travel, independent of parametrization.
(2) Say $h = s$ is the arc-length parametrization, and $J = \mathrm{Im}(h)$. Why is $h \colon I \to J$ a reparametrization? If $g$ is the inverse, define the arc-length parametrization $\beta(s) = \alpha \circ g(s)$.

(3) What is the velocity (which was extrinsic) under arc-length parametrization? $\frac{d\beta}{ds} = \frac{d\alpha}{dt}\frac{dg}{ds} = \frac{d\alpha}{dt}/\frac{dh}{dt} = \frac{d\alpha}{dt}/|\frac{d\alpha}{dt}| = T$, the unit tangent. In particular, unit speed. So arc-length parametrization is canonical: start at the starting point, and travel with unit speed in the given orientation.
(4) Example. Consider the curve $\alpha(t) = (r\cos t, r\sin t, ht)$. Assume $r, h > 0$ constants. Do arc length reparametrization of helix. Useful variable $\omega = \frac{1}{\sqrt{r^2+h^2}}$.
(5) Draw $(r\cos t^2, r\sin t^2, ht^2)$. Different image from the helix. Sharp turn, not regular at $t = 0$.
(6) Draw $(r\cos t^3, r\sin t^3, ht^3)$. Same image as the helix, but still not a reparametrization since not regular at $t = 0$. The change of variable functions are $t^3$ (which is $C^\infty$) and $t^{1/3}$ (which is not $C^1$).
(7) Although arc length parametrization always possible for unit speed curves, very hard in practice. Consider the parabola $y = x^2/2$, find arc length parametrization starting at $(0, 0)$. First find regular parametrization $\alpha(t) = (t, t^2/2)$, then $s(t) = \int_0^t \sqrt{1+t^2}dt = \frac{1}{2}(t\sqrt{1+t^2}+\ln(t+\sqrt{1+t^2}))$. (Do $t = \tan\theta$, then integrate $\sec^3\theta$ by parts.) Then arc length parametrization is $\alpha(t(s))$, but impossible to write down the inverse function $t(s)$.

*Curvature and torsion.*

(1) Nevertheless, only consider unit speed curves $\alpha(s)$ from now. Velocity is unit tangent: $\alpha'(s) = T(s)$. Acceleration $\alpha''(s) = T'(s)$ measures rate of change of $T$, so change in the direction of travel. So curvature $\kappa(s) = |T'(s)|$.
(2) If $\kappa(s) \neq 0$, then $N(s) = T'(s)/\kappa(s)$, the direction of the rate of change. So $T' = \kappa N$. Note, $N$ only defined when curvature is non-zero.
(3) $N$ is perpendicular to $T$, since for any unit vector $v$, $v'$ is perpendicular to $v$ (differentiate $v \cdot v$).
(4) Do helix example. $\kappa = r\omega^2 = \frac{r}{r^2+h^2}$, and $N$ points inwards towards the axis. Special case, $h = 0$, circle of radius $r$. Curvature is $1/r$, smaller circles have larger curvature.
(5) From now on assume $\kappa(s) \neq 0 \forall s$. Needed to make sense of $N$. Then we get a right-handed orthonormal basis $T, N, B$, where $B = T \times N$; explain.
(6) Orthonormal frame, right-handed (positive) vs left-handed (negative). $(i, j, k)$, $(k, j, i)$, $(j, k, i)$ are positive. Cyclic (even) permutations. Form a basis. Any vector can be written uniquely, and the coefficients are given by dot products.
(7) Let's keep differentiating. $N'$ is perpendicular to $N$, so $N' = aT + bB$. Since $N, T$ orthogonal, $a = -\kappa$. Let $b = -\tau$, torsion. (Osculating plane spanned by $T, N$; the plane spanned by $N, B$ is normal plane, and the plane spanned by $T, B$ is rectifying plane. $\tau$ is the rate of rotation of $B$ in the normal plane.)
(8) One more derivative: $B' = \tau N$. ($B'$ perpendicular to $B$, and its coefficient at $N, T$ is negative of the coefficients of $N', T'$ at $B$.)
(9) The data $(\kappa, \tau, T, N, B)$ called Frenet-Serret apparatus. Very important to remember, only considering the Frenet-Serret apparatus for unit speed curve. If not-unit speed, then reparametrize (which might be hard) and then consider this data. Moreover, $N, B, \tau$ only defined if $\kappa \neq 0$.
(10) Calculate $(\kappa, \tau, T, N, B)$ for the helix $(r\cos t, r\sin t, ht)$, $\tau = h\omega^2 = \frac{h}{r^2+h^2}$.
(11) If $\kappa = 0$ on an interval, then $T$ constant, so linear. If $\kappa = 0$ at an isolated point, this story not valid, and the osculating plane can change drastically, $\alpha(t) = (t, e^{-1/t^2}, 0)$ or $(t, 0, e^{-1/t^2})$.
(12) If $\kappa > 0, \tau = 0$ on an interval, then $B$ constant, so particle travels in the plane perpendicular to $B$.
(13) Frenet-Serret equation

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Proof: Decompose a vector along orthonormal basis by dot products. Treating $T, N, B$ as row vectors. So the equation is $3 \times 3 = (3 \times 3)(3 \times 3)$. Also note the matrix is skew-symmetric.

*Fundamental theorem of curves.*

(1) Picard's theorem. $I \subset \mathbb{R}$ open interval around 0, $c \in \mathbb{R}^n$, $A\colon \mathbb{R}^n \times I \to \mathbb{R}^n$, uniformly (in time) bounded partial derivatives wrt space coordinates. Then unique $\alpha\colon I \to \mathbb{R}^n$ with $\alpha(a) = c$ and $d\alpha/dt = A(\alpha(t), t)$. Particle traveling in $\mathbb{R}^n$, initial condition specified, and velocity specified depending on its position and time.
(2) Counterexample $x' = x^{1/3}, x(0) = 0$. No solutions for $t < 0$, two solutions $x = \pm\sqrt{(2t/3)^3}$ for $t > 0$.
(3) Outline of proof; iterations of integrals. $\phi_n(t) = c + \int_a^t A(\phi_{n-1}(t), t)dt$.

(4) Fundamental theorem of curves. Any regular curve with $\kappa > 0$ is uniquely determined by $\kappa$ and $\tau$. More precisely: $0 \in I \subset \mathbb{R}$ ($I$ open interval), $\overline{\kappa} \in C^1(I)$ with $\overline{\kappa}(s) > 0 \forall s$, $\overline{\tau} \in C^0(I)$, $x_0 \in \mathbb{R}^3$, $D, E, F$ right-handed orthonormal basis of $\mathbb{R}^3$. Then unique $C^3$ unit-speed curve $\alpha \colon I \to \mathbb{R}^3$ with $\alpha(0) = x_0$, $(T(0), N(0), B(0)) = (D, E, F)$, $\kappa(s) = \overline{\kappa}(s)$, and $\tau(s) = \overline{\tau}(s)$.

(5) Proof. Picard uniquely specifies the frame $\overline{T}, \overline{N}, \overline{B}$. The function $A \colon \mathbb{R}^9 \times I \to \mathbb{R}^9$ is given by

$$A(\overline{T}_1, \overline{T}_2, \overline{T}_3, \overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{B}_1, \overline{B}_2, \overline{B}_3, s) = \begin{pmatrix} 0 & \overline{\kappa}(s) & 0 \\ -\overline{\kappa}(s) & 0 & -\overline{\tau}(s) \\ 0 & \overline{\tau}(s) & 0 \end{pmatrix} \begin{pmatrix} \overline{T}_1 & \overline{T}_2 & \overline{T}_3 \\ \overline{N}_1 & \overline{N}_2 & \overline{N}_3 \\ \overline{B}_1 & \overline{B}_2 & \overline{B}_3 \end{pmatrix}.$$

So satisfies the Lipshitz condition. For instance

$$\frac{\partial A}{\partial \overline{T}_1} = \begin{pmatrix} 0 & 0 & 0 \\ -\overline{\kappa}(s) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\overline{\kappa}, \overline{\tau} \in C^0$, bounded (on any closed interval in $I$, which is enough). Define $\alpha(s) = x_0 + \int_0^s \overline{T}(s) ds$.

(6) Show $\overline{T}, \overline{N}, \overline{B}$ positive orthonormal. Consider Picard's theorem again with six variables $\overline{x} \cdot \overline{y}$, $x, y \in \{T, N, B\}$. Uniqueness of solution forces orthonormality. Positivity is forced by continuity of $[\overline{T}, \overline{N}, \overline{B}]$—triple scalar product.

(7) Immediate that $\alpha$ is unit speed. To show $\alpha \in C^3$, need $T \in C^2$, so $\overline{\kappa} N \in C^1$, which we have. (This is exactly where we needed $\overline{\kappa}$ to be $C^1$—to ensure the curve is $C^3$, which is needed in order to define torsion.)

(8) Easy to show $\overline{x} = x$ for $x = T, N, \kappa, B, \tau$ in this order.

(9) If $\tau$, $\kappa > 0$ constant, then circular helix. Proof. Just show it satisfies the equation, with $r = \frac{\kappa}{\kappa^2 + \tau^2}$, $h = \frac{\tau}{\kappa^2 + \tau^2}$. ($\tau < 0$ right-handed, $\tau > 0$ left-handed, $\tau = 0$ circle.)

*Non-unit speed curves.*

(1) Regular but not (necessarily) unit speed curve, $\alpha(s(t))$. Derivatives wrt $s$ are prime, wrt $t$ are dots. Chain rule, $\dot{x} = x' \dot{s}$. Let speed $v = \dot{s} = |\dot{\alpha}|$. Running example $\alpha(t) = (t, t^2, t^3)$.

(2) Rest follows by just differentiating $\dot{\alpha} = vT$. ($T = \dot{\alpha}/v$.)

