(Q-1) If $a^{-1}ba = b^{-1}$ and $b^{-1}ab = a^{-1}$ for elements a, b of a group, prove that $a^4 = b^4 = 1$.

(Q-2) Let R be a ring with identity, and let $a \in R$. Suppose there is a unique element a' such that $aa' = 1$. Prove that $a'a = 1$.

(Q-3) Let p be a prime number, and let $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$. $\mathbb{Z}_p$ is a field under the operations of addition and multiplication modulo p.

(a) Show that $0, 1, \ldots, p-1$ are the zeros of $x^p - x$ (considered as a polynomial over $\mathbb{Z}_p$). Conclude that $x^p - x = x(x-1)(x-2)\ldots(x-(p-1)) \pmod{p}$.

(b) *Wilson's theorem.* From part (a), show that

$$(p-1)! = -1 \pmod{p}.$$

(Q-4) Let $n = 2^{p-1}(2^p - 1)$, and suppose that $2^p - 1$ is a prime number. Show that the sum of all (positive) divisors of n, not including n itself, is exactly n. (A number having this property is called a *perfect* number.)

(Q-5) Sum the series $1 + 22 + 333 + \cdots + n(\overbrace{11\ldots1}^{n})$.

(Q-6) A sequence is defined by $a_1 = 2$ and $a_n = 3a_{n-1} + 1$. Find the sum $a_1 + a_2 + \cdots + a_n$.

(Q-7) Verify the following formula:

$$\sum_{k=1}^{n} \sin((2k-1)\theta) = \frac{\sin^2(n\theta)}{\sin\theta}.$$

(Q-8) Sum the following:

(a) $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots + \dfrac{n-1}{n!}$.

(b) $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$.

(c) $\dfrac{2}{1 \times 2 \times 3} + \dfrac{4}{2 \times 3 \times 4} + \dfrac{6}{3 \times 4 \times 5} + \cdots + \dfrac{2n}{n(n+1)(n+2)}$.

(Q-9) Evaluate $\prod_{n=2}^{\infty}(1 - 1/n^2)$.

(Q-10) Let $F_1, F_2, \ldots$ be the Fibonacci sequence. Use the telescoping property to prove

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}.$$