On conjectures of Sharifi

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Abstract

R. Sharifi formulated remarkable conjectures which relate arithmetic of cyclotomic fields to modular curves. We give partial solutions to his conjectures.

Contents.

0 Introduction

0.1. In their proof of Iwasawa main conjecture, Mazur and Wiles [32] used deep relations between modular curves and the arithmetic of cyclotomic fields, extending the earlier work [46] of Ribet.

In the paper [51] etc., Sharifi formulated remarkable conjectures which tell that stronger relations exist between these two subjects.

We give partial solutions to his conjectures.

0.2. Let $p \geq 5$ be a prime number.

The conjectures in Sharifi [51] relate modular curves $X_1(Np^r)$ to the cyclotomic fields $\mathbb{Q}(\zeta_{Np^r})$ (N is prime to p and $r \geq 0$; it is assumed that p does not divide the order $\varphi(N)$ of $(\mathbb{Z}/N\mathbb{Z})^\times$) and our main results stated in section 7.2 consider this relation. For simplicity, in this Introduction, we assume $N = 1$.

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0.3. Ideal class groups of cyclotomic fields.

For $r \geq 1$, let ζ_{p^r} be a primitive p^r-th root of unity, and let $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\}$ be the p-power part of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))$ of the cyclotomic field $\mathbb{Q}(\zeta_{p^r})$.

Let

$$X = \varprojlim_r \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\}$$

where the inverse limit is taken with respect to the norm maps of ideal class groups. Then X is a finitely generated $\mathbb{Z}_p$-module. Let

$$K = \cup_{r\geq 1}\mathbb{Q}(\zeta_{p^r})$$

The action of $\mathrm{Gal}(K/\mathbb{Q})$ on X makes X a Λ-module, where Λ is the completed group algebra $\mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]] = \varprojlim_r \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^r})/\mathbb{Q})]$. We have decompositions

$$X = X^+ \oplus X^-, \quad \Lambda \xrightarrow{\cong} \Lambda^+ \times \Lambda^-,$$

according to the action of the complex conjugation, and $X^{\pm}$ is a $\Lambda^{\pm}$-module.

0.4. Modular curves.

Let $r \geq 1$, and consider the compactified modular curve $X_1(p^r)$ of level p^r. Let H_r be the ordinary part of $H^1(X_1(p^r)(\mathbb{C}), \mathbb{Z}_p) = H^1_{\text{ét}}(X_1(p^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)$ with respect to the dual Hecke operator $T^*(p)$.

Let $\mathfrak{h}_r$ be the subring of $\mathrm{End}_{\mathbb{Z}_p}(H_r)$ generated over $\mathbb{Z}_p$ by the dual Hecke operators $T^*(n) : H_r \to H_r$ $(n \geq 1)$. The Eisenstein ideal $I_r \subset \mathfrak{h}_r$ is the ideal of $\mathfrak{h}_r$ generated by $1 - T^*(p)$ and $1 - T^*(\ell) + \ell\langle\ell\rangle^{-1}$ for prime numbers $\ell \neq p$, where $\langle\ell\rangle \in \mathfrak{h}_r$ is the diamond operator.

Let

$$H = \varprojlim_r H_r, \quad \mathfrak{h} = \varprojlim_r \mathfrak{h}_r, \quad I = \varprojlim_r I_r \subset \mathfrak{h}.$$

Then H/IH and $\mathfrak{h}/I$ are finitely generated as $\mathbb{Z}_p$-modules. By the action of the complex conjugation, we have a decomposition $H = H^+ \oplus H^-$ as an $\mathfrak{h}$-module.

0.5. Homomorphisms ϖ and Υ.

As is explained below, we have homomorphisms

$$\varpi : H^-/IH^- \to X^-, \quad \Upsilon : X^- \to H^-/IH^-$$

which relate modular curves and cyclotomic fields, and which are the subjects of Conjecture 0.11 below.

These homomorphisms are compatible with the Λ^--module structure of X^- and the $\mathfrak{h}$-module structure on H^-/IH^- with respect to the following ring homomorphism $\Lambda^- \to \mathfrak{h}$: Note that we have an isomorphism

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_p^\times, \quad \sigma_c \leftrightarrow c \ (c \in \mathbb{Z}_p^\times), \quad \sigma_c(\zeta_{p^r}) = \zeta_{p^r}^c.$$

Let $\Lambda \to \mathfrak{h}$ be the homomorphism which sends $\sigma_c \in \mathrm{Gal}(K/\mathbb{Q})$ to $c\langle c\rangle \in \mathfrak{h}$, where $\langle c\rangle \in \mathfrak{h}^\times$ is the diamond operator. Then this ring homomorphism factors through the quotient Λ^- of Λ.

0.6. A conjecture of McCallum and Sharifi on the arithmetic of cyclotomic fields.

Before we introduce the definition of the homomorphism ϖ, we review a conjecture of McCallum and Sharifi. They studied the elements

$$\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\} \in \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^- \quad (u, v \in \mathbb{Z}/p^r\mathbb{Z} - \{0\})$$

where

$$\{\ ,\ \} : \mathbb{Z}[1/p, \zeta_{p^r}]^\times \times \mathbb{Z}[1/p, \zeta_{p^r}]^\times \to \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-$$

is a pairing defined by using the cup product of Galois cohomology (see [34], [51]). Note that $1-\zeta_{p^r}^u \in \mathbb{Z}[\zeta_{p^r}, 1/p]^\times$. Basing on their study, McCallum and Sharifi have

Conjecture 0.7. $\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$ *for* $u, v \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ *generate* $Cl(\mathbb{Q}(\zeta_{p^r}))^-/p^r\, Cl(\mathbb{Q}(\zeta_{p^r}))^-$.

0.8. This conjecture gives a new understanding of the important group $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-$. But if we consider only the world of cyclotomic fields, we may not find a strong philosophy which supports this conjecture. However, Sharifi formulated deep conjectures on the relation between the world of cyclotomic fields and the world of modular curves, for which the above conjecture 0.7 is a corollary. Furthermore, by using the relation with modular curves, he proved the above conjecture 0.7 in the case $p < 1000$ ([49]).

0.9. Definition of ϖ.

The theory of modular symbols gives special elements $[u : v]_r$ $(u, v \in \mathbb{Z}/p^r\mathbb{Z} - \{0\})$ of H_r which generate H_r as a $\mathbb{Z}_p$-module (see 2.4.1, 4.4.1).

In [51] 5.7, Sharifi proved that there is a homomorphism $H_r^- \to \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-$ which sends the minus part $[u:v]_r^-$ of $[u:v]_r$ to $\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$ for any $u, v \in \mathbb{Z}/p^r\mathbb{Z} - \{0\}$.

Sharifi conjectured ([51] 5.8) that this map factors through the quotient $H_r^-/I_rH_r^-$ of H_r^-. In this paper, we prove this conjecture, and prove that these homomorphisms for $r \geq 1$ induce a homomorphism $\varpi : H^-/IH^- \to \varprojlim_r \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^- = X^-$.

0.10. Definition of Υ.

We use the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on H.

Let $K = \cup_r \mathbb{Q}(\zeta_{p^r})$ as before, and let L be the largest unramified pro-p abelian extension of K. Then class field theory gives an isomorphism

$$X \cong \mathrm{Gal}(L/K).$$

The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ on H/IH factors through the abelian quotient group $X \cong \mathrm{Gal}(L/K)$. The action of X on the part H^-/IH^- of H/IH is trivial, and for $\sigma \in X$ and $x \in H/IH$, we have $\sigma(x) - x \in H^-/IH^-$.

As is shown in [51], H^+/IH^+ is free of rank 1 as an $\mathfrak{h}/I$-module. Furthermore, we have a canonical basis e of H^+/IH^+ as an $\mathfrak{h}/I$-module (see 6.3.18). We have a homomorphism

$$\Upsilon : X^- \to H^-/IH^- \ ;\ \sigma \mapsto \sigma(e) - e.$$

Conjecture 0.11 (Sharifi [51], Conjecture 5.2 and Remark at the end of section 5)**.**

$$\varpi \circ \Upsilon = 1, \quad \Upsilon \circ \varpi = 1.$$

In fact, as is explained in 7.1.14, this conjecture is slightly stronger than the original one in [51].

0.12. Relation with the conjecture of McCallum and Sharifi.

The conjecture 0.11 implies the conjecture 0.7 of McCallum and Sharifi. In fact, since the projection $X^- \to \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-$ is surjective, the surjectivity of $\varpi : H^-/IH^- \to X^-$ implies that $H_r^- \to \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-$ is surjective. Since H_r^- is generated by $[u:v]_r^-$, we have that $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-$ is generated by $\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$ with $u,v \in \mathbb{Z}/p^r\mathbb{Z}-\{0\}$. It is easy to see that these elements are generated by those for which $u,v \in (\mathbb{Z}/p^r\mathbb{Z})^\times$.

0.13. Relation with Iwasawa theory.

The above conjecture 0.11 of Sharifi gives a deeper understanding of X^- than the Iwasawa main conjecture.

Recall that Iwasawa main conjecture proved by Mazur-Wiles states $X^- \sim \Lambda^-/(\xi)$ where ξ is the p-adic Riemann zeta function, (ξ) is the ideal of Λ^- generated by ξ, and $\sim$ means that these Λ-modules have the same invariants called characteristic ideals. (To simplify the description of this Introduction, here we eliminate the pole of the p-adic Riemann zeta function and denote by ξ a slight modification of the actual p-adic Riemann zeta function. See 4.1.2 for the precise formulation.)

In the course of their proof of Iwasawa main conjecture, Mazur and Wiles showed that X^- is isomorphic to B/IB for some faithful $\mathfrak{h}$-submodule B of H^- via something like the map Υ, that the homomorphism $\Lambda^- \to \mathfrak{h}$ in 0.5 induces an isomorphism $\Lambda^-/(\xi) \cong \mathfrak{h}/I$, and $X^- \cong B/IB \sim \mathfrak{h}/I \cong \Lambda^-/(\xi)$. They did not consider a map like ϖ. Sharifi considers the two maps ϖ and Υ which give a deep understanding of X^-.

We have a canonical isomorphism $H^-/IH^- \cong S_\Lambda/IS_\Lambda$ as $\mathfrak{h}$-modules, where S_Λ is the space of ordinary Λ-adic cusp forms (see section 1.5) and the $\mathfrak{h}$-module structure of S_Λ here is that $T^*(n) \in \mathfrak{h}$ acts on S_Λ as the usual operator $T(n)$ on S_Λ. It is known that S_Λ is the dualizing module of the ring $\mathfrak{h}$ in the sense of commutative ring theory ([19] section 7). Hence the consequence $X^- \cong S_\Lambda/IS_\Lambda$ of Conjecture 0.11 has interesting consequences that the structure of the Λ-module X^- is determined by the ring $\mathfrak{h}$ and that it is described by the space of ordinary Λ-adic cusp forms.

Our results are the following (see section 7.2).

Theorem 0.14. *Let $\xi' \in \Lambda^-$ be the derivative of the p-adic Riemann zeta function ξ (see 7.2.2). Then as a map $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, we have*

$$\xi'\Upsilon \circ \varpi = \xi'.$$

Theorem 0.15. *Assume that either one of the following (i) and (ii) is satisfied.*

(i) The p-adic Riemann zeta function ξ has no multiple zero.