(3) $\ddot{\alpha} = \dot{v}T + v\dot{T} = \dot{v}T + v^2 \kappa N$, $\dot{\alpha} \times \ddot{\alpha} = v^3 \kappa B$. ($\kappa = |\dot{\alpha} \times \ddot{\alpha}|/v^3$, $B = \dot{\alpha} \times \ddot{\alpha}/\kappa v^3$, $N = B \times T$.)

(4) $\dddot{\alpha} = \ddot{v}T + \dot{v}v\kappa N + (\dot{v^2 \kappa})N - v^3 \kappa^2 T - v^3 \kappa \tau B$, $[\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}] = -(v^3 \kappa)^2 \tau$. ($\tau = -[\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}]/(v^3 \kappa)^2$.)

(5) Frenet-Serret equations.

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa v & 0 \\ -\kappa v & 0 & -\tau v \\ 0 & \tau v & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

*Rotation index.*

(1) Define $T, n$ for plane $C^2$ curves, by setting $b = \hat{k}$ and $n = b \times T$. Explain $n = \pm N$, if the latter is defined, and explain the sign. ($n$ is obtained by rotating $T$ by 90°.) Define planar curvature $k_P$ by $T' = k_P n$, which can be negative (turning left vs turning right.) $k_P$ measures the rate of change of direction of $T$.

(2) More precise version of the previous statement. Write $T(s) = (\cos(\theta(s)), \sin(\theta(s)))$, explain how to make sense locally. If $T(s)$ is $C^1$, so is $\theta(s)$. Then $k_P(s) = \theta'(s)$.

(3) Homework problem. Find curve with $\kappa(s) = \frac{1}{1+s^2}$, $\tau(s) = 0$, $x_0 = 0$, $(D, E, F) = (i, j, k)$. Planar motion in $xy$ plane. $T(s) = (\cos\theta(s), \sin(\theta(s)))$, and $\kappa(s) = k_P(s) = \theta'$. (Note $\kappa(s)$ never zero, so binormal $B$ throughout $\hat{k}$ constant, same as $b$.)

(4) After fixing $\theta(0)$, globally define $\theta(s)$ by cutting interval into small pieces where $\theta(s)$ doesn't change by more than 180° (points up, down, right, or left, only). Alternate definition $\theta(s) = \theta(0) + \int_0^s k(s) ds$.

(5) Periodic (closed) curve, $\alpha(s + L) = \alpha(s) \forall s$, period is smallest such $L$ (which is length of the curve $\alpha \colon [0, L] \to \mathbb{R}^2$. Index of periodic curve $\frac{1}{2\pi} \int_0^L k ds$.

(6) Index is total change of $\theta$ divided by $2\pi$ since $\theta' = k_P$. Interesting index examples: circle oriented both ways, clover leaf.

(7) Simple closed curve, no self-intersection, $\alpha(s) \neq \alpha(t) \forall s \neq t \in [0, L)$. Total index is $\pm 1$.

(8) Jordan curve theorem (very hard to prove). Complement of simple closed curve has outside (the non-compact region containing $\infty$) and inside.

(9) Simple closed curve is positively oriented if the inside region is on the left.
(10) Index of positively oriented simple closed curve is 1. After translating if necessary, let $\alpha(0)$ be the lowest point of the curve. That is, if $\alpha(s) = (x(s), y(s))$, it is minimum of $y(s)$, defined since $[0, L]$ is compact. The tangent there is horizontal since $y'(s) = 0$.
(11) Define $(u, v)$ maps to the angle of the vector $\alpha(v) - \alpha(u)$ on the triangle $0 \le u \le v \le L$. Limiting case when $u = v$ is the tangent line. This angle well-defined up to multiples of $2\pi$, but can be well-defined globally by cutting into small triangles, and fixing a starting value, say $(0, 0) \mapsto 0$. Need triangle is simply connected.
(12) The theta difference along the hypotenuse is $2\pi$ times the index, but along but along each of the other two sides is $\pi$, since $\alpha(0) = \alpha(L)$ was the lowest point, so can never turn more than $\pi$. (That is, $(0, L) \mapsto \pi$ and $(L, L) \mapsto 2\pi$.)

*Coordinate patches.*

(1) Just like curves were function $\alpha\colon I \to \mathbb{R}^3$, surface is a function $x\colon U \to \mathbb{R}^3$, where $U$ is an open set in $\mathbb{R}^2$.
(2) What is an open set in $\mathbb{R}^n$? For every point $p \in U$, there exists some $r > 0$, so that $B_r(p) \subset U$. Informally, does not contain any point on the boundary. Open intervals are open sets in $\mathbb{R}$. Complement of open is called closed. (Lots of sets are neither open nor closed.)
(3) Checking whether some function is $C^k$ is local. So makes sense to talk about whether $x$ is $C^k$. We assume $x \in C^k$, and $k \ge 1$ (usually $k \ge 3$).
(4) We also assume $x$ is injective. That is, for simple surfaces like this (also called coordinate patches), do not allow self-intersections.
(5) Finally a regularity condition, just like we assumed $\alpha'(s) \neq 0$ for curves to have a well-defined unit tangent $T$. Let $u^1, u^2$ be the coordinates on $\mathbb{R}^2$, and let $x_i = \frac{\partial x}{\partial u^i}$ (which are 3-dimensional vectors). We assume $x_1 \times x_2 \neq 0$ everywhere. That is, not only are each $x_1$ and $x_2$ non-zero, they are linearly independent.
(6) Geometric meaning. Hold $u_2$ constant, change $u_1$, get a straight line in $U$, produces a curve in the surface, its tangent vector is $x_1$. Similarly, hold $u_1$ constant, change $u_2$, get a different curve, its tangent vector is $u_2$. These curves are called parametric curves. The regularity condition says $x_1$ and $x_2$ are linearly independent (always draw them starting at $x(p)$), that is, they span a 2-dimensional plane, which is called the tangent plane at $p$ (or $x(p)$). What is the unit normal? ($n = \frac{x_1 \times x_2}{|x_1 \times x_2|}$.)
(7) Example is a graph. Let $f\colon U \to \mathbb{R}$, $f \in C^k$. Then graph is a function $U \to \mathbb{R}^3$, $(u^1, u^2) \mapsto (u^1, u^2, f(u^1, u^2))$. Why is this a simple surface? Check $C^k$, injective, and the regularity condition.
(8) Example of an example, graph of $f(u^1, u^2) = \sqrt{1 - (u^1)^2 - (u^2)^2}$. What is the domain? Unit disk, but we want open domain, so open unit disk. The surface is the upper hemisphere.
(9) Another way to visualize the sphere, spherical coordinates. Consider the surface

$$(\theta, \phi) \mapsto (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi).$$

What are the parametric curves? Latitudes (circles, usually not great) and longitudes (half great circles). We need injective, so what is a good domai? $(0, 2\pi) \times (-\pi/2, pi/2)$. What is the image? (Unit sphere minus the poles, as well as the prime meridian.) Finally, check regularity: $|x_\theta \times x_\phi| = \cos\phi > 0$. What is unit normal? Can find it geometrically as well since it points outwards.
(10) Next we study reparametrization. Completely analogous to curves. Consider surfaces $x\colon U \to \mathbb{R}^3$ and $y\colon V \to \mathbb{R}^3$. Reparametrization are $C^k$ functions $h\colon U \to V, g\colon V \to U$, with $h \circ g = \mathrm{Id}_V, g \circ h = \mathrm{Id}_U$ and $x = y \circ h$ (equivalently $y = x \circ g$).
(11) For curves, it was equivalent to saying $h'(s) \neq 0 \forall s$. What about $dh_p$, for $p \in U$? What is $dh_p$? Linear map $\mathbb{R}^2 \to \mathbb{R}^2$, represented by $2 \times 2$ matrix $(\frac{\partial u^i}{\partial u^j})$ since we have basis. This matrix is called the Jacobian. It is non-singular. Has an inverse $df_{h(p)}$ by chain rule, $dg_{h(p)} \circ dh_p = \mathrm{Id}$.
(12) But unlike for curves, this is not enough to check, since inverse function theorem is only local. So to check reparametrization, need to check $h$ is bijective (with inverse say $g$), $h \in C^k$, $dh_p$ is non-singular everywhere. No need to check $g \in C^k$, since that is a local statement and follows from inverse function theorem. (This will actually be automatic if both $x$ and $y$ are regular, by Implicit function theorem.)

(13) Also unlike curves, no canonical parametrization. So to define any invariants of surfaces, first need to parametrize, then define it in terms of that parametrization, and then check independence of parametrization.

(14) We have so far: Tangent plane, unit normal. Check independence under reparametrization. Draw picture of $x_i = \frac{\partial x}{\partial u^i}$ and $y_j = \frac{\partial y}{\partial v^j}$ on the same surface. Why is plane spanned by $y_1, y_2$ same as plane spanned by $x_1, x_2$? How are they related?

(15) Since $y = x \circ g$, chain rule,

$$y_i = \frac{\partial y}{\partial v^i} = \sum \frac{\partial x}{\partial u^j}\frac{\partial u^j}{\partial v^i} = \sum x_j \frac{\partial u^j}{\partial v^i}$$

(in the double index summation notation). In terms of matrices $2 \times 3 = (2 \times 2)(2 \times 3)$,

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{\partial u^1}{\partial v^1} & \frac{\partial u^1}{\partial v^2} \\ \frac{\partial u^2}{\partial v^1} & \frac{\partial u^2}{\partial v^2} \end{pmatrix}$$

(16) To check same tangent plane, just need to check same normal vector. So

$$y_1 \times y_2 = \det(J)(x_1 \times x_2).$$

Recall $\det(J) \neq 0$, so simultaneously checks that the condition of regularity is preserved under transformation ($x_1 \times x_2 \neq 0$ implies $y_1 \times y_2 \neq 0$), as well as, unit normal preserved up to sign (depending on whether $\det(J) > 0$ or $< 0$), and hence the tangent plane is also preserved.

(17) What is a tangent vector $X$ (usually capital letters)? By definition it is a linear combination of $x_1$ and $x_2$. To preserve double index notation, write $\vec{X} = \sum_i X^i \vec{x}_i$.

