Name: ____

Math 100 : Problem Solving
Final Exam
Instructor: Ciprian Manolescu

You have 180 minutes.

Each problem is worth 10 points.
No books, notes or calculators are allowed.

1. Use induction on n to prove that

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{n^2}\right) \leq 5\left(1-\frac{1}{n}\right)$$

for all $n \geq 2$.

2. If $a, b, c > 0$ are required to satisfy $a^2 + b^2 + c^2 = 1$, what is the maximum of the expression $a + 2b + 2c$?

3. Find all pairs of integers (x, y) such that $4x - 3y = 1$.

4. Let a, b, c be the roots of the equation

$$x^3 + 2x^2 - 9x - 1 = 0.$$

Write down a cubic equation whose roots are ab, bc, and ac.

5. Let F_n be the Fibonacci numbers, defined by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

(a) Show that

$$\frac{1}{F_{n-1}F_{n+1}} = \frac{1}{F_{n-1}F_n} - \frac{1}{F_nF_{n+1}}$$

for $n \geq 2$.

(b) Sum the infinite series:

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}}.$$

6. Multiple choice: Circle the correct answer. (No justifications necessary in this problem.)

(a) The number of all functions $f : \{1, 2, 3\} \to \{1, 2, \dots, 50\}$ is

(A) $3 \cdot 50$; (B) $50 \cdot 49 \cdot 48$; (C) $\binom{50}{3}$; (D) 3^{50}; (E) 50^3.

(b) The number of injective functions $f : \{1, 2, 3\} \to \{1, 2, \dots, 50\}$ is

(A) 0; (B) $50 \cdot 49 \cdot 48$; (C) $\binom{50}{3}$; (D) $\binom{52}{50}$; (E) $\binom{52}{3}$.

(c) The number of surjective functions $f : \{1, 2, 3\} \to \{1, 2, \dots, 50\}$ is

(A) 0; (B) $50 \cdot 49 \cdot 48$; (C) $\binom{50}{3}$; (D) $50^3 - 50 \cdot 49^3$; (E) $50^3 - 50 \cdot 49^3 + \binom{50}{2} \cdot 48^3$.

(d) The number of strictly increasing functions $f : \{1, 2, 3\} \to \{1, 2, \dots, 50\}$ is

(A) 0; (B) $50 \cdot 49 \cdot 48$; (C) $\binom{50}{3}$; (D) $\binom{52}{50}$; (E) $\binom{52}{3}$.

7. Solve the recurrence relation $9a_n = 6a_{n-1} - a_{n-2}$ with initial conditions $a_0 = 6, a_1 = 5$.

8. Let $ABCD$ be a square with center O and side length d, and let P be a point in the plane with $|PO| = r$. Express the quantity

$$|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$$

in terms of d and r.

9. Let a, b, c be real numbers satisfying the condition:

$$a \cdot \frac{\pi}{2} + b \cdot \frac{\pi^2}{8} + c \cdot \frac{\pi^3}{24} = 1.$$

Show that the equation

$$\cos x = a + bx + cx^2$$

has at least one real solution.

10. Find

$$\lim_{n\to\infty}\left(\frac{1}{4n+1}+\frac{1}{4n+2}+\cdots+\frac{1}{5n}\right)$$

Do not write on this page.

1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
5		out of 10 points
6		out of 10 points
7		out of 10 points
8		out of 10 points
9		out of 10 points
10		out of 10 points
Total		out of 100 points