(ii) The class of $(1-T^*(p))\{0,\infty\} \in H^-$ generates $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as an $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module.

Here $\{0,\infty\}$ denotes the path from the cusp 0 to the cusp ∞.

Then we have:

(1) $\varpi \circ \Upsilon = 1$. $\Upsilon \circ \varpi \equiv 1$ modulo the p-primary torsion of H^-/IH^-.

(2) $X^- \cong (H^-/IH^-)/(tor)$ where (tor) is the p-primary torsion part of H^-/IH^-.

(3) The conjecture 0.7 is true.

Theorem 0.16. *Assume that $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by one element as a module over $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then the conjecture 0.7 is true.*

Remark 0.17. (1) In all known examples, ξ has no multiple zero (see the sentences after 3.1 of Greenberg [15]).

(2) The assumption (ii) in Theorem 0.15 is closely related to the work [49] of Sharifi. In that work, he proved the conjecture 0.7 under an assumption which is slightly stronger than (ii). Our proof of Theorem 0.15 assuming (ii) follows his method in [49].

(3) We can prove the above theorems for each component of Λ^- corresponding to an odd power of the Teichmüller character. See section 7.2.

(4) Sharifi formulated his conjectures also for $\mathbb{Q}(\zeta_{Np^r})$ and $X_1(Np^r)$ $(r \geq 1)$ with fixed N such that p does not divide $\varphi(N)$. In this paper, we study this generalized situation. See section 7.2 for our results.

0.18. We introduce another conjecture of Sharifi in [51]. Let

$$\begin{aligned}\mathcal{C} &= \Big(\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \{1-\zeta_{p^r}^a, 1-\zeta_{p^r}\} \cdot [a]\Big)_{r \geq 1} \\ &\in X^-[[\mathbb{Z}_p^\times]] := \varprojlim_r \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^-/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))^- \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[(\mathbb{Z}/p^r\mathbb{Z})^\times].\end{aligned}$$

On the other hand, let

$$\mathcal{L} = \Big(\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} T^*(p)^{-r}[1:a]_r \cdot [a]\Big)_{r \geq 1} \in H[[\mathbb{Z}_p^\times]] := \varprojlim_r H_r \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[(\mathbb{Z}/p^r\mathbb{Z})^\times]$$

be the p-adic L function in two variables of Mazur-Kitagawa for modular forms ([30], [24]; See section 4.4 of this paper for a review).

Conjecture 0.19. *The homomorphism*

$$\Upsilon : X^-[[\mathbb{Z}_p^\times]] \to (H^-/IH^-)[[\mathbb{Z}_p^\times]]$$

sends $\mathcal{C}$ to the minus part $\mathcal{L}^- \bmod I$ of $\mathcal{L} \bmod I$.

From the above Theorem 0.15, we can deduce

Theorem 0.20. *Under the assumption of Theorem 0.15, the classes of $\Upsilon(\mathcal{C})$ and $\mathcal{L}^-$ in $((H^-/IH^-)/(tor))[[\mathbb{Z}_p^\times]]$ coincide.*

0.21. For an earlier work on conjectures of Sharifi, see Busuioc [4]. We heard that G. Stevens has some results on conjectures of Sharifi, especially on Conjecture 5.8 in Sharifi [51].

0.22. We give rough sketches of our proofs of the theorems 0.14 – 0.16.

We use a commutative diagram

$$\begin{array}{ccccccc} H^- & \xrightarrow{(1)} & ''K_2'' & \xrightarrow{(2)} & X^- & \xrightarrow{\Upsilon} & H^-/IH^- \\ & & \downarrow & & & & \downarrow \xi' \\ & & S_\Lambda & \to & S_\Lambda/IS_\Lambda & \cong & H^-/IH^-. \end{array}$$

Here:

$''K_2''$ denotes a certain p-adic completion of the inverse system of K_2 of $X_1(p^r)$.

The homomorphism (1) "sends" (more precisely, the map (1) is the inverse limit of the maps in finite levels which send) $[u:v]_r^- \in H_r^-/IH_r^-$ to the Beilinson element $\{g_{0,u/p^r}, g_{0,v/p^r}\}$ in $''K_2''$, where $g_{\alpha,\beta}$ ($(\alpha,\beta) \in (\frac{1}{p^r}\mathbb{Z}/\mathbb{Z})^2 - \{(0,0)\}$) is the function on $X_1(p^r)$ called a Siegel unit. See section 3.3 for details. In the introduction of his paper [51], Sharifi expected that Beilinson elements would be useful for the study of his conjectures. In this paper, we try to realize his expectation.

The homomorphism (2) is the evaluation at the ∞-cusp of the modular curve. It "sends" $\{g_{0,u/p^r}, g_{0,v/p^r}\}$ to $\{1-\zeta_{p^r}^u, 1-\zeta_{p^r}^v\}$. See section 5.2 for details. The composition of (1) and (2) is the map ϖ. Hence the composition of the upper rows is $\Upsilon \circ \varpi$.

S_Λ is the space of ordinary Λ-adic cusp forms. The left vertical arrow is a kind of p-adic logarithm (see section 4.3 for details). The right vertical arrow is the multiplication by ξ'.

The commutativity of the square is proved by the study of Galois cohomology in section 9 (the derivative ξ' appears in the study in section 9.3). On the other hand, one can prove that the composition $H^- \to {''K_2''} \to S_\Lambda \to S_\Lambda/IS_\Lambda \cong H^-/IH^-$ is ξ' times the natural projection (see section 8.1). From these, we have Theorem 0.14 (see section 10.1 for details). In fact, in the actual construction of the map (1), some denominator appears (the map (1) is not defined integrally), and this is the reason why we need $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ in Theorem 0.14.

Next, Theorem 0.15 under the assumption (i) is deduced from Theorem 0.14 as follows. By the assumption (i), ξ' is invertible in $\Lambda^-/(\xi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathfrak{h}/I \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and hence we have $\Upsilon \circ \varpi = 1$ on $H^-/IH^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. From this and by some arguments (see 7.1.3), we obtain $\varpi \circ \Upsilon = 1$ on X^-. This shows the surjectivity of ϖ and hence we can deduce the conjecture 0.7 by the argument in 0.12.

Theorem 0.16 is also deduced from Theorem 0.14. See section 10.2.

Our proof of Theorem 0.15 under the assumption (ii) is a sort of modification of the method of Sharifi in [49].

0.23. This paper contains some general results which we hope to be useful not only for the applications to the conjectures of Sharifi. For example, in sections 3.2 and 3.3, we study Beilinson elements for Hida family as in Ochiai [37]. In section 4.4, we compare the L-functions of Mazur-Kitagawa in two variables with the p-adic L-functions in two variables obtained by using Beilinson elements, also as in Ochiai [37]. Our results seem not to be covered by those in [37] in the points that we use Beilinson elements associated to more general modular symbols $[u:v]$, and that we use a new pairing $((\ ,\ ))$ defined in section 1.9 to compare these two kinds of p-adic L-functions in two variables.

0.24. The plan of this paper is as follows.

After preparations in sections 1-4, we give the definition of the map ϖ (resp. Υ) in section 5 (resp. section 6). (From section 6, as is explained in section 6.1, we fix a character θ and study the θ-component.)

In section 7, we introduce conjectures of Sharifi and state our results on his conjectures.

In sections 8–10, we prove our results on his conjectures.

In section 11, we describe some relation to Iwasawa theory of modular forms.

0.25. We thank Romyar Sharifi for stimulating discussions and very helpful comments. We thank John Coates for his consistent encouragement, and Masato Kurihara for a lot of precious advice.

0.26. Notation and convention

Let $\bar{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. For each prime number ℓ, let $\bar{\mathbb{Q}}_\ell$ be an algebraic closure of $\bar{\mathbb{Q}}_\ell$. We fix an embedding $\bar{\mathbb{Q}} \to \mathbb{C}$ and an embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\ell$ for each prime number ℓ.

For $n \geq 1$, let $\zeta_n = e^{2\pi i/n} \in \bar{\mathbb{Q}} \subset \mathbb{C}$.

We fix a prime number $p \geq 5$. Our assumption $p \geq 5$ comes from the fact that in many places in this paper, we use results of H. Hida and M. Ohta in which they assume $p \geq 5$. (Results in §3.1 hold also for $p = 2, 3$ and results in §4.2 hold also for $p = 3$.)

In this paper in the places where we present an isomorphism $M \cong M(r)$ for a $\mathbb{Z}_p$-module M and $r \in \mathbb{Z}$ without telling which isomorphism we take, they are given by the base $(\zeta_{p^n})_n$ of $\mathbb{Z}_p(1)$.

1 Preliminaries on modular curves and modular forms

In this section 1, we review modular curves and modular forms. We also supply a new deifnition $((\ ,\ ))$ (section 1.6), an improvement (1.7.12), and a new result (section 1.8) as well. A reason why this section 1 is long is that the formulations in the works which we refer to in this paper differ in delicate ways depending on the authors, and so we have to present our formulation carefully.

1.1 Modular curves

For generality of modular curves, see for example [23].

1.1.1. For integers $m, M \geq 1$ such that $m + M \geq 5$ we have the modular curve $X(m, M)$ with cusps, which is a proper curve over $\mathbb{Z}$, and the modular curve $Y(m, M)$ without cusps, which is an open subscheme of $X(m, M)$. They are characterized as follows. Over $\mathbb{Z}[1/mM]$, $Y(m, M) \otimes \mathbb{Z}[1/mM]$ is the moduli space of triples (E, e_1, e_2) with E an elliptic curve and e_1, e_2 sections of E such that $me_1 = Me_2 = 0$ and such that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z} \to E$; $(a, b) \mapsto ae_1 + be_2$ is injective. The scheme $X(m, M)$ (resp. $Y(m, M)$) is the integral closure of the projective j-line $\mathbb{P}^1_{\mathbb{Z}}$ (resp. the affine j-line $\mathrm{Spec}(\mathbb{Z}[j])$) in $Y(m, M) \otimes \mathbb{Z}[1/mM]$. (Cf. [6], [23].)

1.1.2. If $m|m'$ and $M|M'$, we have the canonical morphisms $X(m',M') \to X(m,M)$ and $Y(m',M') \to Y(m,M)$. Over $\mathbb{Z}[1/m'M']$, it corresponds to the morphism of moduli functors $(E,e_1,e_2) \mapsto (E,e_1',e_2')$ where $e_1' = (m'/m)e_1$, $e_2' = (M'/M)e_2$.