(18) Alternate description, tangent vectors are velocity vectors of curves through that point. That is, fix $p \in U$, let $q = x(p)$ be its image on the surface, and let $\alpha\colon I \to U$ be a curve through $p$ (with $\alpha(0) = p$). Then composite $x \circ \alpha$ is a curve on the surface through $q$. We claim, the velocity of this curve is a tangent vector. Chain rule,

$$\frac{d(x \circ \alpha)}{dt}|_{t=0} = \frac{\partial x}{\partial u^1}|_p \frac{du^1}{dt}|_{t=0} + \frac{\partial x}{\partial u^2}|_p \frac{du^2}{dt}|_{t=0} = x_1 \frac{du^1}{dt} + x_2 \frac{du^2}{dt}$$

is a linear combination of $x_1$ and $x_2$.

*Surfaces.*

(1) Definition of $C^k$ surface, example is $S^2 \subset \mathbb{R}^3$. Recall simple surface, but $S^2$ cannot be covered by a single coordinate chart. So need a bunch.

(2) Digression about topology. Need a notion of open sets, example $\mathbb{R}^n$. Subset of a topological space is a topological space. Do examples of open sets in $S^2$. One extreme example is whole $S^2$ is open in $S^2$, but not in $\mathbb{R}^3$.

(3) Surface is a subset $S \subset \mathbb{R}^3$, and a collection (could be infinite, indeed usually uncountable) of coordinate charts $x\colon U \to \mathbb{R}^3$ (which are all $C^k$, injective, regular), satisfying the following.

(4) Each $x$ maps to $S$ (that is $x(U) \subset S$), and the map $x\colon U \to S$ is a homeomorphism. (This means $x^{-1}\colon x(U) \to U$ is also continuous.)

(5) For each $p \in S^2$, there is some coordinate chart $x$ with $p \in x(U)$. That the coordinate charts cover the whole surface.

(6) Finally, the different charts are compatible. So if we have two charts $x\colon U \to S$, $y\colon V \to S$, we get a bijection $y^{-1} \circ x\colon x^{-1}(x(U) \cap y(V)) \to y^{-1}(x(U) \cap y(V))$. We require this to be a $C^k$ reparametrization, that is, each of the maps $y^{-1} \circ x$ and $x^{-1} \circ y$ are $C^k$.

(7) This compatibility allows us to define tangent space $T_pS$—a two dimensional vector space—at each point $p \in S$ (usually drawn at $p$). Choose a coordinate chart $x$ that covers $p$ (which exists), define tangent space for $x$ as the linear span of $x_1$ and $x_2$, and then check it is well-defined, that is, independent of choices. So if $y$ is another coordinate chart also covering $p$, then the span of $y_1$ and $y_2$ is the same space. That is true, since they are reparametrizations of $x(U) \cap y(V)$.

(8) Example of surface: $S^2$. We have already seen many parametrizations of $S^2$, like upper hemisphere (recall, $x(u^1, u^2) = (u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2})$ or spherical coordinates. Can cover $S^2$ with six such hemispheres, but let's do something different.

(9) Stereographic projection from the unit sphere minus north pole $(0,0,1)$ to the plane (draw picture); inverse is a coordinate chart. Equation of straight line from $(v^1, v^2, 0)$ to $(0,0,1)$ is $(tv^1, tv^2, 1-t)$, so when on sphere $t = \frac{2}{(v^1)^2+(v^2)^2+1}$, so coordinate chart given by

$$y(v^1, v^2) = \Big(\frac{2v^1}{(v^1)^2+(v^2)^2+1}, \frac{2v^2}{(v^1)^2+(v^2)^2+1}, \frac{1-(v^1)^2-(v^2)^2}{(v^1)^2+(v^2)^2+1}\Big)$$

which is $C^\infty$ since we are not dividing by 0.

(10) So $x$ and $y$ cover $S^2$, so just need to check overlap condition. The domain and range of $y^{-1} \circ x$ are the punctured disk and complement of unit disk.

(11) Write down $y^{-1} \circ x$, $(u^1, u^2) \mapsto \big(\frac{u^1}{1-\sqrt{1-(u^1)^2-(u^2)^2}}, \frac{u^1}{1-\sqrt{1-(u^1)^2-(u^2)^2}}\big)$, (figure out $x^{-1}$ by drawing the straight line $(tp, tq, 1+t(q-1))$ from $(0,0,1)$ to the point $(p,q,r)$ on the sphere), from punctured disk and its inverse $x^{-1} \circ y$, $(v^1, v^2) \mapsto \big(\frac{2v^1}{(v^1)^2+(v^2)^2+1}, \frac{2v^2}{(v^1)^2+(v^2)^2+1}\big)$, from the complement of the disk. Check both are $C^\infty$, so reparametrization, so defined $S^2$ as a surface.

*First fundamental form.*

(1) First recall linear map, $V \cong \mathbb{R}^n$ is $n$-dimensional vector space (for us, $n=2$), $f\colon V \to V$ is linear if $f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) \forall \lambda, \mu, u, v$. If we choose a basis $\{x_1, \ldots, x_n\}$, then $f$ determined by its value on $x_i$, so if $f(x_i) = \sum f_i^j x_j$, then $f$ is uniquely determined by matrix $(f_i^j)$. Conversely, a matrix $A$ determines a linear map $v \mapsto Av$.

(2) Bilinear map is something completely different. Map $g\colon V \times V \to \mathbb{R}$ which is linear in each factor, $g(\lambda u + \mu v, w) = \lambda g(u,w) + \mu g(v,w)$ and similarly. If we choose a basis, also determined by a matrix $g_{ij} = g(x_i, x_j)$. Conversely, matrix $B$ determines a bilinear map $(u,v) \mapsto u^T B v$.

(3) Matrix represents bilinear form if both indices subscripts, but a linear map if one index subscript one index superscript.

(4) Symmetric, positive definite ($g(v,v) > 0 \forall v \neq 0$) bilinear map is called an inner product. Corresponding matrix is symmetric and positive definite.

(5) Back to surfaces. Given a surface $S$ and point $p \in S$, have defined 2-dimensional tangent space $T_pS$. One of the first examples of an abstract vector space, does not come with natural basis. (We can choose a coordinate chart $x$, and will get a basis $x_1, x_2$, but some other coordinate chart will give some other basis.) Nevertheless has an inner product, first fundamental form, $g(u,v) = \langle u, v\rangle = I(u,v) = u \cdot v$. We defined without parametrizing, so property of surface (independent of parametrization). If we fix a patch, $g$ represented by matrix $(g_{ij})$ where $g_{ij} = x_i \cdot x_j$.

(6) Example $S = \mathbb{R}^2$, parametrize as $x(u^1, u^2) = (u^1, u^2)$, $g$ is the identity matrix. Identical for the cylinder $x(\theta, z) = (\cos\theta, \sin\theta, z)$.

(7) The sphere $S^2$ with spherical coordinates $x(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$, $0 < \theta < 2\pi$, $-\pi/2 < \phi < \pi/2$. We get

$$g = \begin{pmatrix} \cos^2\phi & 0 \\ 0 & 1 \end{pmatrix}$$

(8) We are studying local properties of surfaces. Local means depends only on a neighborhood, like $C^k$, tangents, normals, curvature, most things in the course so far. Indeed only global thing so far is total index of simple closed curves. Property means independent of parametrization. Example, tangent space $T_pS$, normal $\pm n$, first fundamental form $g$. When did we check $g$ is independent of parametrization? ($g$ was defined without using parametrization.) Of course, if we choose a parametrization, $g$ is given a matrix $(g_{ij})$ with $g_{ij} = x_i \cdot x_j$, and if we change parametrization, we will get a different matrix which is related to this by multiplying with $J^t$ and $J$ on two sides.

(9) Local properties are classified into intrinsic or extrinsic. Intrinsic means can access them if you are a two dimensional creature living on the surface, cannot look up. (In Flatland, the protagonist is a triangle living in $\mathbb{R}^2$). So $T_pS$ is intrinsic, since tangent vectors are just directions of travel, and length is how fast you are traveling. So is $g$, since $u \cdot v = |u||v|\cos\theta$, and we can measure lengths and angles. But normal is extrinsic, since we cannot look up.

(10) Technical definition of intrinsic, anything that depends only on $g$ (since lengths and angles are all that we can measure); everything else is extrinsic. So how do we prove $\pm n$ is extrinsic?

(11) Parametrize cylinder, $x(u^1,u^2) = (\cos u^1, \sin u^1, u^2)$, compute first fundamental form, same as that of plane. So locally looks the same (isometric) to the plane. (Globally different, since on the cylinder you can keep travelling in certain directions and come back.) But for the plane $n$ is constant, but for the cylinder $n$ changes, so $n$ is extrinsic.

(12) Is $S^2$ locally isometric to $\mathbb{R}^2$? Different first fundamental form $\begin{pmatrix} \cos^2\phi & 0 \\ 0 & 1 \end{pmatrix}$, so can say nothing yet. (Different parametrization might have produced the same first fundamental form.) Indeed, we will see later that $S^2$ is not locally isometric to the plane—that is, if you are a 2-dimensional creature living on $S^2$, just by measuring lengths and angles locally, you should be able to tell. This is how people figured out that Earth is not flat.

*Curves on surfaces*

(1) Particle travelling on $S^2$, longitude, latitude at time $t$ is given by $(a(t), b(t))$. What is distance travelled in time 0 to $t$? Reasonable real-world problem, we know GPS coordinates. So $(a(t), b(t))$ gives a curve $\alpha\colon I \to \mathbb{R}^2$ to $(\theta,\phi)$ plane, and $x\colon \mathbb{R}^2 \to S^2$ is the spherical coordinates chart, and the curve is $x\circ\alpha$. So

$$\begin{aligned} v &= \frac{d(x\circ\alpha)}{dt} = x_1\frac{da}{dt} + x_2\frac{db}{dt} \\ |v|^2 = g(v,v) &= g_{11}\Big(\frac{da}{dt}\Big)^2 + 2g_{12}\frac{da}{dt}\frac{db}{dt} + g_{22}\Big(\frac{db}{dt}\Big)^2 \\ &= \cos^2(b(t))\Big(\frac{da}{dt}\Big)^2 + \Big(\frac{db}{dt}\Big)^2 \end{aligned}$$

and distance travelled is $\int_0^t v dt$.

