HW 2

(Q-1) Suppose that a, b, c are positive numbers. Prove that:
- $(a+b)(b+c)(c+a) \geq 8abc$.
- $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$.
- If $a+b+c=1$, then $ab+bc+ca \leq \frac{1}{3}$.

(Q-2) For $0 < a < b$, show that

$$(n+1)(b-a)a^n < b^{n+1} - a^{n+1} < (n+1)(b-a)b^n.$$

(Q-3) If a, b, c are positive numbers, prove that

$$(a^2b + b^2c + c^2a)(a^2c + b^2a + c^2b) \geq 9a^2b^2c^2.$$

(Q-4) Suppose $a_1, \ldots, a_n$ are positive numbers and $b_1, \ldots, b_n$ is a rearrangement of $a_1, \ldots, a_n$. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

(Q-5) For each integer $n > 2$, prove that
- $n! < (\frac{n+1}{2})^n$,
- $1 \times 3 \times 5 \times \cdots \times (2n-1) < n^n$.

(Q-6)
- Let $x_i > 0$ for $i = 1, 2, \ldots, n$, and let $p_1, p_2, \ldots, p_n$ be positive integers. Prove that

$$(x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n})^{1/(p_1+\cdots+p_n)} \leq \frac{p_1x_1 + \ldots p_nx_n}{p_1 + \cdots + p_n}.$$

- Prove the same result as in the previous part holds even when the p_i's are positive rational numbers.

(Q-7) Use Cauchy-Schwarz inequality to prove the following:
- If $p_1, \ldots, p_n, x_1, \ldots, x_n$ are $2n$ positive numbers,

$$(p_1x_1 + \cdots + p_nx_n)^2 \leq (p_1 + \cdots + p_n)(p_1x_1^2 + \cdots + p_nx_n^2).$$

- If a, b, c are positive numbers,

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2.$$

(Q-8) Prove that for $0 < a < 1$,

$$(1+x)^a \leq 1 + ax, \qquad x \geq -1.$$

How should the inequality go when $a < 0$, or when $a > 1$?

(Q-9) Prove that

$$\frac{x}{1+x} < \log(1+x) < \frac{x(x+2)}{2(x+1)}, \qquad x > 0.$$

(Q-10) Prove that

$$\frac{\sin a}{\sin b} < \frac{a}{b} < \frac{\tan a}{\tan b}, \qquad 0 < b < a < \frac{\pi}{2}.$$