1.1.3. For $M \geq 3$, let $X(M) = X(M,M)$, $Y(M) = Y(M,M)$. For $M \geq 4$, let $X_1(M) = X(1,M)$, $Y_1(M) = Y(1,M)$.

1.1.4. The group $GL(2,\mathbb{Z}/M\mathbb{Z})$ acts on $X(M)$ and on $Y(M)$ in the way that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $Y(M)\otimes\mathbb{Z}[1/M] \to Y(M)\otimes\mathbb{Z}[1/M]$ sends the class of (E,e_1,e_2) to the class of (E,e_1',e_2') where

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

The modular curve $X(m,M)$ (resp. $Y(m,M)$) ($m \geq 1, M \geq 1, m+M \geq 5$) is the quotient of $X(L)$ (resp. $Y(L)$), where $m|L$, $M|L$, by action of the subgroup $G_L(m,M)$ of $GL(2,\mathbb{Z}/L\mathbb{Z})$ defined by

$$G_L(m,M) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}/L\mathbb{Z})$$
$$\mid\ a \equiv 1 \bmod m, b \equiv 0 \bmod m, c \equiv 0 \bmod M, d \equiv 1 \bmod M\}.$$

If $m|L$, $M|L$, we will identify $X_1(M)\otimes\mathbb{Z}[1/L,\zeta_m]$ (resp. $Y_1(M)\otimes\mathbb{Z}[1/L,\zeta_m]$) with the quotient of $X(L)\otimes\mathbb{Z}[1/L]$ (resp. $Y(L)\otimes\mathbb{Z}[1/L]$) by the action of

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}/L\mathbb{Z}) \mid c \equiv 0 \bmod M, d \equiv 1 \bmod M, ad-bc \equiv 1 \bmod m\}.$$

It is a quotient of $X(m,M)\otimes\mathbb{Z}[1/L]$ (resp. $Y(m,M)\otimes\mathbb{Z}[1/L]$) if $m|M$.

1.1.5. Let $\mathcal{H}$ be the upper half plane.

We have the canonical map

$$\mathcal{H} \to Y(m,M)(\mathbb{C})$$

which sends $\tau \in \mathcal{H}$ to the class of (E,e_1,e_2) where E is the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ over $\mathbb{C}$ and $e_1 = \tau/m, e_2 = 1/M (\mathrm{mod}\ \mathbb{Z}\tau+\mathbb{Z})$.

The action of $SL(2,\mathbb{Z})$ on $\mathcal{H}$ commutes with the action of $GL(2,\mathbb{Z}/M\mathbb{Z})$ on $Y(M)$.

1.1.6. In the paper [22] to which we refer often in this paper, the notation $X(m,M)$ (resp. $Y(m,M)$) was used for $X(m,M)\otimes\mathbb{Q}$ (resp. $Y(m,M)\otimes\mathbb{Q}$).

1.1.7. Cusp forms and modular forms of weight 2. The space $S_2(m,M)_{\mathbb{Q}}$ of cusp forms of weight 2 on $X(m,M)_{\mathbb{Q}} = X(m,M)\otimes\mathbb{Q}$ is the space $\Gamma(X(m,M)_{\mathbb{Q}},\Omega^1)$, where Ω^1 is the sheaf of differential forms on $X(m,M)_{\mathbb{Q}}$. The space $M_2(m,M)_{\mathbb{Q}}$ of modular forms of weight 2 on $X(m,M)_{\mathbb{Q}}$ is the space $\Gamma(X(m,M)_{\mathbb{Q}},\Omega^1(\log\{\text{cusps}\}))$ of differential forms which may have logarithmic poles along the cusps.

Let $S_2(M)_{\mathbb{Q}} = S_2(1,M)_{\mathbb{Q}}$, $M_2(M)_{\mathbb{Q}} = M_2(1,M)_{\mathbb{Q}}$.

1.2 Hecke operators

1.2.1. Consider the modular curves $X(m, M)$ and $Y(m, M)$. We review Hecke operators $T(n)$ and dual Hecke operators $T^*(n)$ ($n \geq 1$, $(n, m) = 1$), and diamond operators, which act on the cohomology groups $H^i(X(m, M)(\mathbb{C}), \mathbb{Z})$, $H^i(Y(m, M)(\mathbb{C}), \mathbb{Z})$, the space of cusp forms $S_2(m, M)_{\mathbb{Q}}$, the space of modular forms $M_2(m, M)_{\mathbb{Q}}$, and the K-groups $K_i(X(m, M) \otimes \mathbb{Z}[1/L])$ and $K_i(Y(m, M) \otimes \mathbb{Z}[1/L])$ ($L \geq 1$), etc.

In this paper, dual Hecke operators are used more than Hecke operators.

1.2.2. Diamond operator $\langle a \rangle$ ($a \in (\mathbb{Z}/L\mathbb{Z})^\times$ where $m|L$ and $M|L$). It is an automorphism of $Y(m, M)$ (or of $X(m, M)$) induced by the automorphism $\begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$ of $Y(L)$ (or of $X(L)$).

If $m|M$, in the identification of $Y_1(M) \otimes \mathbb{Q}(\zeta_m)$ with a quotient of $Y(m, M)$ (1.1.4), the automorphism of $\langle a \rangle \otimes 1$ of $Y_1(M) \otimes \mathbb{Q}(\zeta_m)$ is compatible with $\langle a \rangle$ of $Y(m, M)$.

1.2.3. Let $m \geq 1, M \geq 1$, $m + M \geq 5$, and let ℓ be a prime number. Then the Hecke operator $T(\ell)$ and the dual Hecke operator $T^*(\ell)$ are defined as follows.

Let $Y(m(\ell), M)$ (resp. $Y(m, M(\ell))$) be the quotient of $Y(L)$ where $m\ell|L$ and $M|L$ (resp. $m|L$ and $M\ell|L$) by the subgroup of $G_L(m, M)$ consisting of all elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $b \equiv 0 \bmod m\ell$ (resp. $c \equiv 0 \bmod M\ell$). Let

$$\psi_\ell : Y(m, M(\ell)) \to Y(m, M)$$

be the unique morphism which is compatible with the map $\mathcal{H} \to \mathcal{H}$; $\tau \mapsto \ell\tau$. Let $\pi : Y(m, M(\ell)) \to Y(m, M)$ be the canonical projection. We will use later the following moduli interpretations of $Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell]$ and $\psi_\ell, \pi : Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell] \to Y(m, M) \otimes \mathbb{Z}[1/mM\ell]$. Over $\mathbb{Z}[1/mM\ell]$, $Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell]$ represents the isomorphism classes of quadruples (E, e_1, e_2, C) where (E, e_1, e_2) is as in the moduli description of $Y(m, M) \otimes \mathbb{Z}[1/mM\ell]$, C is a subgroup scheme of E which is étale locally isomorphic to $\mathbb{Z}/M\ell\mathbb{Z}$ and étale locally generated by some e_2' such that $e_2 = \ell e_2'$, π represents $(E, e_1, e_2, C) \mapsto (E, e_1, e_2)$, and ψ_ℓ represents $(E, e_1, e_2, C) \mapsto (E/(MC), e_1 \bmod MC, e_2' \bmod MC)$.

Then

$$T(\ell) = (\psi_\ell)_* \pi^*, \quad T^*(\ell) = \pi_* \psi_\ell^*.$$

1.2.4. For an integer $n = \ell^e$ with ℓ a prime number which does not divide m and with $e \geq 0$, $T(n)$ and $T^*(n)$ are defined as follows. If ℓ divide M,

$$T(\ell^e) = T(\ell)^e, \quad T^*(\ell^e) = T^*(\ell)^e.$$

If ℓ does not divide M, they are defined inductively by $T(1) = T^*(1) = 1$ and

$$T(\ell^{e+2}) = T(\ell)T(\ell^{e+1}) + T(\ell^e)\langle \ell \rangle \cdot \ell, \quad T^*(\ell^{e+2}) = T^*(\ell)T^*(\ell^{e+1}) + T^*(\ell^e)\langle \ell \rangle^{-1} \cdot \ell.$$

For an integer $n = \prod_i \ell_i^{e(i)}$ where ℓ_i are distinct prime numbers which do not divide m,

$$T(n) = \prod_i T(\ell_i^{e(i)}), \quad T^*(n) = \prod_i T^*(\ell_i^{e(i)}).$$

We have

$$T(n_1)T(n_2) = T(n_2)T(n_1), \quad T^*(n_1)T^*(n_2) = T^*(n_2)T^*(n_1),$$
$$T(n)\langle a\rangle = \langle a\rangle T(n), \quad T^*(n)\langle a\rangle = \langle a\rangle T^*(n)$$

$(n_1, n_2, n \geq 1,\ (a, M) = 1)$. We have

$$T(n) = T^*(n)\langle n\rangle \quad \text{if } (n, mM) = 1.$$

1.2.5. $T(n)$ and $T^*(n)$ are the transposes of each other in the Poincaré duality $H^1(X(m,M)(\mathbb{C}),\mathbb{Z})\times H^1(X(m,M)(\mathbb{C}),\mathbb{Z}) \to \mathbb{Z}$ and in the Poincaré duality $H^1(Y(m,M)(\mathbb{C}),\mathbb{Z})\times H^1_c(Y(m,M)(\mathbb{C}),\mathbb{Z}) \to \mathbb{Z}$. Here H^1_c is the cohomology with compact supports.

From here, we consider the case $m = 1$.

1.2.6. Let $\mathfrak{h}(M)_{\mathbb{Z}}$ (resp. $\mathfrak{H}(M)_{\mathbb{Z}}$) be the subring of $\mathrm{End}_{\mathbb{Z}}(H^1(X_1(M)(\mathbb{C}),\mathbb{Z}))$ (resp. $\mathrm{End}_{\mathbb{Z}}(H^1(Y_1(M)(\mathbb{C}),\mathbb{Z}))$ generated over $\mathbb{Z}$ by $T^*(n)$ $(n \geq 1)$ and $\langle n\rangle$ $((n, M) = 1)$.

They are commutative rings. They act also on $S_2(M)_{\mathbb{Q}}$ (resp. $M_2(M)_{\mathbb{Q}}$).

We have a canonical ring homomorphism $\mathfrak{H}(M)_{\mathbb{Z}} \to \mathfrak{h}(M)_{\mathbb{Z}}$ by restricting $T^*(n)$ and $\langle n\rangle$ on $H^1(Y_1(M)(\mathbb{C}),\mathbb{Z})$ to $H^1(X_1(M)(\mathbb{C}),\mathbb{Z})$.

1.2.7. The p-adic (dual) Hecke algebra $\mathfrak{h}(M)_{\mathbb{Z}_p} = \mathfrak{h}(M)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (resp. $\mathfrak{H}(M)_{\mathbb{Z}_p} = \mathfrak{H}(M)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$) is generated by $T^*(n)$ $(n \geq 1)$ over $\mathbb{Z}_p$. This is because for a prime number ℓ which does not divide M, we have $\langle \ell\rangle^{-1} = \ell^{-1}(T^*(\ell^2) - T^*(\ell)^2)$.

1.2.8. For a scheme C over $\mathbb{Q}$, we will denote by $H^m_{\text{ét}}(C)$ the étale cohomology $H^m_{\text{ét}}(C \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)$.

1.2.9. As a module over $\mathfrak{h}(M)_{\mathbb{Q}}$, $H^1(X_1(M)(\mathbb{C}),\mathbb{Q})$ is a free module of rank 2. The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $H^1_{\text{ét}}(X_1(M)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1(X_1(M)(\mathbb{C}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ over $\mathfrak{h}(M)_{\mathbb{Q}_p}$ has the following properties (1) and (2).