(2) An aside. Notice, for us, usually $g$ is given by a diagonal matrix, that is $x_1$ and $x_2$ are perpendicular. This is because we are choosing parametrizations carefully. This is not true for arbitrary parametrizations.

(3) So that was an example of a curve on a surface. So we have a curve $\alpha = (\alpha^1,\alpha^2)\colon I \to U \subset \mathbb{R}^2$, and a coordinate patch $x\colon U \to S \subset \mathbb{R}^3$, and by composing we get a curve $x\circ\alpha\colon I \to \mathbb{R}^3$ on the surface $S$.

(4) We have already seen the tangent vector to the curve $\sum x_i \frac{d\alpha^i}{dt}$, a linear combination of $x_1$ and $x_2$, is in $T_pS$.

(5) As before, from now on, we will impose the curve is unit speed, and write $s$ instead of $t$. That is a messy condition,

$$\sum g_{ij}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt} = \begin{pmatrix} \frac{d\alpha^i}{dt} & \frac{d\alpha^j}{dt} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \frac{d\alpha^i}{dt} \\ \frac{d\alpha^j}{dt} \end{pmatrix} = 1.$$

The unit tangent then is given by $T = \frac{d(x\circ\alpha)}{ds} = \sum x_i \frac{d\alpha^i}{ds}$.

(6) Next for unit speed curves, we can compute curvature $kN = dT/ds$. If $n$ is unit normal to surface, $T$ is unit normal to curve, then set $S = n \times T$; $n, T, S$ form a positive orthonormal frame. The curvature $kN$ has no component along $T$, so we can write $\kappa N = \kappa_g S + \kappa_n n$.

(7) So total curvature

$$\kappa N = \frac{dT}{ds} = \sum x_i \frac{d^2\alpha^i}{ds^2} + \sum x_{ij}\frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds}.$$

Chain rule again, $dx_i/ds = \sum x_{ij} d\alpha^j/ds$, here $x_{ij}$ is the second derivative, and if $x \in C^2$ (which we assume from now on) $x_{ij} = x_{ji}$.

(8) We want to break $\kappa N = \kappa_n n + \kappa_g S$ into normal and geodesic curvature.

(9) The vector $x_i$ has no normal component, but what about $x_{ij}$? Can have both normal and tangential component, so can write

$$x_{ij} = L_{ij} n + \sum \Gamma^k_{ij} x_k,$$

this is the definition of $L_{ij}$ (second fundamental form, will come back to it later) and $\Gamma^k_{ij}$ (Christoffel symbols).

(10) Therefore, we get

$$\begin{aligned}\kappa_n &= \sum L_{ij} \frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds} \\ \kappa_g S &= \sum x_i \frac{d^2\alpha^i}{ds^2} + \sum \Gamma_{ij}^k x_k \frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds} \\ &= \sum_k x_k \big(\frac{d^2\alpha^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds}\big).\end{aligned}$$

(Note triple summation.)

(11) So all that remains is to compute $L_{ij}$ and $\Gamma_{ij}^k$. For $L_{ij}$, just dot with $n$, and we get $L_{ij} = x_{ij} \cdot n$.

(12) $\Gamma_{ij}^k$ is trickier. We can dot with the tangent vectors,

$$x_{ij} \cdot x_m = \sum \Gamma_{ij}^k g_{km}$$

and we are trying to find $\Gamma_{ij}^k$, so this is like solving a linear equation. So we need to multiply with the inverse matrix for $g$.

(13) Since $g$ is positive definite, $\det(g) > 0$, so has inverse,

$$g^{-1} = (g^{ij}) = \frac{1}{\det(g)} \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix}$$

(replace each element by its signed minor, take transpose, and divide by determinant).

(14) What is matrix multiplication? If $AB = C$, and $A = (a_{ij})$, $B = (b^{ij})$, $C = (c_i^j)$, $c_i^j = \sum a_{ik} b^{kj}$. Since $g.g^{-1}$ is the identity matrix, we get $\sum g_{ik} g^{kj} = \delta_i^j$, where $\delta_i^j$ is the Kronecker delta, that is, $(\delta_i^j)$ is the identity matrix.

(15) So now multiply with $g^{-1}$. How do we do that? (Multiply with $g^{ml}$ and sum over $m$.)

$$\sum x_{ij} \cdot x_m g^{ml} = \sum \Gamma_{ij}^k g_{km} g^{ml} = \sum \Gamma_{ij}^k \delta_k^l = \Gamma_{ij}^l.$$

(16) Compute for curves on plane, $b = \hat{k} = n$, so $\kappa_n = 0$ and $S = N$, so $\kappa_g = k$, the signed curvature for curves in the plane.

(17) The other extreme case, the helix $(r\cos t, r\sin t, ht)$ on the radius-$r$ cylinder. Not unit speed, so reparametrize $(r\cos(\omega s), r\sin(\omega s), h\omega s)$, for $\omega = 1/\sqrt{r^2+h^2}$. Parametrize cylinder

$$x(u^1, u^2) = (r\cos u^1, r\sin u^1, u^2),$$

so $x_1 = (-r\sin u^1, r\cos u^1, 0)$ and $x_2 = (0, 0, 1)$ (these are not unit vectors) and $n = (\cos u^1, \sin u^1, 0)$ outward pointing. The curve can be written $x \circ \alpha$, where $\alpha(s) = (\omega s, h\omega s)$ is a straight line. Recall the unit tangent and curvature. They are

$$\begin{aligned}T &= d(x\circ\alpha)/ds = (-r\omega\sin(\omega s), r\omega\cos(\omega s), h\omega) \\ \kappa N &= dT/ds = (-r\omega^2\cos(\omega s), -r\omega^2\sin(\omega s), 0) = -r\omega^2 n,\end{aligned}$$

so entire curvature is in the normal direction, that is, $\kappa_n = -r\omega^2$ and $\kappa_g = 0$.

(18) Geodesic curvature measures curvature inside the surface, and normal curvature measures how the surface is curved in $\mathbb{R}^3$. So curves in $\mathbb{R}^2$ had no normal curvature, and this curve in the cylinder does. If a unit speed curve has $\kappa_g = 0$, it is called a geodesic, which corresponds to a straight line inside the surface. So this shows that the helix is a geodesic in the cylinder. So if you are a 2-dimensional creature living on the cylinder, these helices seem like straight lines to you.

(19) What are geodesics on $S^2$? (Great circles.) What about other lines of latitude (smaller circles)? Consider a latitude in the northern hemisphere. Are you turning left or right? Consider an extreme case, a small latitude around the north pole.

(20) Let us compute $\kappa_g$ and $\kappa_n$ for this latitude on $S^2$. Spherical coordinates again $x(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. We will do the calculation entirely in local coordinates. Consider the latitude at $\phi_0$. Recall $g =$

$\begin{pmatrix} \cos^2\phi & 0 \\ 0 & 1 \end{pmatrix}$. Recall

$$x_1 = (-\sin\theta\cos\phi, \cos\theta\cos\phi, 0)$$
$$x_2 = (-\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi)$$

and $n$ is outward pointing (just geometrically), so

$$n = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi).$$

(21) If we write the curve as $x \circ (t, \phi_0)$, then square of speed is $g_{11} = \cos^2\phi_0$, so not unit speed. So let $\alpha(s) = (s/\cos\phi_0, \phi_0)$, then $x \circ \alpha$ is unit speed latitude. Note $d\alpha^1/ds = 1/\cos\phi_0$ and $d\alpha^2/ds = 0$, and all higher derivative are zero.
(22) Unit tangent is $\sum x_i d\alpha^i/ds = x_1/\cos\phi_0$.
(23) Normal curvature is $\kappa_n = \sum L_{ij}(d\alpha^i/ds)(d\alpha^j/ds) = L_{11}\cos^2\phi_0$. Calculate

$$x_{11} = (-\cos\theta\cos\phi, -\sin\theta\cos\phi, 0),$$

so $L_{11} = x_{11} \cdot n = -\cos^2\phi_0$, so $\kappa_n = -1$ (independent of $\phi_0$).
(24) Geodesic curvature is

$$\kappa_g S = \sum x_k\big(d^2\alpha^k/ds^2 + \sum \Gamma^k_{ij}(d\alpha^i/ds)(d\alpha^j/ds)\big) = \frac{1}{\cos^2\phi_0}(x_1\Gamma^1_{11} + x_2\Gamma^2_{11}).$$

