

HW 6

- (Q-1) The number 3 can be expressed as a sum of one or more positive integers, taking order into account, in four ways, namely, 3, 1 + 2, 2 + 1, and 1 + 1 + 1. Show that any positive integer n can be so expressed in 2^{n-1} ways.
- (Q-2) In how many ways can 10 be expressed as a sum of 5 *nonnegative* integers, when order is taken into account? (Hint: Find an equivalent problem in which the phrase “5 nonnegative integers” is replaced by “5 positive integers”.)
- (Q-3) Given n objects arranged in a row. A subset of these objects is called *unfriendly* if no two of its elements are consecutive. Show that the number of unfriendly subsets with k elements is $\binom{n-k+1}{k}$. (Hint: Adopt an idea similar to that used in Larson 1.3.6.)
- (Q-4) Let a_1, a_2, \dots, a_n be a permutation of the set $S_n = \{1, 2, \dots, n\}$. An element i of S_n is called a fixed point of this permutation if $a_i = i$. A *derangement* of S_n is a permutation of S_n having no fixed points. Let g_n be the number of derangements of S_n . Show that

$$g_1 = 0, \quad g_2 = 1,$$

and

$$g_n = (n-1)(g_{n-1} + g_{n-2}), \quad \text{for } n > 2.$$

(Hint: a derangement either interchanges the first element with another element, or it doesn't.)

- (Q-5) Sum: $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n \binom{n}{n}$.
- (Q-6) Sum: $1 \times 2 \binom{n}{2} + 2 \times 3 \binom{n}{3} + \dots + (n-1)n \binom{n}{n}$.
- (Q-7) Sum: $\binom{n}{1} + 2^2 \binom{n}{2} + 3^2 \binom{n}{3} + \dots + n^2 \binom{n}{n}$.
- (Q-8) Sum: $\binom{n}{1} - 2^2 \binom{n}{2} + 3^2 \binom{n}{3} - \dots + (-1)^{n+1} n^2 \binom{n}{n}$.
- (Q-9) Prove $\binom{r}{0} \binom{s}{n} + \binom{r}{1} \binom{s}{n+1} + \binom{r}{2} \binom{s}{n+2} + \dots + \binom{r}{r} \binom{s}{n+r} = \binom{r+s}{s-n}$.
- (Q-10) Prove $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$.