(1) For a prime number ℓ which does not divide Mp, the action is unramified at ℓ, and we have

$$\det(1 - Fr_\ell^{-1}u) = 1 - T(\ell)u + \ell\langle \ell\rangle u^2 = 1 - \langle \ell\rangle T^*(\ell)u + \ell\langle \ell\rangle u^2.$$

Here Fr_ℓ is the arithmetic Frobenius of ℓ.

(2) The determinant of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is $\kappa(\sigma)^{-1}\langle \sigma\rangle^{-1}$. Here $\kappa : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the cyclotomic character, and $\langle \sigma\rangle$ denotes $\langle a\rangle$ for the $a \in (\mathbb{Z}/M\mathbb{Z})^\times$ such that $\sigma(\zeta_M) = \zeta_M^a$.

This is well known.

1.2.10. Assume $p|M$. In this paper, the ordinary component $H^1_{\text{ét}}(X_1(M))^{\text{ord}}$ (resp. $H^1_{\text{ét}}(Y_1(M))^{\text{ord}}$) of $H^1_{\text{ét}}(X_1(M))$ (resp. $H^1_{\text{ét}}(Y_1(M))$) is defined to be the part on which the action of $T^*(p)$ (not $T(p)$) is bijective. For $x \in H^1_{\text{ét}}(Y_1(M))$ (resp. $H^1_{\text{ét}}(Y_1(M))$), the ordinary component $x^{\text{ord}} \in H^1_{\text{ét}}(X_1(M))^{\text{ord}}$ (resp. $H^1_{\text{ét}}(Y_1(M))^{\text{ord}}$) of x is given as $x^{\text{ord}} = \lim_{n\to\infty} T^*(p)^{n!}x$.

We denote by $\mathfrak{h}(M)^{\text{ord}}_{\mathbb{Z}_p}$ (resp. $\mathfrak{H}(M)^{\text{ord}}_{\mathbb{Z}_p}$) the image of $\mathfrak{h}(M)_{\mathbb{Z}_p}$ (resp. $\mathfrak{H}(M)_{\mathbb{Z}_p}$) in the endomorphism ring of $H^1_{\text{ét}}(X_1(M))^{\text{ord}}$ (resp. $H^1_{\text{ét}}(Y_1(M))^{\text{ord}}$).

1.2.11. The ordinary part for $T(p)$ is defined also. In this paper, we mainly consider the ordinary part for $T^*(p)$.

1.3 Cusps of $X_1(M)$

We recall descriptions of cusps of $X_1(M)$ and the relations of Hecke operators at cusps.

1.3.1. Let P_M be the set of all pairs $(a,b) \in \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ such that a,b generate $\mathbb{Z}/M\mathbb{Z}$ as an ideal.

1.3.2. Over $\mathbb{C}$, we have

$$\begin{aligned}\{\text{cusp of } X_1(M)(\mathbb{C})\} &= \Gamma_1(M)\backslash \mathbb{P}^1(\mathbb{Q}) \overset{(1)}{=} \Gamma_1(M)\backslash(SL(2,\mathbb{Z})/\{\pm 1\})/\begin{pmatrix}1 & \mathbb{Z}\\ 0 & 1\end{pmatrix} \\ &\overset{(2)}{=} P_M/\sim\end{aligned}$$

where $/\sim$ is the quotient by the following equivalence relation $\sim$: $(a,b)\sim(a',b')$ if and only if $a' = \epsilon a$ and $b' \equiv \epsilon b \mod a$ with $\epsilon \in \{\pm 1\}$. Here the identification (1) sends the class of $g = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$ to the class of $a/c = g\infty \in \mathbb{P}^1(\mathbb{Q})$, and the identification (2) sends the class of $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$ to the class of $(c,d) \in P_M$.

In particular, $(0,1) \in P_M$ corresponds to the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$ and $(1,0) \in P_M$ corresponds to $0 \in \mathbb{P}^1(\mathbb{Q})$. The cusps of $X_1(M)(\mathbb{C})$ corresponding to $(0,b) \in P_M$ with $b \in (\mathbb{Z}/M\mathbb{Z})^\times$ are called ∞-cusps. But "the" ∞-cusp means the cusp $\infty \in \mathbb{P}^1(\mathbb{Q})$. The cusps of $X_1(M)(\mathbb{C})$ corresponding to $(a,b) \in P_M$ with $a \in (\mathbb{Z}/M\mathbb{Z})^\times$ (which depend only on a) are called 0-cusps.

1.3.3. All cusps of $X_1(M)$ are rational over $\mathbb{Q}(\zeta_M)$. More precisely, if $(a,b) \in P_M$ and if R denotes the positive divisor of M such that $(a) = (R)$ as an ideal of $\mathbb{Z}/M\mathbb{Z}$, then the residue field of the cusp of $X_1(M)$ determined by (a,b) is $\mathbb{Q}(\zeta_R) \cap \mathbb{R}$ if $M = R$ or if $M = 2R$, and is $\mathbb{Q}(\zeta_R)$ otherwise.

In particular, the residue fields of ∞-cusps (resp. 0-cusps) of $X_1(M)$ are $\mathbb{Q}(\zeta_M) \cap \mathbb{R}$ (resp. $\mathbb{Q}$).

1.3.4. An algebraic description of cusps is as follows. Let $M \geq 4$.

Let $\mathbb{Z}[1/M, \zeta_M][[q^{1/M}]][q^{-1}]$ be the formal power series ring in one variable $q^{1/M}$. ($q^{1/M}$ is the variable and q is understood as the M-th power of $q^{1/M}$.)

For $(a,b) \in P_M$, let

$$\infty_M(a,b) : \mathrm{Spec}(\mathbb{Z}[1/M, \zeta_M][[q^{1/M}]][q^{-1}]) \to Y_1(M) \otimes \mathbb{Z}[1/M]$$

be the morphism corresponding to the M-torsion point $q^{a/M}\zeta_M^b \bmod q^{\mathbb{Z}}$ of the q-Tate elliptic curve E_q over $\mathbb{Z}[1/M, \zeta_M][[q^{1/M}]][q^{-1}]$. This morphism $\infty_M(a,b)$ gives the cusp of $X_1(M) \otimes \mathbb{Q}(\zeta_M)$ corresponding to $(a,b) \in P_M$ (1.3.2).

For $c \in (\mathbb{Z}/M\mathbb{Z})^\times$, we have $\langle c\rangle \circ \infty_M(a,b) = \infty_M(ac, bc)$.

1.3.5. Let $S = \mathrm{Spec}(\mathbb{Z}[\zeta_M][[q^{1/M}]][1/q])$. Let ℓ be a prime number. We review the relation between cusps and dual Hecke operators $T^*(\ell)$ which we will use in section 5.

(1) Assume $\ell \not| M$. Then we have the following commutative diagram in which the left square is cartesian.

$$\begin{array}{ccccc} S & \leftarrow & S\coprod S' & \rightarrow & S\coprod S \\ \downarrow & & \downarrow & & \downarrow \\ Y_1(M) & \leftarrow & Y(1,M(\ell)) & \stackrel{\psi_\ell}{\rightarrow} & Y_1(M) \end{array}$$

Here $S' = \mathrm{Spec}(\mathbb{Z}[\zeta_M][[q^{1/M\ell}]][1/q])$. The left vertical arrow is $\infty_M(a,b)$. The right vertical arrow is $(\infty_M(a',b), \infty_M(a,b'))$ where $a', b' \in \mathbb{Z}/M\mathbb{Z}$, $a = \ell a'$, $b = \ell b'$. The left upper horizontal arrow is the canonical one, and the right horizontal arrow is (ψ_ℓ, α), where ψ_ℓ corresponds to the homomorphism $\mathbb{Z}[\zeta_M][[q^{1/M}]][1/q] \to \mathbb{Z}[\zeta_M][[q^{1/M}]][1/q]$; $q^{1/M} \mapsto q^{\ell/M}$ of degree ℓ and α corresponds to the isomorphism $\mathbb{Z}[\zeta_M][[q^{1/M}]][1/q] \to \mathbb{Z}[\zeta_M][[q^{1/M\ell}]][1/q]$ which sends $q^{1/M}$ to $q^{1/M\ell}$. The middle vertical arrow is (for the moduli interpretation of $Y(1,M(\ell))$ in 1.2.3)

$$((E_q, q^{a/M}\zeta_M^b, C), (E_q, q^{a/M}\zeta_M^b, C')),$$

where the ℓ-torsion part of C (resp. C') is generated by ζ_ℓ (resp. $q^{1/\ell}$).

Let $R \geq 1$ be the divisor of M such that $(a) = (R)$ as an ideal of $\mathbb{Z}/M\mathbb{Z}$.

(2) Assume $\ell | M$ but $\ell \not| R$. Assume $b = \ell b'$ for some $b' \in \mathbb{Z}/M\mathbb{Z}$. Then we have the following commutative diagram in which the left square is cartesian.

$$\begin{array}{ccccc} S & \leftarrow & S & = & S \\ \downarrow & & \downarrow & & \downarrow \\ Y_1(M) & \leftarrow & Y(1,M(\ell)) & \stackrel{\psi_\ell}{\rightarrow} & Y_1(M) \end{array}$$

where the left vertical arrow is $\infty_M(a,b)$, the right vertical arrow is $\infty_M(a,b')$, the middle vertical arrow is (for the moduli interpretation of $Y(1,M(\ell))$ in 1.2.3) $(E_{q^\ell}, q^{\ell a/M}\zeta_M^b, C)$ where C is the cyclic subgroup of E_{q^ℓ} of order $M\ell$ generated by $q^{a/M}\zeta_M^{b'} \bmod q^{\ell\mathbb{Z}}$, the left upper horizontal arrow is associated to $q^{1/M} \mapsto q^{\ell/M}$, the left lower horizontal arrow is the canonical projection, and the right lower horizontal arrow is ψ_ℓ.

(3) Assume $\ell | R$. Then we have the following commutative diagram in which the left square is cartesian.

$$\begin{array}{ccccc} S & \leftarrow & \coprod_{a'} S & \rightarrow & \coprod_{a'} S \\ \downarrow & & \downarrow & & \downarrow \\ Y_1(M) & \leftarrow & Y(1,M(\ell)) & \stackrel{\psi_\ell}{\rightarrow} & Y_1(M) \end{array}$$

where a' ranges over all elements of $\mathbb{Z}/M\mathbb{Z}$ such that $\ell a' = a$, the left vertical arrow is $\infty_M(a,b)$, the right vertical arrow at a' is $\infty_M(a',b)$, the middle vertical arrow at a' is $(E_q, q^{a/M}\zeta_M^b, C)$ where C is the cyclic subgroup of E_q of order $M\ell$ generated by $q^{a'/M}\zeta_{M\ell}^{\tilde{b}} \bmod q^{\mathbb{Z}}$ where $\tilde{b}$ is any lifting of b to $\mathbb{Z}/M\ell\mathbb{Z}$, the left upper horizontal arrow is the canonical one, the right upper horizontal arrow is associated to $q^{1/M} \to q^{\ell/M}$, the left lower horizontal arrow is the canonical projection, and the right lower horizontal arrow is ψ_ℓ.