Note $\Gamma^k_{ij} = \sum x_{ij} \cdot x_l g^{lk}$, and $g^{-1} = \begin{pmatrix} 1/\cos^2\phi & 0 \\ 0 & 1 \end{pmatrix}$, so $\Gamma^1_{11} = x_{11} \cdot x_1/\cos^2\phi_0 = 0$, $\Gamma^2_{11} = x_{11} \cdot x_2 = \cos\phi_0\sin\phi_0$. So $\kappa_g S = \tan\phi_0 x_2$. Note $x_2$ is a unit vector, so $S = x_2$ (geometrically), so $\kappa_g = \tan\phi_0$.
(25) Recall $\kappa_n = \sum L_{ij}(d\alpha^i/ds)(d\alpha^j/ds)$ and $\kappa_g S = \sum x_k\big(d^2\alpha^k/ds^2 + \sum \Gamma^k_{ij}(d\alpha^i/ds)(d\alpha^j/ds)\big)$. Why are they independent of parametrization (depend only on curve and surface)? Note, $T$ and $\kappa N$ depend only on the curve (which is unit speed), and $n$ (up to a sign) depends only on the surface (independent of parametrization), so $S = n \times T$ is also independent (up to the same sign), and so $\kappa_n = \kappa N \cdot n$ and $\kappa_g = \kappa N \cdot S$ are independent as well (up to the same sign).
(26) Geodesic curvature is internal curvature, and normal curvature measures how the surface is curved in $\mathbb{R}^3$. Second is extrinsic. Recall the helix in cylinder had $\kappa_n = \pm r/(r^2+h^2)$ (and $\kappa_g = 0$), but every curve in the plane has $\kappa_n = 0$, but plane and cylinder are locally isometric (same first fundamental form).
(27) More importantly, geodesic curvature is instrintic, that is, depends only on $g$. (If you are creature on the surface, you can measure it.) Clearly $x_k$ and derivatives of $\alpha^i$ are intrinsic (can create a coordinate chart instrinsically), so only $\Gamma^k_{ij}$ remain.
(28) It is defined as the coefficient $x_{ij} = L_{ij}n + \sum \Gamma^k_{ij}x_k$, so by dotting with $x_l$ and multiplying by the inverse matrix $\Gamma^k_{ij} = x_{ij} \cdot x_l g^{lk}$ (raising an index). So need to show $x_{ij} \cdot x_l$ are instrinsic.
(29) Even though $x_{ij}$ is not a tangent vector in general, can compute this by the following trick. Consider $g_{ij} = x_i \cdot x_j$, $g_{il} = x_i \cdot x_l$ and $g_{jl} = x_j \cdot x_l$, and differentiate

$$\partial g_{ij}/\partial u^l = x_{il} \cdot x_j + x_{jl} \cdot x_i \qquad \partial g_{il}/\partial u^j = x_{ij} \cdot x_l + x_{jl} \cdot x_i \qquad \partial g_{jl}/\partial u^i = x_{ij} \cdot x_l + x_{il} \cdot x_j$$

(using commutativity of dot product, as well as of second derivatives). So

$$\Gamma^l_{ij} = \frac{1}{2}\big(\partial g_{il}/\partial u^j + \partial g_{jl}/\partial u^i - \partial g_{ij}/\partial u^l\big)$$

is intrinsic (it is not a property of surfaces though, that is, it depends on the parametrization).

*Geodesics.*

(1) So can talk about straight lines on the surface, geodesics, as an intrinsic concept. Unit speed curve with $\kappa_g = 0$. That is, if you are creature on the surface, you know if you are travelling in a straight line. (For instance, travelling along a latitude in the northern hemisphere is not straight.)
(2) Fundamental theorem of geodesics. Locally around each point and in each direction, there is a unique geodesic. Only a local statement, global existence not guaranteed (consider punctured plane); issue is non-compactness.
(3) So let $p \in S$ and $T \in T_pS$ unit vector. Since local, choose a coordinate chart $x\colon U \to S$. Then want to show there exists unique $\alpha\colon I \to U$ with $x \circ \alpha(0) = p$, $d(x \circ \alpha)/ds|_{s=0} = T$ and $x \circ \alpha$ a geodesic.

(4) To be a geodesic need to unit speed and have $\kappa_g = 0$. The second condition states

$$0 = \kappa_g S = \sum x_k\big(d^2\alpha^k/ds^2 + \sum \Gamma^k_{ij}(d\alpha^i/ds)(d\alpha^j/ds) = 0$$

and since $x_1, x_2$ are linearly independent, we get two equations

$$d^2\alpha^1/ds^2 + \sum \Gamma^1_{ij}(x\circ\alpha(s))\frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds} = 0$$
$$d^2\alpha^2/ds^2 + \sum \Gamma^2_{ij}(x\circ\alpha(s))\frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds} = 0.$$

This looks like second order ODE, but can convert to first order ODE with four variables $w^1 = d\alpha^1/ds, w^2 = d\alpha^2/ds, w^3 = \alpha^1, w^4 = \alpha^2$. That is, we are solving for the following system (which is equivalent to the given system):

$$\begin{aligned} dw^1/ds &= -\sum_{1\le i,j\le 2} \Gamma^1_{ij}(w^3,w^4)w^iw^j \\ dw^2/ds &= -\sum_{1\le i,j\le 2} \Gamma^2_{ij}(w^3,w^4)w^iw^j \\ dw^3/ds &= w^1 \\ dw^4/ds &= w^2 \end{aligned}$$

By Picard's theorem, there is a unique solution (locally). We have to fix the initial conditions. The initial conditions of $w^3, w^4$ state the curve starts at $p$, and the initial condition for $w^1, w^2$ state the curve has initial tangent vector $T$.

(5) So why is the above curve geodesic, that is, why is it unit speed? At $s = 0$, it was unit speed, but why does it stay unit speed?

(6) Recall the derivation of velocity and acceleration

$$v = \frac{d(x\circ\alpha)}{ds} = \sum x_i\frac{d\alpha^i}{ds}$$
$$a = \frac{dv}{ds} = n\sum L_{ij}\frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds} + \sum x_k\big(\frac{d^2\alpha^k}{ds^2} + \sum \Gamma^k_{ij}\frac{d\alpha^i}{ds}\frac{d\alpha^k}{ds}\big)$$

does not require $x\circ\alpha$ to be unit speed. So our curve $x\circ\alpha$ has acceleration $a$ in the direction of $n$, so is perpendicular to $v$, therefore $d(v\cdot v)/ds = 0$, that is, the speed of the curve remains unchanged. Since it was unit speed at $s = 0$, it stays unit speed.

(7) If there exists a shortest curve between two points $p, q \in S$, then it is geodesic. (There need not exist a shortest curve, even when there are geodesics connecting the two points; consider non-antipodal $p, q \in S^2$, and puncture it at a point on the short geodesic joining $p$ to $q$.)

(8) Proof is by variation of parameters locally. If a length minimizing curve has $\kappa_g \neq 0$, then we can get a shorter curve by pushing the curve towards the direction it is turning in. Consider a point with $\kappa_g \neq 0$, choose local coordinates, and construct a non-zero perturbation (which is zero outside a neighborhood) of the curve in the direction of $\kappa_g S$.

(9) First step, choose a bump function. The function $f(t) = e^{-t^{-2}}$ for $t > 0$ and $f(t) = 0$ for $t \le 0$ is $C^\infty$. The function $f(t-a)f(b-t)$ is a $C^\infty$ non-negative function with support $(a, b)$.

(10) $x\circ\alpha$ is the given unit speed curve (length minimizing), and say $\kappa_g(s_0) \neq 0$. Then it is non-zero on an $\epsilon$ neighborhood around $s_0$ and let $\lambda$ be a bump function supported on that neighborhood. Let $\lambda\kappa_g S = \sum x_k v^k$, and consider the curve

$$\alpha_t(s) = (\alpha^1(s) + tv^1(s), \alpha^2(s) + tv^2(s)).$$

Note $\partial(x\circ\alpha_t)/\partial t|_{t=0} = \sum x_k v^k = \lambda\kappa_g S$.

(11) We have one curve for each $t$. These curves $x\circ\alpha_t(s)$ are not unit speed, but still as regular as the original curve ($t = 0$).

(12) The length of the curve is given by

$$L(t) = \int_I \sqrt{\frac{d(x\circ\alpha_t)}{ds}\cdot\frac{d(x\circ\alpha_t)}{ds}}ds$$

and since $t = 0$ is a minimum for $L(t)$, $dL/dt|_{t=0} = 0$. But it equals

$$\int_I \frac{\partial}{\partial t}\sqrt{\frac{d(x\circ\alpha_t)}{ds}\cdot\frac{d(x\circ\alpha_t)}{ds}}|_{t=0}ds \qquad \text{(can change order since } C^2\text{)}$$

$$= \int_I \frac{1}{2\sqrt{\frac{d(x\circ\alpha_t)}{ds}\cdot\frac{d(x\circ\alpha_t)}{ds}}} 2\frac{d^2(x\circ\alpha_t)}{dsdt}\cdot\frac{d(x\circ\alpha_t)}{ds}|_{t=0}ds \qquad \text{(can change order since } C^2\text{)}$$

$$= \int_I \frac{d^2(x\circ\alpha_t)}{dsdt}\cdot\frac{d(x\circ\alpha_t)}{ds}|_{t=0}ds \qquad \text{(at } t=0\text{, the curve is unit speed)}$$

$$= \int_I \frac{d}{ds}(\frac{d(x\circ\alpha_t)}{dt}\cdot\frac{d(x\circ\alpha_t)}{ds})|_{t=0} - \frac{d(x\circ\alpha_t)}{dt}\cdot\frac{d^2(x\circ\alpha_t)}{ds^2}|_{t=0}ds \qquad \text{(just product rule)}$$

$$= \frac{d(x\circ\alpha_t)}{dt}\cdot\frac{d(x\circ\alpha_t)}{ds}|_{t=0,\partial I} - \int_I (\lambda\kappa_g S)\cdot(\kappa_g S + \kappa_n n)ds \qquad \text{(integrate wrt } s\text{)}$$

$$= 0 - \int_I \lambda\kappa_g^2 ds < 0 \qquad (\lambda = 0 \text{ on } \partial I).$$

*Parallel translation.*

(1) Another aspect of geodesics is parallel translation. If $\gamma(t)$ is a curve in $\mathbb{R}^2$, and we have a vector at some point on $\gamma$, we can translate it along $\gamma$. That is, keep the vector remains parallel as you walk along $\gamma$.
(2) So how to define it in arbitrary surfaces? Let $\gamma(t)$ be a (not necessarily unit-speed) curve on $S$, and $X(t)$ be a vector field along $\gamma$, that is, at each point $\gamma(t)$, we have a tangent vector $X(t) \in T_{\gamma(t)}S$. Then $X$ is parallel along $\gamma$, if $dX/dt$ has no tangential component, that is, if it is only in the normal direction. So parallel translation can change $X(t)$, but the change is zero in the tangential direction.
(3) Parallel translation doesn't change lengths, $\frac{dX\cdot X}{dt} = 2\frac{dX}{dt}\cdot X = 0$, and angles, $\frac{dX\cdot Y}{dt} = \frac{dX}{dt}\cdot Y + X\cdot\frac{dY}{dt} = 0$.
(4) Examples. If $\gamma(s)$ is geodesic, then unit tangent is parallel translation. Not true if $\gamma$ is not geodesic, like a circle in $\mathbb{R}^2$. Also, if the vector has constant length and constant angle with the tangent, then is parallel along $\gamma(s)$. (Again only if $\gamma$ is a geodesic.)
(5) Parallel translate a vector in a closed loop in $\mathbb{R}^2$, and you come backto the starting vector. Not true if the surface is curved. Consider geodesic triangle in $S^2$ joining points $P, Q$ on the equator and the north pole, and parallel translate south-pointing vector. Angle changes by $\theta$, where $\theta$ actually happens to be the area of the triangle.

