NAME:

MATH 100 Final

December 13, 2017

- You have 3 hours.
- This is a non-collaborative closed-book exam. You are not allowed to use books, notes, or any electronic devices (such as calculators, phones, computers) during the exams.
- There are a total of 10 problems and a total of 22 pages.
- You need to justify all answers.
- Write your solutions in the space below the questions. If you need more space use the back of the page.
- Do not forget to write your name in the space above.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Do not write in this box.

(Q-1) Prove by induction on $n \geq 1$ that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

(Q-2) If $a, b, c \in \mathbb{R}$ are required to satisfy $a^2 + b^2 + c^2 = 1$, what is the maximum of the expression $3a + 4b + 12c$? When is it attained?

(Q-3) Find all pairs of integers (x, y) satisfying $7x - 4y = 1$.

(Q-4) Prove that the polynomial $x^{3a} + x^{3b+1} + x^{3c+2}$ is divisible by $x^2 + x + 1$ for all natural numbers a, b, c.

(Q-5) Sum the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}.$$

(Q-6) What is the probability that the geometric mean of two random numbers x, y chosen (uniformly and independently) from $[0, 1]$ is greater than 0.5?

(Q-7) Let ABC be an equilateral triangle of side length d, and let O be its centroid. Let P be a point at distance r from O. Calculate

$$PA^2 + PB^2 + PC^2$$

in terms of d and r.

(Q-8) Count the number of functions $\{1, 2, 3\} \to \{1, 2, \ldots, 100\}$ that are:

(a) Injective.

(b) Strictly increasing.

(c) Increasing, but not necessarily strictly.

(Q-9) Solve the recurrence relation

$$a_n = a_{n-1} + a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$.

(Q-10) Calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2 + n^2}.$$