Name: _____

Math 100 : Problem Solving
Final Exam
Instructor: Ciprian Manolescu

You have 180 minutes.

Each problem is worth 10 points.
No books, notes or calculators are allowed.

1. Prove by induction on $n \geq 1$ that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

2. (a) Find the minimum of the expression

$$(a + b)(b + 2c)(c + 5d)(a + 10d)$$

over all possible $a, b, c, d > 0$ such that $abcd = 500$.

(b) For what values of a, b, c, d is the minimum attained?

3. Find all pairs of integers (x, y) such that $9x + 5y = 2$.

4. For what values of $a, b \in \mathbb{R}$ is the polynomial $P(x) = x^{50} + ax + b$ divisible by $(x+1)(x+2)$?

5. Calculate the infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}.$$

6. (a) How many functions $f : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$ are there?

(b) How many of these functions are one-to-one (injective)?

7. Pick two real numbers $x, y \in [0, 10]$ at random. What is the probability that $x+y \le 12$?

8. Let ABC be a triangle and let $D \in BC, E \in AC, F \in AB$ be such that

$$\frac{BD}{BC} = \frac{CE}{AC} = \frac{AF}{AB} = \frac{1}{3}.$$

Show that the triangles ABC and DEF have the same centroid.

9. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = \sqrt{f(x)^2 + f(y)^2}$$

for all $x, y \in \mathbb{R}$.

10. Calculate the integral

$$\int_0^1 \frac{e^{x+3}}{e^{x+3} + e^{4-x}}\,dx.$$

1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
5		out of 10 points
6		out of 10 points
7		out of 10 points
8		out of 10 points
9		out of 10 points
10		out of 10 points
Total		out of 100 points