*Second fundamental form.*

(1) Define directional derivative, $p \in S, T \in T_pS, f\colon S \to \mathbb{R}$, then differentiate $f$ in the direction of $T$. Choose a path $x\circ\alpha$ through $p$ (say $x\circ\alpha(0) = p$), and $Xf = d(f\circ x\circ\alpha)/dt|_{t=0}$. Unfortunate notation $Xf$, does not mean multiplication.
(2) If $X = \sum X^i x_i$, then $v = \sum x_i d\alpha^i/dt|_{t=0}$, so $d\alpha^i/dt|_{t=0} = X^i$. Also by chain rule, $Xf|_p = \sum \frac{\partial f\circ x}{\partial u^i}X^i$, so directional derivative does not depend on $\alpha$. (This definition depends on the coordinate chart $x$, the first definition depends on the curve $\alpha$, they are the same, so $Xf$ independent of either.)
(3) The second definition also shows $Xf$ is linear in $X$, that is $(\lambda X)f = \lambda(Xf)$ and $(X+Y)f = Xf + Yf$.
(4) Extends for a vector valued function $\vec{f} = (f_1, f_2, f_3)$, since it is just three scalar valued functions: $X\vec{f} = (Xf_1, Xf_2, Xf_3)$.
(5) Unit normal $n$ is a nice vector valued function on the surface. Only well-defined up to a sign, but on each coordinate chart can define it to be $(x_1\times x_2)/|x_1\times x_2|$, so well-defined on each chart. So can consider directional derivative of $n$. Define the Weingarten map $L\colon T_pS \to T_pS$, $L(X) = -Xn$, (minus sign just for convenience), measures how the normal changes as we move in the $X$ direction.
(6) Linear since directional derivative is linear. Maps to the tangent space since $Xn$ is perpendicular to $n$ (same proof: differentiate $n\cdot n = 1$ in the $X$ direction).
(7) This is a property of surface since independent of coordinate charts. (Only the sign of $n$ depends on coordinate charts; so $L$ well-defined up to a sign.)
(8) As linear map, once we fix a coordinate chart $x$ (and hence basis $x_1, x_2$), can write $L(x_i) = \sum L_i^j x_j$, so the $2\times 2$ matrix $(L_i^j)$ determines $L$.

(9) Not a coincidence that we are using the same letter to denote $L_{ij} = x_{ij} \cdot n$. Those $L_{ij}$ can be used to define the second fundamental form (not as fundamental as the first one), a symmetric bilinear form (not definite) $T_pM \times T_pM \to \mathbb{R}$. By bilinearity, only need to define it on basis vectors, set $II(x_i, x_j) = L_{ij}$.

(10) An aside, once we fix a basis, $2 \times 2$ matrices can be used to define bilinear forms like $g_{ij}$ or $L_{ij}$ (both subscripts), or linear maps like $L_i^j$ (one subscript, one superscript).

(11) The $L_i^j$ and $L_{ij}$ are related by raising and lowering indices, always using the first fundamental form $g$. We saw this once for Christoffel symbols. $L_{ij} = \sum L_i^k g_{kj}$, so just matrix multiplication (since $II(X,Y) = X^t(L_{ij})Y = \langle X, LY\rangle = X^t(g_{ij})(L_i^j)Y$, we have $(L_i^j)$ to the right of $(g_{ij})$ below),

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix}$$

and after multiplying by the inverse matrix $g^{-1} = (g^{kj})$, $L_i^j = \sum L_{ik} g^{kj}$. In summation notation, $\sum L_{ij} g^{jm} = \sum L_i^k g_{kj} g^{jm} = \sum L_i^k \delta_k^m = L_i^m$. (The $L_{ij}$ and $g_{ij}$ matrices are symmetric, not necessarily the $L_i^j$ matrix.)

(12) We gave so far: Bilinear form, $T_pS \times T_pS \to \mathbb{R}$, first fundamental form $g$, $g(X,Y) = \langle X, Y\rangle = X \cdot Y$ (so surface property). After choosing coordinate chart $x$, $g(x_i, x_j) = x_i \cdot x_j = g_{ij}$ (both subscripts), and if $X = \sum_i X^i x_i, Y = \sum_i Y^i x_i$, then $g(X,Y) = \sum X^i Y^j g_{ij}$.

(13) Linear map, $T_pS \to T_pS$, Weingarten map $L$, $L(X) = -Xn$ (so surface property, up to a sign). After choosing coordinate chart $x$, $L(x_i) = -\frac{\partial n}{\partial u_i} = \sum L_i^j x_j$ (one subscript one superscript), and if $X = \sum_i X^i x_i$, then $L(X) = \sum X^i L_i^j x_j$.

(14) Bilinear form, $T_pS \times T_pS \to \mathbb{R}$, second fundamental form $II$. After choosing coordinate chart $x$, $L(x_i, x_j) = L_{ij} = x_{ij} \cdot n$ (both subscripts), and if $X = \sum_i X^i x_i, Y = \sum_i Y^i x_i$, then $II(X,Y) = \sum X^i Y^j L_{ij}$. Why is this also a surface property (independent of charts)?

(15) Recall there is a relation between the three. $L_{ij} = \sum L_i^k g_{kj}$ (matrix multiplication). So $II(x_i, x_j) = \langle L(x_i), x_j\rangle$ (another way to write matrix multiplication), and by bilinearity $II(X,Y) = \langle LX, Y\rangle = \langle X, LY\rangle$ (similarly). So $II$ is determined by $g$ and $L$, both are surface properties, so is $II$.

(16) Geometric meaning. $g$ is first fundamental form, measuring lengths and angles; $L$ is Weingarten map, measuring how the normal is changing in different directions; $II$ is trickier. If $X = \sum X^i x_i$ is unit length ($\langle X, X\rangle = 1$), and $x \circ \alpha$ any curve with tangent vector $X$ (so unit speed at $p$, could be the geodesic in the direction of $X$, get $d\alpha^i/dt|_p = X^i$), then $II(X,X) = \sum L_{ij} X^i X^j$ is the normal curvature.

(17) Compute for radius-$r$ cylinder $x(u^1, u^2) = (r\cos u^1, r \sin u^1, u^2)$. Get

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad L = \begin{pmatrix} -1/r & 0 \\ 0 & 0 \end{pmatrix} \qquad II = \begin{pmatrix} -r & 0 \\ 0 & 0 \end{pmatrix}$$

Indeed $(L_{ij}) = (L_i^j)(g_{ij})$, so no need to compute all three. Usually $L_{ij}$ easiest to compute.

(18) It is isometric to $\mathbb{R}^2$, there is a different coordinate system which gives the same first fundamental form as $\mathbb{R}^2$ (just scale $u^1$ by $r$). What is $L$ and $II$ for the plane ($L = 0$, so is $II(X,Y) = \langle LX, Y\rangle$), but both non-zero for cylinder. So $L$ and $II$ extrinsic (a 2-dimensional creature on a cylinder locally thinks it looks like $\mathbb{R}^2$, so cannot compute either $L$ or $II$).

(19) An aside: why are we studying all these objects? We want to study surface properties (which is covered by lots of coordinate patches), so we need to define invariants that do not depend on the coordinate charts. Those are very hard to come by, so far, we only got $g, L, II$ (and even then $II$ is defined in terms of $g$ and $L$). Additionally we also want intrinsic properties, so that 2-dimensional creatures on the surface can access them. So our current state of knowledge is

| | property | not property |
|---|---|---|
| intrinsic | $g$ (and $T_pS$) | $\Gamma_{ij}^k$ |
| extrinsic | $n, L, II$ (up to signs) | |

(Nothing in the bottom-right square, because that got has nothing to do with surfaces.) Also, $\kappa_g$ is instrinsic, $\kappa_n$ is extrinsic, but those are properties of curves on surfaces, not properties of just surfaces. In particular, we haven't got too many instrinsic surface properties; haven't got a single one that can tell us that the sphere is not locally isometric to the plane. By the end of the course, we will get that.

*Principal curvatures.*

(1) Separate question. Fix $p \in S, X \in T_pS$ with $\langle X, X\rangle = 1$ (unit length, direction at $p$). Consider curves through $p$ with velocity $X$ at $p$, for example geodesics. Their geodesic curvatures could be zero, but their normal curvatures are dictated: $II(X, X)$. So what normal curvatures do we get? What is the maximum and minimum normal curvatures?

(2) Do it for the cylinder. Normal curvature in $u^2$-direction is 0, and in $u^1$-direction is $1/r$. In other directions, we can take the curve to be the geodesic helix $(r\cos\omega s, r\sin\omega s, h\omega s)$ which has curvature $\frac{r}{r^2+h^2}$. So maximum curvature is $1/r$ and minimum is 0.

(3) What to do in general? We are trying to find the maximum and minimum of the normal curvature $\kappa_n = II(X, X)$ as $X$ varies, subject to the condition $\langle X, X\rangle = 1$. So we do Lagrange multipliers.