In these (1)-(3), $T^*(\ell)$ is compatible with v_*u^* where u (resp. v) is the right (resp. left) upper horizontal arrow of the diagram.

1.4 Another model $X_1'(M)$

1.4.1. Let $X_1'(M)$ be the quotient of $X(L)$ $(M|L, L \geq 3)$ by the subgroup

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}/L\mathbb{Z}) \mid a \equiv 1 \bmod M, c \equiv 0 \bmod M\}.$$

(Recall that the corresponding condition for $X_1(M)$ was $c \equiv 0 \bmod M$ and $d \equiv 1 \bmod M$.) Define similarly an open set $Y_1'(M)$ of $X_1'(M)$ as a quotient of $Y(L)$.

For $M \geq 4$, $Y_1'(M) \otimes_{\mathbb{Z}} \mathbb{Z}[1/M]$ is the moduli space over $\mathbb{Z}[1/M]$ of an elliptic curve E with an injective homomorphism $\mathbb{Z}/M\mathbb{Z}(1) \to E$.

1.4.2. The schemes $X_1'(M)$ and $X_1(M)$ are related by the two isomorphisms

$$v_M : X_1'(M) \otimes \mathbb{Z}[1/M, \zeta_M] \xrightarrow{\cong} X_1(M) \otimes \mathbb{Z}[1/M, \zeta_M], \quad w_M : X_1'(M) \cong X_1(M).$$

The isomorphism v_M is given by the identifications

$$\begin{aligned} & X_1'(M) \otimes \mathbb{Z}[1/M, \zeta_M] \\ &= \text{the quotient of } X(M) \otimes \mathbb{Z}[1/M] \text{ by } \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}/L\mathbb{Z}) \mid c \equiv 0, a \equiv d \equiv 1 \bmod M\} \\ &= X_1(M) \otimes \mathbb{Z}[1/M, \zeta_M]. \end{aligned}$$

The induced isomorphism $Y_1'(M) \otimes \mathbb{Z}[1/M, \zeta_M] \cong Y_1(M) \otimes \mathbb{Z}[1/M, \zeta_M]$ corresponds to the isomorphism of moduli functors given by $(E, \alpha) \mapsto (E, \beta)$, where E is an elliptic curve, $\alpha : \mathbb{Z}/M\mathbb{Z}(1) \to E$ and $\beta : \mathbb{Z}/M\mathbb{Z} \to E$ are injective homomorphisms, and $\alpha(\zeta_M^a) = \beta(a)$ for $a \in \mathbb{Z}/M\mathbb{Z}$.

The morphism $v_M : Y_1'(M)(\mathbb{C}) \xrightarrow{\cong} Y_1(M)(\mathbb{C})$ is regarded as the identity map of $\Gamma_1(M) \backslash \mathcal{H}$.

The isomorphism $w_M : Y_1'(M) \xrightarrow{\cong} Y_1(M)$ is the unique isomorphism which gives over $\mathbb{Z}[1/M]$ the isomorphism of moduli functors $(E, \alpha) \to (E/C, \beta)$ ($\alpha : \mathbb{Z}/M\mathbb{Z}(1) \to E$, $\beta : \mathbb{Z}/M\mathbb{Z} \to E$), where C is the image of α, and β sends 1 to the image of an M-division point e of E such that the Weil pairing sends $(\alpha(\zeta_M), e)$ to ζ_M.

Via $\mathcal{H} \to X_1'(M)(\mathbb{C})$ and $\mathcal{H} \to X_1(M)(\mathbb{C})$, w_M corresponds to $\mathcal{H} \to \mathcal{H}$; $\tau \mapsto -1/M\tau$.

This w_M is called the Atkin-Lehner involution.

1.4.3. Via the isomorphism w_M, $T(n)$ (resp. $T^*(n)$) on $X_1'(M)$ $(n \geq 1)$ corresponds to $T^*(n)$ (resp. $T(n)$) on $X_1(M)$, and $\langle a \rangle$ $(a \in (\mathbb{Z}/M\mathbb{Z})^\times)$ on $X_1'(M)$ corresponds to $\langle a \rangle^{-1}$ on $X_1(M)$.

1.4.4. Note that we have two isomorphisms

$$v_M, w_M : H^1_{\text{ét}}(X_1(M)) \xrightarrow{\cong} H^1_{\text{ét}}(X_1'(M)).$$

This isomorphism v_M is regarded as the identity map of $H^1(\Gamma_1(M) \backslash \mathcal{H}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Lemma 1.4.5. *(1) The isomorphism $v_M : H^1(X_1(M)) \xrightarrow{\cong} H^1(X'_1(M))$ preserves the actions of Hecke operators, dual Hecke operators and diamond operators, but changes the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *as follows. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and for $x \in H^1(X_1(M))$, we have*

$$v_M(\sigma x) = \langle \sigma \rangle \sigma v_M(x)$$

where $\langle \sigma \rangle$ is as in 1.2.9.

(2) The isomorphism $w_M : H^1(X_1(M)) \xrightarrow{\cong} H^1(X'_1(M))$ preserves the actions of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$*, but exchanges the action of $T(n)$ on one side with the action of $T^*(n)$ on the other, and the action of $\langle a \rangle$ ($a \in (\mathbb{Z}/M\mathbb{Z})^\times$) on one side with the action of $\langle a \rangle^{-1}$ on the other.*

Proof. (1) The statement about Hecke (resp, dual Hecke) operators and diamond operators is clear. We consider the Galois action. Let σ and a be as above. The automorphism $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ on $X(M)$ induces $1 \otimes \sigma$ on $X'_1(M) \otimes \mathbb{Q}(\zeta_M)$ but induces $\langle a \rangle \otimes \sigma$ on $X_1(M) \otimes \mathbb{Q}(\zeta_M)$. Hence Via $v_M : X'_1(M) \otimes \mathbb{Q}(\zeta_M) \cong X_1(M) \otimes \mathbb{Q}(\zeta_M)$, $1 \otimes \sigma$ on $X'_1(M) \otimes \mathbb{Q}(\zeta_M)$ corresponds to $\langle a \rangle \otimes \sigma$ on $X_1(M) \otimes \mathbb{Q}(\zeta_M)$. The statement about the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ follows from this.

(2) follows from 1.4.3. □

1.5 p-adic relation of modular forms of different weights

Modular forms which appear in this paper are mainly of weight 2. However in several places in sections 4, 7, 8, we use the fact that modular forms weight 2 are related p-adically to modular forms of higher weights. We review this in this section 1.5. We also review ordinary Λ-adic modular forms.

1.5.1. Fix an integer $N \geq 1$ which is prime to p. Let

$$H := \varprojlim_r H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}}, \quad \tilde{H} := \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}},$$

$$\mathfrak{h} = \varprojlim_r \mathfrak{h}(Np^r)^{\text{ord}}_{\mathbb{Z}_p}, \quad \mathfrak{H} = \varprojlim_r \mathfrak{H}(Np^r)^{\text{ord}}_{\mathbb{Z}_p}.$$

Then $\mathfrak{h}$ acts on H and $\mathfrak{H}$ acts on $\tilde{H}$. We have a surjective ring homomorphism $\mathfrak{H} \to \mathfrak{h}$ defined by restricting the actions on $\tilde{H}$ to H.

1.5.2. Let

$$\Lambda := \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times]].$$

We have canonical ring homomorphisms

$$\Lambda \to \mathfrak{h}, \quad \Lambda \to \mathfrak{H}$$

which send the group element $[a]$ of Λ ($a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$) to the diamond operator $\langle a \rangle$.

1.5.3. For a commutative ring R, let $Q(R)$ be the total quotient ring of R defined by

$$Q(R) = \{a/b \mid a, b \in R, b \text{ is a non-zero-divisor of } R\}.$$

1.5.4. Regard $1+p\mathbb{Z}_p$ as a subgroup of $\mathbb{Z}_p^\times \times \{1\} \subset \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and regard $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ as a subring of Λ. Then the homomorphisms $\mathbb{Z}_p[[1+p\mathbb{Z}_p]] \to \mathfrak{h}$ and $\mathbb{Z}_p[[1+p\mathbb{Z}_p]] \to \mathfrak{H}$ are finite flat ([19]). From this we have

$$Q(\mathfrak{h}) \cong \mathfrak{h} \otimes_\Lambda Q(\Lambda), \quad Q(\mathfrak{H}) \cong \mathfrak{H} \otimes_\Lambda Q(\Lambda).$$

Furthermore, H and $\tilde{H}$ are finitely generated free $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$-modules.

Proposition 1.5.5. *(Hida [17] section 1.) For any $r \geq 1$, we have isomorphisms*

$$\mathfrak{h} \otimes_\Lambda \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] \xrightarrow{\cong} \mathfrak{h}(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p}, \quad \mathfrak{H} \otimes_\Lambda \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] \xrightarrow{\cong} \mathfrak{H}(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p}.$$

1.5.6. Let $k \geq 2$, $M \geq 4$. Let

$$V_k(X_1(M))_\mathbb{Z} := H^1(X_1(M)(\mathbb{C}), j_*\mathcal{F}_{\mathbb{Z},k-2}) \overset{\subset}{\to} V_k(Y_1(M))_\mathbb{Z} := H^1(Y_1(M)(\mathbb{C}), \mathcal{F}_{\mathbb{Z},k-2}),$$

where $\mathcal{F}_{\mathbb{Z},r}$ is the r-th symmetric power $\mathrm{Sym}^r(R^1f_*\mathbb{Z})$ $(r \geq 0)$ with $f : E \to Y_1(M) \otimes \mathbb{Z}[1/M]$ the universal elliptic curve, and $j : Y_1(M) \to X_1(M)$ is the inclusion map. These are the weight k-versions of $V_2(X_1(M))_\mathbb{Z} = H^1(X_1(M)(\mathbb{C}), \mathbb{Z})$, $V_2(Y_1(M))_\mathbb{Z} = H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$.

For any commutative ring R, let $V_k(X_1(M))_R := V_k(X_1(M))_\mathbb{Z} \otimes_\mathbb{Z} R$, $V_k(Y_1(M))_R := V_k(Y_1(M))_\mathbb{Z} \otimes_\mathbb{Z} R$.

Then $V_k(X_1(M))_{\mathbb{Z}_p}$ and $V_k(Y_1(M))_{\mathbb{Z}_p}$ are understood as étale cohomology groups, and hence $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on them.

Let $S_k(M)_\mathbb{Q}$ (resp. $M_k(M)_\mathbb{Q}$) be the space of cusp forms (resp. modular forms) on $X_1(M)_\mathbb{Q}$ of weight k. It is a subspace of $\Gamma(Y, \Omega^1_{Y/\mathbb{Q}} \otimes_{\mathcal{O}_Y} \mathrm{coLie}(E)^{\otimes(k-2)})$ for $Y = Y_1(M)_\mathbb{Q}$ where $\mathrm{coLie}(E)$ is the $\mathcal{O}_Y$-dual of the Lie algebra of the universal elliptic curve E over Y.