(4) We have $p \in S$, and we are trying to understand how much the surface is locally curved at $p$. One thing we can try is look at all directions at $p$, that is, $X \in T_p(S)$ with $g(X, X) = 1$ (there is an $S^1$-worth of directions), and compute normal curvatures in those directions. Doesn't matter what curve you choose (could be the geodesic), the normal curvature $\kappa_n$ is forced to be $II(X, X)$. So we can try to understand what are the possible $\kappa_n$ as $X$ varies (as a measure of how curved the surface is). In particular, we ask, what are the maximum and minimum $\kappa_n$?

(5) Choose coordinate chart $x$, and write $X = \sum X^i x_i$. Then $X^i$'s are the variables (everything else constant). Optimizing $II(X, X) = \sum L_{ij}X^iX^j$ subject to $g(X, X) = \sum g_{ij}X^iX^j = 1$. Let

$$f(X^1, X^2, \lambda) = II(X, X) - \lambda(g(X, X) - 1) = \sum L_{ij}X^iX^j - \lambda(\sum g_{ij}X^iX^j - 1).$$

$\partial f/\partial\lambda = 0$ is just the constraint $g(X, X) = 1$. But we also get

$$\frac{\partial f}{\partial X^i} = 2\sum L_{ij}X^j - 2\lambda g_{ij}X^j.$$

Let $Y = \sum Y^i x_i$ be any other vector. Multiply with $Y^i$ and add to get

$$\sum L_{ij}Y^iX^j - 2\lambda g_{ij}Y^iX^j = II(Y, X) - \lambda\langle Y, X\rangle = \langle Y, LX - \lambda X\rangle = 0.$$

That is, if the direction $X$ maximizes or minimizes $\kappa_n$, then $\langle Y, LX - \lambda X\rangle = 0$ for all $Y$. Since $g$ is positive definite, this implies $LX - \lambda X = 0$ (take $Y = LX - \lambda X$), so $X$ is an eigenvector and $\lambda$ is an eigenvalue. Moreover, for this $X$, $\kappa_n = II(X, X) = \langle LX, X\rangle = \lambda$.

(6) Linear algebra: linear maps $f\colon V \to V$ need not have all (or for that matter, any) eigenvalues real, and even if it does, the eigenvectors need not span $V$. But $L$ is self-adjoint, that is, with respect to the inner product, $\langle LX, Y\rangle = \langle X, LY\rangle$, and for self-adjoint linear maps, all eigenvalues are real, and the eigenvectors span $V$. (Spectral theorem.) So did Lagrange multipliers and spectral theorem in same lecture—the course really is a combination of multivariable calculus and linear algebra.

(7) Can do geometrically. What are the normal curvatures as $X$ varies? So we have a function $\kappa_n\colon S^1 \to \mathbb{R}$. Two cases.

(8) The function is constant $\lambda$. Example? $\mathbb{R}^2$ and $S^2$. So every direction $X$ is a maximum, so $LX = \lambda X$, so every vector is an eigenvector with same eigenvalue $\lambda$, which is the normal curvature in any direction.

(9) Non-constant. Then the function has a maximum and minimum ($S^1$ compact with no boundary), say $\kappa_1 > \kappa_2$. So these are eigenvalues of $L$, with eigenvectors $Y, Z$ (alternate proof of spectral theorem in this case). Moreover, since $L$ has at most two eigenvalues, these are all, so $\kappa_n$ has only one maximum and one minimum. So maximum $\kappa_n$ in $\pm Y$ direction, and minimum $\kappa_n$ in $\pm Z$ direction (draw picture with $\pm Y, \pm Z$ in $S^1$, and explain how curvature increases/decreases around $S^1$).

(10) Principal curvatures $\kappa_1, \kappa_2$ (maximum and minimum normal curvatures) and principal directions $Y, Z$. They are orthogonal: $\kappa_1 g(Y, Z) = g(LY, Z) = g(Y, LZ) = \kappa_2 g(Y, Z)$, so $g(Y, Z) = 0$.

(11) Example: cylinder. $\kappa_1 = 1/r$, $\kappa_2 = 0$, and principal directions $x_1$ and $x_2$ (indeed orthogonal).

(12) $L$ is a surface property (independent of coordinate charts, up to sign), so are its eigenvalues $\kappa_1, \kappa_2$. More useful are the trace $\kappa_1 + \kappa_2$ and determinant $\kappa_1\kappa_2$, which all are automatically surface properties as well. Moreover, even the sign of determinant is well-defined.

(13) Mean (average) curvature $H = (\kappa_1 + \kappa_2)/2 = \text{trace}/2$ and Gaussian curvature $K = \kappa_1\kappa_2 = \det$. ($H$ is also average over the circle, if anyone asks.) These therefore, are also independent of coordinate charts. But if $L = (L_i^j)$ in terms of some coordinate chart, then trace is $L_1^1 + L_2^2$ and determinant is $L_1^1L_2^2 - L_1^2L_2^1$.

(14) Are they intrinsic or extrinsic? Again consider the cylinder vs the plane, which are locally isometric (locally indistinguishable by a 2-dimensional creature living on the surface). But $L$ is different (non-zero

vs zero), so are $\kappa_1, \kappa_2$ $(0, 1/r$ vs $0, 0)$, so is $H$ $(1/2r$ vs $0)$, so these are all extrinsic. However, the Gaussian curvature.

(15) And that is Gauss' Theorema Egregium: $K$ is somehow instrinsic, and measures how much the surface is curved. Even though it is defined in terms of $L$ which is extrinsic, it is a surface property (independent of charts) that can be computed purely in terms of the first fundamental form $g$, and so can be computed by creatures living on the surface.

*Theorema Egregium.*

(1) Gaussian curvature $K$ is intrinsic.
(2) Fix coordinate chart $x$, and consider the double derivative, written in terms of the basis $n, x_1, x_2$.

$$x_{ij} = L_{ij} n + \sum \Gamma^l_{ij} x_l.$$

Notice, both sides are symmetric in $i, j$, that is, unchanged if we switch $i$ and $j$.

(3) Differentiate wrt $u^k$,

$$\begin{aligned} x_{ijk} &= \frac{\partial L_{ij}}{\partial u^k} n + L_{ij} \frac{\partial n}{\partial u^k} + \sum \frac{\partial \Gamma^l_{ij}}{\partial u^k} x_l + \sum \Gamma^l_{ij} x_{lk} \\ &= \frac{\partial L_{ij}}{\partial u^k} n - L_{ij} L(x_k) + \sum \frac{\partial \Gamma^l_{ij}}{\partial u^k} x_l + \sum \Gamma^l_{ij} \big( L_{lk} n + \sum \Gamma^m_{lk} x_m \big) \\ &= \big( \frac{\partial L_{ij}}{\partial u^k} + \sum \Gamma^l_{ij} L_{lk} \big) n - \sum L_{ij} L^l_k x_l + \sum \frac{\partial \Gamma^l_{ij}}{\partial u^k} x_l + \sum \Gamma^l_{ij} \Gamma^m_{lk} x_m \\ &= \big( \frac{\partial L_{ij}}{\partial u^k} + \sum \Gamma^l_{ij} L_{lk} \big) n + \sum x_l \big( - L_{ij} L^l_k + \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \sum \Gamma^m_{ij} \Gamma^l_{mk} \big) \end{aligned}$$

(4) Now, even though the left side is symmetric wrt $i, j, k$ $(x \in C^3)$, the right side is not symmetric in $j, k$ (it is still symmetric in $i, j$). So we get the same thing if we switch $j, k$.

$$\begin{aligned} &\big( \frac{\partial L_{ij}}{\partial u^k} + \sum \Gamma^l_{ij} L_{lk} \big) n + \sum x_l \big( - L_{ij} L^l_k + \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \sum \Gamma^m_{ij} \Gamma^l_{mk} \big) \\ = &\big( \frac{\partial L_{ik}}{\partial u^j} + \sum \Gamma^l_{ik} L_{lj} \big) n + \sum x_l \big( - L_{ik} L^l_j + \frac{\partial \Gamma^l_{ik}}{\partial u^j} + \sum \Gamma^m_{ik} \Gamma^l_{mj} \big). \end{aligned}$$

(5) Equating the normal components, we get one equation (Cadazzi-Mainardi). But more interesting, equating the tangential components, we get

$$L_{ik} L^l_j - L_{ij} L^l_k = \frac{\partial \Gamma^l_{ik}}{\partial u^j} + \sum \Gamma^m_{ik} \Gamma^l_{mj} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} - \sum \Gamma^m_{ij} \Gamma^l_{mk} = R^l_{ijk}$$

(6) Define $R\colon T_pS \times T_pS \times T_pS \to T_pS$, trilinear map, $R(x_j, x_k, x_i) = \sum R^l_{ijk} x_l$, Riemannian curvature tensor. From the left hand side, we get $R(x_j, x_k, x_i) = II(x_i, x_k) L(x_j) - II(x_i, x_l) L(x_k)$, so $R(Y, Z, X) = II(X, Z) L(Y) - II(X, Y) L(Z)$ is a surface property (independent of coordinate charts), and from right hand side, we get $R$ is intrinsic.
(7) Gauss didn't really consider $R$, that was his student Riemann. Gauss considered the Gaussian curvature $K$, but that follows from the more general Riemannian curvature tensor. Lower an index and write $R_{ijkm} = \sum R^l_{ijk} g_{lm}$, which is also intrinsic. (This represents a quadrilinear map $T_pS \times T_pS \times T_pS \times T_pS \to \mathbb{R}$.) This is given by

$$R_{ijkm} = \sum R^l_{ijk} g_{lm} = \sum L_{ik} L^l_j g_{lm} - \sum L_{ij} L^l_k g_{lm} = L_{ik} L_{jm} - L_{ij} L_{km}$$

In particular, for $i = k = 1$ and $j = m = 2$, $R_{1212} = \det(L_{ij})$, which is therefore, intrinsic. Hence $K = \det(L) = R_{1212}/det(g)$ is also intrinsic. (We already know $K$ is a surface property since it is the determinant of the Weingarten map.)