For a commutative ring R over $\mathbb{Q}$, let $S_k(M)_R = S_k(M)_\mathbb{Q} \otimes_\mathbb{Q} R$, $M_k(M)_R = M_k(M)_\mathbb{Q} \otimes_\mathbb{Q} R$. We have the period map

$$\mathrm{per} : M_k(M)_\mathbb{C} \to V_k(Y_1(M))_\mathbb{C}$$

which induces

$$\mathrm{per} : S_k(M)_\mathbb{C} \to V_k(X_1(M))_\mathbb{C}.$$

These maps are injective and induce the Eichler-Shimura isomorphisms

$$S_k(M)_\mathbb{C} \oplus \overline{S_k(M)_\mathbb{C}} \xrightarrow{\cong} V_k(X_1(M))_\mathbb{C} \quad M_k(M)_\mathbb{C} \oplus \overline{S_k(M)_\mathbb{C}} \xrightarrow{\cong} V_k(Y_1(M))_\mathbb{C},$$

where $\overline{(-)}$ denotes the complex conjugation.

Like in the case $k = 2$, Hecke operators $T(n)$ $(n \geq 1)$, dual Hecke operators $T^*(n)$ $(n \geq 1)$, and diamond operators $\langle n \rangle$ $(n \geq 1, (n, M) = 1)$ act on $V_k(X_1(M))_\mathbb{Z}$, $V_k(Y_1(M))_\mathbb{Z}$, $S_k(M)_\mathbb{Q}$, and $M_k(M)_\mathbb{Q}$, and these actions are compatible with the Eichler-Shimura isomorphisms.

The different weights are connected p-adically when we go to the inverse limit of level Np^r $(r \to \infty)$ as follows.

1.5.7. (Shimura [53], Hida [18].) We have isomorphisms

$$\pi_k : H = \varprojlim_r V_2(X_1(Np^r))^{\mathrm{ord}}_{\mathbb{Z}_p} \xrightarrow{\cong} \varprojlim_r V_k(X_1(Np^r))^{\mathrm{ord}}_{\mathbb{Z}_p}(k-2),$$

$$\pi_k : \tilde{H} = \varprojlim_r V_2(Y_1(Np^r))^{\mathrm{ord}}_{\mathbb{Z}_p} \xrightarrow{\cong} \varprojlim_r V_k(Y_1(Np^r))^{\mathrm{ord}}_{\mathbb{Z}_p}(k-2)\ ;$$

$$x \mapsto xe_2^{k-2} \quad (\text{we regard } e_2^{k-2} \in \varprojlim_r H^0(X_1(Np^r), j_*(\mathcal{F}_{k-2}/p^r\mathcal{F}_{k-2})(k-2))).$$

This isomorphism π_k preserves the actions of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Furthermore,

$$\pi_k \circ T^*(n) = T^*(n) \circ \pi_k \quad (n \geq 1),$$

$$\pi_k \circ \langle (a,b) \rangle = a^{2-k} \langle (a,b) \rangle \circ \pi_k \quad (a \in \mathbb{Z}_p^\times, b \in (\mathbb{Z}/N\mathbb{Z})^\times).$$

Proposition 1.5.8. *(Hida [17] section 1, Ohta [40] Proposition 2.3.3.) For any $k \geq 2$ and $r \geq 1$, the isomorphisms π_k in 1.5.7 induce isomorphisms*

$$H \otimes_\Lambda \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] \xrightarrow{\cong} V_k(X_1(Np^r))^{\mathrm{ord}}_{\mathbb{Z}_p}(k-2), \quad \tilde{H} \otimes_\Lambda \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] \xrightarrow{\cong} V_k(Y_1(Np^r))^{\mathrm{ord}}_{\mathbb{Z}_p}(k-2).$$

Here $\Lambda \to \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ is the composition

$$\Lambda \xrightarrow{tw_{2-k}} \Lambda \to \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$$

in which the first arrow sends the group element $[(a,b)]$ ($a \in \mathbb{Z}_p^\times$, $b \in (\mathbb{Z}/N\mathbb{Z})^\times$) of Λ (this element is identified with diamond operator) to $a^{2-k}[(a,b)]$ and the second arrow is the canonical projection.

1.5.9. Next we consider modular forms. Assume $M \geq 4$.

Following Ohta [38] (2.1.1), we define the Atkin-Lehner operator $w_M : M_k(M)_{\mathbb{C}} \to M_k(M)_{\mathbb{C}}$ by

$$(w_M(f))(\tau) = (-1)^k M^{1-k} \tau^{-k} f(-1/M\tau) \quad (f \in M_k(M)_{\mathbb{C}},\ \tau \in \mathcal{H}).$$

In the case $k = 2$, via the Eichler-Shimura isomorphism, this w_M is compatible with $w_M : H^1(\Gamma_1(M) \setminus \mathcal{H}, \mathbb{Z}) \to H^1(\Gamma_1(M) \setminus \mathcal{H}, \mathbb{Z})$ induced by the map $\Gamma_1(M) \setminus \mathcal{H} \to \Gamma_1(M) \setminus \mathcal{H}\ ;\ \tau \mapsto -1/M\tau$. For a general k, via the Eichler-Shimura isomorphism, this w_M is compatible with the homomorphism

$$w_M : V_k(Y_1(M))_{\mathbb{Q}} \to V_k(Y_1(M))_{\mathbb{Q}}$$

defined as follows. Let E be the universal elliptic curve $\Gamma_1(M) \backslash ((\mathcal{H} \times \mathbb{C})/\sim)$ over $\Gamma_1(M) \setminus \mathcal{H}$, where $\sim$ is the equivalence relation defined by $(\tau, z) \sim (\tau', z') \Leftrightarrow \tau = \tau', z \equiv z'$ mod $\mathbb{Z}\tau + \mathbb{Z}$ and $\Gamma_1(M)$ acts on $\mathcal{H} \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, z) \mapsto (\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d})$$

preserving the equivalence. Then the map $E \to E$ induced by $\mathcal{H} \times \mathbb{C} \to \mathcal{H} \times \mathbb{C}\ ;\ (\tau, z) \mapsto (-1/M\tau, z/\tau)$ induces a homomorphism $w'_M : V_k(Y_1(M))_{\mathbb{Z}} \to V_k(Y_1(M))_{\mathbb{Z}}$. We define $w_M = (-1)^k M^{2-k} w'_M$.

1.5.10. Let

$$S_k(M)_{\mathbb{Z}} = \{f \in S_k(M)_{\mathbb{Q}} \mid a_n(w_M(f)) \in \mathbb{Z} \text{ for all } n \geq 1\},$$
$$M_k(M)_{\mathbb{Z}} = \{f \in M_k(M)_{\mathbb{Q}} \mid a_n(w_M(f)) \in \mathbb{Z} \text{ for all } n \geq 1\},$$

where a_n is the n-th coefficient of the q-expansion (at the ∞-cusp). Then $S_k(M)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = S_k(M)_{\mathbb{Q}}$ (resp. $M_k(M)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = M_k(M)_{\mathbb{Q}}$), and $S_k(M)_{\mathbb{Z}}$ (resp. $M_k(M)_{\mathbb{Z}}$) is stable in $S_k(M)_{\mathbb{Q}}$ (resp. $M_k(M)_{\mathbb{Q}}$) under the actions of $T^*(n)$ ($n \geq 1$) and $\langle a \rangle$ ($a \in (\mathbb{Z}/M\mathbb{Z})^\times$) (Hida [17], §1).

1.5.11. (Hida [19] section 7, Wiles [60], Ohta [38] section 2.) For $r \geq 1$, consider the injective homomorphisms

$$S_k(Np^r)_{\mathbb{Z}} \to \mathbb{Z}[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]]q \subset \mathbb{Z}[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]],$$
$$M_k(Np^r)_{\mathbb{Z}} \to \mathbb{Z}[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]] + \mathbb{Q}[(\mathbb{Z}/Np^r\mathbb{Z})^\times] \subset \mathbb{Q}[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]];$$
$$f \mapsto \sum_{a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times} (q\text{-expansion of } \langle a \rangle^{-1} T(p)^r w_{Np^r}(f)) \cdot [a].$$

For $r \geq 1$, the trace map $S_k(Np^{r+1})_{\mathbb{Q}} \to S_k(Np^r)_{\mathbb{Q}}$ (resp. $M_k(Np^{r+1})_{\mathbb{Q}} \to M_k(Np^r)_{\mathbb{Q}}$) sends $S_k(Np^{r+1})_{\mathbb{Z}}$ to $S_k(Np^r)_{\mathbb{Z}}$ (resp. $M_k(Np^{r+1})_{\mathbb{Z}}$ to $M_k(Np^r)_{\mathbb{Z}}$). Via the above homomorphism, this trace map is compatible with the canonical projection $\mathbb{Q}[(\mathbb{Z}/Np^{r+1}\mathbb{Z})^\times][[q]] \to \mathbb{Q}[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]]$. When we go to the inverse limit of the trace map, the image of $\varprojlim_r M_k(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p} \to \varprojlim_r \mathbb{Q}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]]$ is contained in $\Lambda[[q]] + Q(\Lambda) = \{\sum_{n=0}^\infty a_n q^n \mid a_n \in \Lambda \text{ for } n \geq 1, a_0 \in Q(\Lambda)\}$. Let $S_{k,\Lambda}$ be the image of the injection $\varprojlim_r S_k(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p} \to \Lambda[[q]]q \subset \Lambda[[q]]$ and let $M_{k,\Lambda}$ be the image of the injection $\varprojlim_r M_k(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p} \to Q(\Lambda)[[q]]$.

Then we have

Fact. For $r \in \mathbb{Z}$, let $tw_r : Q(\Lambda)[[q]] \xrightarrow{\cong} Q(\Lambda)[[q]]$ be the ring isomorphism induced from the ring isomorphism $tw_r : \Lambda \xrightarrow{\cong} \Lambda$; $[(a,b)] \mapsto a^r[(a,b)]$ ($a \in \mathbb{Z}_p^\times$, $b \in (\mathbb{Z}/N\mathbb{Z})^\times$). Then $tw_{k-2} : Q(\Lambda)[[q]] \to Q(\Lambda)[[q]]$ induces isomorphisms $S_{2,\Lambda} \xrightarrow{\cong} S_{k,\Lambda}$ and $M_{2,\Lambda} \xrightarrow{\cong} M_{k,\Lambda}$. Furthermore, the composition

$$\pi_k : \varprojlim_r M_2(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p} \xrightarrow{\cong} M_{2,\Lambda} \xrightarrow{tw_{k-2}} M_{k,\Lambda} \xrightarrow{\cong} \varprojlim_r M_k(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p}$$

satisfies

$$\pi_k \circ T^*(n) = T^*(n) \circ \pi_k \quad (n \geq 1),$$
$$\pi_k \circ \langle (a,b) \rangle = a^{2-k} \langle (a,b) \rangle \circ \pi_k \quad (a \in \mathbb{Z}_p^\times, b \in (\mathbb{Z}/N\mathbb{Z})^\times).$$

We denote $S_{2,\Lambda}$ (resp. $M_{2,\Lambda}$) simply by S_Λ (resp. M_Λ), and call an element of this space an ordinary Λ-adic cusp (resp. modular) form.

1.5.12. We define an $\mathfrak{h}$-module structure of S_Λ and an $\mathfrak{H}$-module structure of M_Λ by identifying S_Λ with the inverse limit of $S_2(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p}$ and by identifying M_Λ with the inverse limit of $M_2(Np^r)^{\mathrm{ord}}_{\mathbb{Z}_p}$. The actions of $T^*(n) \in \mathfrak{H}$ and $\langle a \rangle \in \mathfrak{H}$ ($a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$) on M_Λ defined by this structure coincide with the usual actions of $T(n)$ and $\langle a \rangle^{-1}$ on M_Λ, respectively. This is because we use w_{Np^r} in the definition of $\varprojlim_r M_k(Np^r)_{\mathbb{Z}_p} \to Q(\Lambda)[[q]]$, which exchanges $T^*(n)$ and $T(n)$. We have the similar fact for the action of $\mathfrak{h}$ on S_Λ.

1.6 Λ-adic Poincaré duality

We describe the Λ-adic Poincaré duality of Ohta defined in section 4 of [38] (see 1.6.3 below) following [51] section 4, and define a variant (1.6.6) of it.

Let $N \geq 1$ be prime to p, and let $\Lambda := \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$.

1.6.1. Let $r \geq 1$. Let

$$(\ , \ ) : H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Z}) \times H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Z}) \to \mathbb{Z}$$

be the usual pairing of Poincaré duality.

1.6.2. Consider the pairing

$$(\ , \ )_{\Lambda, Np^r} : H^1_{\text{ét}}(X_1(Np^r)) \times H^1_{\text{ét}}(X_1(Np^r)) \to \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times],$$
$$(x, y) \mapsto \sum_{a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times} (x, \langle a \rangle v_{Np^r}^{-1} w_{Np^r} T^*(p)^r y) \cdot [a].$$

Here w_{Np^r} and v_{Np^r} are as in section 1.4.

1.6.3. The pairings in 1.6.2 for $r \geq 1$ induce a pairing

$$H \times H \to \Lambda \ ; \ (x, y) \mapsto ((x_r, y_r)_{\Lambda, Np^r})_{r \geq 1},$$

where x_r (resp. y_r) denotes the image of x (resp. y) in $H^1_{\text{ét}}(X_1(Np^r))$.

1.6.4. The pairing $(\ , \ )_\Lambda : H \times H \to \Lambda$ has the following properties.

(1) For any $a \in \mathfrak{h}$, we have

$$(ax, y)_\Lambda = (x, ay)_\Lambda.$$

(2) For any $a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, we have

$$(\langle a \rangle x, y)_\Lambda = (x, \langle a \rangle y)_\Lambda = [a](x, y)_\Lambda.$$

(3) For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we have

$$(\sigma x, \sigma y)_\Lambda = \kappa(\sigma)^{-1} \langle \sigma \rangle^{-1} (x, y)_\Lambda.$$

(4) We have an isomorphism

$$H \xrightarrow{\cong} \mathrm{Hom}_\Lambda(H, \Lambda) \ ; \ x \mapsto (y \mapsto (x, y)_\Lambda).$$

1.6.5. Let

$$(-, -)_{\Lambda, 0} : H \times H \to \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$$

be the composition

$$H \times H \to \Lambda \to \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$$

where the second arrow sends $[a]$ ($a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$) to $[a]$ if $a \in (1 + p\mathbb{Z}_p) \times \{1\} \subset \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and to 0 otherwise.

Then $(\ , \ )_{\Lambda, 0}$ is a perfect pairing of finitely generated free $\mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$-modules.

If $p \nmid \varphi(N)$, then the pairing $(\ , \ ) : H \times H \to \Lambda$ itself is a perfect pairing of finitely generated projective Λ-modules.

Proposition 1.6.6. *(1) We have a pairing*

$$((-,-))_\Lambda : H \times H \longrightarrow S_\Lambda$$

defined by

$$((x,y))_\Lambda = \sum_{n=1}^{\infty} (x, T^*(n)y)_\Lambda \cdot q^n.$$

(2) This pairing satisfies

$$((ax,y))_\Lambda = ((x,ay))_\Lambda = a((x,y))_\Lambda, \quad ((x,y))_\Lambda = -((y,x))_\Lambda$$

for $x, y \in H$ and $a \in \mathfrak{h}$. Here and in the following (3), the $\mathfrak{h}$-module structure of S_Λ is as in 1.5.12.

(3) This pairing is perfect in the following sense. We have an isomorphism

$$H \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{h}}(H, S_\Lambda) \ ; \ x \mapsto (y \mapsto ((x,y))_\Lambda).$$

Proof. By the duality theory of Hida [19] section 7.3, Theorem 5, we have an isomorphism

$$S_\Lambda \xrightarrow{\cong} \mathrm{Hom}_\Lambda(\mathfrak{h}, \Lambda) \ ; \ f \mapsto (h \mapsto a_1(hf)).$$

The inverse map is given by $b \mapsto \sum_{n=1}^{\infty} b(T^*(n))q^n \in S_\Lambda$ ($b \in \mathrm{Hom}_\Lambda(\mathfrak{h}, \Lambda)$). For $x, y \in H$, we have an element of $\mathrm{Hom}_\Lambda(\mathfrak{h}, \Lambda)$ defined by $a \mapsto (x, ay)_\Lambda$. By the above isomorphism of Hida, we have the pairing in (1).

(2) and (3) follow from the properties of the pairing of $(-,-)_\Lambda$ in 1.6.4. □

1.6.7. Let

$$\tilde{H}_c = \varprojlim_r H_c^1(Y_1(Np^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)^{\mathrm{ord}}.$$

Here H_c^1 is the cohomology with compact supports.

We have similarly a pairing

$$(-,-)_\Lambda : \tilde{H} \times \tilde{H}_c \to \Lambda\,.$$

This pairing also has the properties corresponding to (1)–(3) in 1.6.4. By using the duality $M_\Lambda \xrightarrow{\cong} \mathrm{Hom}_\Lambda(\mathfrak{H}, \Lambda)$; $f \mapsto (h \mapsto a_1(hf))$ of Hida ([19] section 7.3, Theorem 5), we have a pairing

$$((-,-))_\Lambda : \tilde{H} \times \tilde{H}_c \to M_\Lambda$$

characterized by the property $a_1(h \cdot ((x,y))_\Lambda) = (x, hy)_\Lambda$ ($h \in \mathfrak{H}_\theta$, $x \in \tilde{H}$, $y \in \tilde{H}_c$). This pairing also has the properties corresponding to (2) and (3) in Proposition 1.6.6.

1.6.8. We consider the relation between weight 2 and weight $k \geq 2$. Let

$$(\ ,\ ) \ : \ V_k(X_1(Np^r))_{\mathbb{Z}} \times V_k(X_1(Np^r))_{\mathbb{Z}} \to \mathbb{Q}$$

be the pairing induced by the pairing

$$\mathcal{F}_{\mathbb{Z},k-2} \times \mathcal{F}_{\mathbb{Z},k-2} \to \mathbb{Q} \ ; \ (x_1 \ldots x_{k-2}, y_1 \ldots y_{k-2}) \mapsto (k-2)!^{-1} \sum_\sigma (x_1 \cup y_{\sigma(1)}) \ldots (x_{k-2} \cup y_{\sigma(k-2)})$$

$(x_i, y_i \in R^1 f_*\mathbb{Z})$ with the notation as in 1.5.6, where σ ranges over all bijections $\{1, \dots, k-2\} \to \{1, \dots, k-2\}$ and $\cup$ is the cup product $R^1 f_*\mathbb{Z} \times R^1 f_*\mathbb{Z} \to R^2 f_*\mathbb{Z} \cong \mathbb{Z}$.

For $r \geq 1$ and $k \geq 2$, define

$$(\ , \ )_{k,r} : V_k(X_1(Np^r))_{\mathbb{Z}} \times V_k(X_1(Np^r))_{\mathbb{Z}} \to \mathbb{Q}[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$$

$$(x, y) \mapsto \sum_{a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times} (x, \langle a \rangle v_{Np^r}^{-1} w_{Np^r} T^*(p)^r y) \cdot [a].$$

Let

$$((\ , \ ))_{k,r} : V_k(X_1(Np^r))_{\mathbb{Z}} \times V_k(X_1(Np^r))_{\mathbb{Z}} \to S_k(Np^r)_{\mathbb{Q}}$$

be the pairing which sends (x, y) to $f \in S_k(Np^r)_{\mathbb{Q}}$ such that in the map $S_k(Np^r) \to \mathbb{Q}[(\mathbb{Z}/Np^r\mathbb{Z})^\times][[q]]$ in 1.5.11, f corresponds to $\sum_{n=1}^{\infty} (x, T^*(n)y)_{k,r} q^n$. The unique existence of such f follows from duality of Hida ([19] section 5.3)

$$S_k(Np^r)_{\mathbb{Q}} \cong \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}_k(Np^r)_{\mathbb{Q}}, \mathbb{Q}).$$

Here $\mathfrak{h}_k(Np^r)_{\mathbb{Q}}$ is the $\mathbb{Q}$-algebra of Hecke operators acting on $S_k(Np^r)$, and a homomorphism of $\mathbb{Q}$-modules $h : \mathfrak{h}_k(Np^r)_{\mathbb{Q}} \to \mathbb{Q}$ corresponds to $\sum_{n=1}^{\infty} h(T^*(n)) q^n \in S_k(Np^r)_{\mathbb{Q}}$.

1.6.9. By Ohta [38] Theorem 4.2.5, we have

$$(\pi_{k,r}(x), \pi_{k,r}(y))_{r,k} = \pi_{k,r}((x, y)_\Lambda) \quad \text{for } x, y \in H$$

where the two $\pi_{k,r}$ on the left hand side denote the homomorphism $H \to V_k((X_1(Np^r))_{\mathbb{Z}_p}(k-2) \cong V_k(X_1(Np^r))_{\mathbb{Z}_p}$, and $\pi_{k,r}$ on the right hand side denotes the composition $\Lambda \xrightarrow{\cong} \Lambda \to \mathbb{Q}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ in which the first arrow sends the group element $[a]$ $(a = (b, c) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times)$ to $b^{k-2}[a]$ and the second arrow is the canonical projection.

From this, we have

Proposition 1.6.10.

$$((\pi_{k,r}(x), \pi_{k,r}(y)))_{r,k} = \pi_{k,r}(((x, y))_\Lambda) \quad \textit{for } x, y \in H$$

where the two $\pi_{k,r}$ on the left hand side denote the homomorphism

$$H \to V_k(X_1(Np^r))_{\mathbb{Z}_p}(k-2) \cong V_k(X_1(Np^r))_{\mathbb{Z}_p}$$

induced by π_k in 1.5.7, and $\pi_{k,r}$ on the right hand side denotes the composition $S_\Lambda \xrightarrow{\cong} S_{k,\Lambda} \cong \varprojlim_n S_k(Np^n)_{\mathbb{Z}_p} \to S_k(Np^r)_{\mathbb{Z}_p}$ where the first arrow is tw_{k-2} (1.5.11) and the last arrow is the canonical projection.

1.7 Λ-adic Eichler-Shimura isomorphism

We review the theory Λ-adic Eichler-Shimura isomorphisms obtained in Ohta [38], [40]. Our formulation is slightly different from that of Ohta: We use $X_1(M)$ though he used $X_1'(M)$, so the Galois action on the étale cohomology H^1 is changed. We slightly improve his theory by using the functor D defined in 1.7.4 (see 1.7.12).