(8) These concepts generalize to higher dimensions as well. We always have a Riemannian curvature tensor, which is a fundamental measure of how the manifold is curved. Somewhat less information is contained in Gaussian curvature, which is called sectional curvature.
(9) Geometric meaning of Riemannian curvarture tensor $R(x_j, x_k, x_i)$ (usually written $R(x_j, x_k) x_i$, since $x_j, x_k$ play a different role than $x_k$). Consider $u^j$ and $u^k$ parametric curves, with tangent vectors $x_j$ and $x_k$ (draw), and consider a parallelogram of side lengths $t$ and $s$ ($u^j, u^k$ parametric curves ensure

Lie bracket of the 2 vector fields is 0, otherwise need another term). Now parallely translate $x_i$ (or any other vector) along the parallelogram; you are off by an error $\vec{e}$ when you come back, and consider the infinitesimal error $\lim_{t,s\to 0} \vec{e}/ts$, that is the Riemannian curvature tensor $R(x_j, x_k)x_i$. This is an intrinsic description, and is a measurement how much the surface is curved.

*Area*

(1) One unfinished end. Given a coordinate patch, the area form is $dA = |x_1 \times x_2| du^1 du^2$. (This was done in multivariable calculus: If we move $du$ units in the $u$ direction, and $dv$ units in the $v$ direction, $du, dv$ small, then cover a parallelogram $x_1 du, x_2 dv$, whose area is $dA = |x_1 \times x_2| dudv$.) Also note, $|x_1 \times x_2| = \sqrt{\det(g)}$.
(2) Change of coordinates preserve area. We have already seen $|y_1 \times y_2| = |J||x_1 \times x_2|$ where $J$ is the $2 \times 2$ Jacobian. Recall how area form in the plane changes by Jacobian under change of coordinates.
(3) Example: upper hemisphere. Easiest with spherical coordinates $(r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)$, with $0 < \theta < 2\pi$ and $0 < \phi < \pi/2$.
(4) Re-do as a graph $(u, v, \sqrt{1 - u^2 - v^2})$ with $u^2 + v^2 < 1$. For the integral, need to change to polar coordinates $u = r\cos\theta, v = r\sin\theta$, so the calculation is similar. (Using Jacobian $|\partial(u,v)/\partial(r,\theta)| = r$, the area form in the plane is $dudv = rdrd\theta$.)

*Review.*

(1) Saddle and paraboloid $z = ax^2/2 + by^2/2$. Draw $a = 2, b = -2$ and $a = 2, b = 1/2$.
(2) Parametrize $x(u,v) = (u, v, au^2/2 + bv^2/2)$, with $u$ as first variable, $v$ as second.
(3) $x_1 = (1, 0, au), x_2 = (0, 1, bv), x_1 \times x_2 = (-au, -bv, 1), n = \frac{1}{\sqrt{a^2u^2+b^2v^2+1}}(-au, -bv, 1)$.

$$(g_{ij}) = \begin{pmatrix} 1 + a^2u^2 & abuv \\ abuv & 1 + b^2v^2 \end{pmatrix} \qquad (L_{ij}) = \begin{pmatrix} \frac{a}{\sqrt{a^2u^2+b^2v^2+1}} & 0 \\ 0 & \frac{b}{\sqrt{a^2u^2+b^2v^2+1}} \end{pmatrix}$$

Notice $x_1$ and $x_2$ are not orthogonal, that is, the $u, v$ parametric curves are not orthogonal. What are the parametric curves? Draw.
(4) Also, $\det g = |x_1 \times x_2|^2 = a^2u^2 + b^2v^2 + 1$, and $\det(L_{ij}) = \frac{ab}{a^2u^2+b^2v^2+1}$, so we can immediately calculate Gaussian curvature $K = \det(L_{ij})/\det g = \frac{ab}{(a^2u^2+b^2v^2+1)^2}$. Notice $K > 0$ for paraboloid and $K < 0$ for saddle. Explain geometrically (principal curvatures in same or opposite directions).
(5) Moreover, this gives,

$$(g^{ij}) = \begin{pmatrix} \frac{1+b^2v^2}{a^2u^2+b^2v^2+1} & -\frac{abuv}{a^2u^2+b^2v^2+1} \\ -\frac{abuv}{a^2u^2+b^2v^2+1} & \frac{1+a^2u^2}{a^2u^2+b^2v^2+1} \end{pmatrix} \qquad \begin{pmatrix} L_1^1 & L_2^1 \\ L_1^2 & L_2^2 \end{pmatrix} = (g^{ij})(L_{ij}) = \begin{pmatrix} \frac{a(1+b^2v^2)}{(a^2u^2+b^2v^2+1)^{3/2}} & -\frac{ab^2uv}{(a^2u^2+b^2v^2+1)^{3/2}} \\ -\frac{a^2buv}{(a^2u^2+b^2v^2+1)^{3/2}} & \frac{b(1+a^2u^2)}{(a^2u^2+b^2v^2+1)^{3/2}} \end{pmatrix}$$

Notice $(L_i^j)$ is not a symmetric matrix.
(6) This gives the Weingarten map

$$L(x_1) = -\partial n/\partial u = L_1^1 x_1 + L_1^2 x_2 = \frac{a(1+b^2v^2)}{(a^2u^2+b^2v^2+1)^{3/2}} x_1 - \frac{a^2buv}{(a^2u^2+b^2v^2+1)^{3/2}} x_2$$
$$L(x_2) = -\partial n/\partial v = L_2^1 x_1 + L_2^2 x_2 = -\frac{ab^2uv}{(a^2u^2+b^2v^2+1)^{3/2}} x_1 + \frac{b(1+a^2u^2)}{(a^2u^2+b^2v^2+1)^{3/2}} x_2$$

In theory, one can compute this directly by differentiating $n = \frac{1}{\sqrt{a^2u^2+b^2v^2+1}}(-au, -bv, 1)$ wrt $u, v$, but then, we have to write it in terms of the basis vectors $x_1, x_2$, so this is much faster.
(7) This is getting complicated, so now let's just concentrate at the origin: $u = v = 0$. There, we get

$$(L_i^j) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

so the eigenvalues (principal curvatures) $\kappa_1, \kappa_2$ are $a, b$, and mean curvature $H = \frac{a+b}{2}$ and Gaussian curvature $K = ab$. Moreover, the eigenvectors are $(1,0) = x_1$ and $(0,1) = x_2$, so the maximum/minimum normal curvatures are in $x_1, x_2$ direction, given by the parametric curves (which are parabolas, draw).
(8) Moreover, the parametric curves at origin are geodesics (by symmetry), but hard to parametrize them by arc length, and hard to find other geodesics. Actually for the saddle, can find two more: (straight

lines $x = \pm\sqrt{b/ay}$). Straight lines have normal curvature zero, so the Gaussian curvature (product of maximum and minimum) should be non-positive.

(9) Let's compute directly that $\kappa_n$ in the $x_1, x_2$ direction at the origin are $a, b$. Let us compute normal curvature of the parabola $z = ax^2/2, y = 0$.

(10) First compute total curvature $\kappa N$. Parametrize (not unit speed) $\alpha(t) = (t, 0, at^2/2)$. Recall,

$$\begin{aligned}
\dot\alpha &= vT \\
\ddot\alpha &= \dot vT + v\dot T = \dot vT + v^2T' = \dot vT + v^2\kappa N \\
\dot\alpha \times \ddot\alpha &= v^3\kappa B && \text{so can compute } \kappa, B \text{, but wait} \\
(\dot\alpha \times \ddot\alpha) \times \dot\alpha &= v^4\kappa N \\
\kappa N &= (\dot\alpha \times \ddot\alpha) \times \dot\alpha/v^4 \\
&= ((1,0,at) \times (0,0,a)) \times (1,0,at)/(1+a^2t^2)^2 \\
&= (0,-a,0) \times (1,0,at)/(1+a^2t^2)^2 = a(-at,0,1)/(1+a^2t^2)^2.
\end{aligned}$$

Normal (at $u = t, v = 1$) is $(-at, 0, 1)/\sqrt{1+a^2t^2}$. So $\kappa_n = \kappa N \cdot n = a/(1+a^2t^2)^{3/2}$. Also, the entire curvature is in normal direction, that is, $\kappa_n n = \kappa N$, so $\kappa_g S = \kappa N - \kappa_n n = 0$, as we checked earlier. Moreover, normal curvature at origin is $a$, which we also checked earlier (one of the principal curvatures). Similarly, we can also do the other parabola.

(11) An alterante way to compute $\kappa_n$.

$$\begin{aligned}
\kappa_n &= \sum L_{ij}\frac{d\alpha^i}{ds}\frac{d\alpha^j}{ds} \\
&= \frac{1}{(ds/dt)^2}\sum L_{ij}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt} \\
&= \frac{1}{1+a^2t^2}L_{11} && (\alpha^1 = u = t, \alpha^2 = v = 0) \\
&= a/(1+a^2t^2)^{3/2}
\end{aligned}$$

So this is much quicker, particularly for parametric curves. No such easy chain rule for geodesic curvature $\kappa_g$. So to compute it, use $\kappa_g S = \kappa N - \kappa_n n$.

(12) Now let us do integration. Assume $a, b > 0$ (paraboloid), and let $S$ be the surface below the plane $z = 1$, that is, $au^2/2 + bv^2/2 < 1$ is the open set $U$ in the parametric plane that parametrizes $S$. What is the total area of $S$?

(13) The area form is $dA = |x_1 \times x_2|dudv = \sqrt{\det(g)}dudv$. So total area is

$$\int_U \sqrt{a^2u^2 + b^2v^2 + 1}\,dudv.$$

(14) What is the total Gaussian curvature? (This is an important quantity, appears in Gauss-Bonnet theorem in 120B.) So integrate $K$ wrt $dA$ (don't forget the $|x_1 \times x_2|$ term),

$$\int_U \frac{ab}{(a^2u^2 + b^2v^2 + 1)^{3/2}}dudv.$$