Let $M \geq 4$ and assume p divides M.

1.7.1. For a smooth proper curve C over $\mathbb{Q}_p$, we have an isomorphism

$$\Gamma(C, \Omega^1_C) \cong D^0_{dR}(H^1(C \otimes \bar{\mathbb{Q}}_p, \mathbb{Q}_p)(1))$$

where D_{dR} is the de Rham functor of Fontaine with filtration $(D^i_{dR})_i$ ([8]). If we take $X_1(M) \otimes \mathbb{Q}_p$ as C, $\Gamma(C, \Omega^1_C)$ is identified with $S_2(M)_{\mathbb{Q}_p} = S_2(M)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Hence we have an isomorphism

$$S_2(M)_{\mathbb{Q}_p} \cong D^0_{dR}(H^1_{\text{ét}}(X_1(M))(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

For a smooth proper curve C over $\mathbb{Q}_p$ and for a dense open subscheme C° of C, if we denote $C - C^\circ$ by Σ, we have an isomorphism

$$\Gamma(C, \Omega^1_C(\log \Sigma)) \cong D^0_{dR}(H^1(C^\circ \otimes \bar{\mathbb{Q}}_p, \mathbb{Q}_p)(1)).$$

If we take $X_1(M) \otimes \mathbb{Q}_p$ as C and $Y_1(M) \otimes \mathbb{Q}_p$ as C°, $\Gamma(C, \Omega^1_C(\log(\Sigma)))$ is identified with $M_2(M)_{\mathbb{Q}_p}$. Hence we have an isomorphism

$$M_2(M)_{\mathbb{Q}_p} \cong D^0_{dR}(H^1_{\text{ét}}(X_1(M))(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

These induce isomorphisms of the ordinary parts

$$S_2(M)^{\text{ord}}_{\mathbb{Q}_p} \cong D^0_{dR}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p),$$

$$M_2(M)^{\text{ord}}_{\mathbb{Q}_p} \cong D^0_{dR}(H^1_{\text{ét}}(Y_1(M))^{\text{ord}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

1.7.2. We have exact sequences of finitely generated free $\mathbb{Z}_p$-modules endowed with actions of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and Hecke operators

$$0 \to H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub}} \to H^1_{\text{ét}}(X_1(M))^{\text{ord}} \to H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}} \to 0$$

$$0 \to H^1_{\text{ét}}(Y_1(M))^{\text{ord}}_{\text{sub}} \to H^1_{\text{ét}}(Y_1(M))^{\text{ord}} \to H^1_{\text{ét}}(Y_1(M))^{\text{ord}}_{\text{quo}} \to 0$$

characterized by the following properties: The actions of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1)$ and $H^1_{\text{ét}}(Y_1(M))^{\text{ord}}_{\text{quo}}(1)$ are unramified, and for an element σ of the inertia subgroup of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, the actions of σ on $H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub}}$ and $H^1_{\text{ét}}(Y_1(M))^{\text{ord}}_{\text{sub}}$ coincide with the actions of $\langle\sigma\rangle^{-1}$ (1.2.9).

The canonical map $H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub}} \to H^1_{\text{ét}}(Y_1(M))^{\text{ord}}_{\text{sub}}$ is an isomorphism.

We have isomorphisms

$$D^0_{dR}(H^1_{\text{ét}}(X_1(M)))^{\text{ord}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \xrightarrow{\cong} D_{dR}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p),$$

$$D^0_{dR}(H^1_{\text{ét}}(Y_1(M))^{\text{ord}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \xrightarrow{\cong} D_{dR}(H^1_{\text{ét}}(Y_1(M))^{\text{ord}}_{\text{quo}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

1.7.3. For a topological abelian group A such that $A \xrightarrow{\cong} \varprojlim_{A'} A/A'$ where A' ranges over all open subgroups of A, and for a topological abelian group B having the same property as A, we define the topological tensor product $A\hat{\otimes}B$ by

$$A\hat{\otimes}B = \varprojlim_{A',B'} A/A' \otimes_{\mathbb{Z}} B/B'$$

where A' (resp. B') ranges over all open subgroups of A (resp. B).

1.7.4. For a pro-p abelian group T endowed with a continuous unramified action of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, let

$$D(T) = (T\hat{\otimes}W(\bar{\mathbb{F}}_p))^{Fr_p=1} = \{x \in T\hat{\otimes}W(\bar{\mathbb{F}}_p) \mid (Fr_p \otimes Fr_p)(x) = x\}.$$

Here the topological tensor product is defined with respect to the original topology of T and the p-adic topology of $W(\bar{\mathbb{F}}_p)$, and Fr_p is the arithmetic Frobenius (the element of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$, where $\mathbb{Q}_p^{\mathrm{ur}} \subset \bar{\mathbb{Q}}_p$ is the maximal unramified extension of $\mathbb{Q}_p$, which acts on the residue field $\bar{\mathbb{F}}_p$ of $\mathbb{Q}_p^{\mathrm{ur}}$ by $x \mapsto x^p$). Then $D(T)$ is also a pro-p abelian group, and we have a canonical isomorphism

$$D(T)\hat{\otimes}W(\bar{\mathbb{F}}_p) \xrightarrow{\cong} T\hat{\otimes}W(\bar{\mathbb{F}}_p).$$

1.7.5. For a pro-p abelian group T endowed with a continuous unramified action of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, let $\varphi : D(T) \to D(T)$ be the homomorphism induced by $1 \otimes Fr_p$ on $T\hat{\otimes}W(\bar{\mathbb{F}}_p)$. It is a bijection. It is also induced by $Fr_p^{-1} \otimes 1$ on $T\hat{\otimes}W(\bar{\mathbb{F}}_p)$. We can recover T from $D(T)$ and $\varphi : D(T) \to D(T)$ as $T = \{x \in D(T)\hat{\otimes}W(\bar{\mathbb{F}}_p) \mid (\varphi \otimes Fr_p)(x) = x\}$.

Proposition 1.7.6. *There is a functorial isomorphism of pro-p abelian groups $D(T) \cong T$. In other words, D is isomorphic to the forgetful functor as a functor from the category of pro-p-abelian groups endowed with continuous unramified actions of* $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ *to the category of pro-p-abelian groups.*

Proof. For each $n \geq 1$, take an element α_n of $W(\mathbb{F}_{p^n})$ satisfying the following conditions (i) and (ii).

(i) For each $n \geq 1$, we have an isomorphism $\mathbb{Z}_p[\mathrm{Gal}(L_n/\mathbb{Q}_p)] \xrightarrow{\cong} W(\mathbb{F}_{p^n})$; $g \mapsto g\alpha_n$, where L_n is the field of fractions of $W(\mathbb{F}_{p^n})$.

(ii) If $m, n \geq 1$ and $n|m$, the trace map $W(\mathbb{F}_{p^m}) \to W(\mathbb{F}_{p^n})$ sends α_m to α_n.

The functorial isomorphism $T \xrightarrow{\cong} D(T)$ is given as follows. Assume first that T is finite. Then there is $n \geq 1$ such that the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on T factors through $\mathrm{Gal}(L_n/\mathbb{Q}_p)$. For this n, we define the isomorphism $T \xrightarrow{\cong} D(T)$ as $x \mapsto \sum_{i\in\mathbb{Z}/n\mathbb{Z}} Fr_p^i(\alpha_n) \otimes Fr_p^i(x)$. This isomorphism is independent of the choice of n. For general T, we define the isomorphism $T \xrightarrow{\cong} D(T)$ as the inverse limit of these isomorphisms for finite quotients of T. $\square$

1.7.7. Since the actions of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)$ and on $H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)$ are unramified, we have

$$D_{dR}(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) \cong D_{crys}(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) \cong D(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p,$$

$$D_{dR}(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) \cong D_{crys}(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p) \cong D(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

1.7.8. Hence we have isomorphisms

$$S_2(M)^{\mathrm{ord}}_{\mathbb{Q}_p} \cong D(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

$$M_2(M)^{\mathrm{ord}}_{\mathbb{Q}_p} \cong D(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proposition 1.7.9. *Let $S_2(M)_{\mathbb{Z}_p}$ and $M_2(M)_{\mathbb{Z}_p}$ be as in section 1.5. Then the isomorphisms in 1.7.8 induce isomorphisms*

$$S_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \xrightarrow{\cong} D(H^1_{\acute{e}t}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)), \quad M_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \xrightarrow{\cong} D(H^1_{\acute{e}t}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)).$$

Proof. Let $\mathbb{C}_p$ be the completion of $\bar{\mathbb{Q}}_p$, and let $O_{\mathbb{C}_p}$ be the valuation ring of $\mathbb{C}_p$. Ohta ([40]) proved that (the isomorphisms in 1.7.8) $\otimes_{\mathbb{Q}_p}\mathbb{C}_p$ induce isomorphisms

$$S_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p} \xrightarrow{\cong} H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}} \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p},$$

$$M_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p} \xrightarrow{\cong} H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}} \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p}.$$

Hence the isomorphisms in 1.7.8 induce an isomorphism from

$$S_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} = (S_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap (S_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p})$$

onto

$$D(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) = (D(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap (D(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p})$$

and an isomorphism from

$$M_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} = (M_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap (M_2(M)^{\mathrm{ord}}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p})$$

onto

$$D(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) = (D(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap (D(H^1_{\text{ét}}(Y_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} O_{\mathbb{C}_p}).$$

□

1.7.10. Now fix $N \geq 1$ which is prime to p. We take the inverse limit of the above story at level $M = Np^r$ ($r \to \infty$). Let H, $\tilde{H}$, $\mathfrak{h}$, $\mathfrak{H}$, Λ, S_Λ, M_Λ be as in section 1.5.

We have exact sequences

$$0 \to H_{\mathrm{sub}} \to H \to H_{\mathrm{quo}} \to 0, \quad 0 \to \tilde{H}_{\mathrm{sub}} \to \tilde{H} \to \tilde{H}_{\mathrm{quo}} \to 0$$

where

$$H_{\mathrm{sub}} := \varprojlim_r H^1_{\text{ét}}(X_1(Np^r))^{\mathrm{ord}}_{\mathrm{sub}} \quad H_{\mathrm{quo}} := \varprojlim_r H^1_{\text{ét}}(X_1(Np^r))^{\mathrm{ord}}_{\mathrm{quo}},$$

$$\tilde{H}_{\mathrm{sub}} := \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r))^{\mathrm{ord}}_{\mathrm{sub}} \quad \tilde{H}_{\mathrm{quo}} := \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r))^{\mathrm{ord}}_{\mathrm{quo}}.$$

The canonical injection $H_{\mathrm{sub}} \to \tilde{H}_{\mathrm{sub}}$ is an isomorphism.

From 1.7.9 and 1.7.6, we obtain

Proposition 1.7.11. *(1) We have canonical isomorphisms (Λ-adic Eichler-Shimura isomorphisms)*

$$S_\Lambda \cong D(H_{\mathrm{quo}}(1)), \quad M_\Lambda \cong D(\tilde{H}_{\mathrm{quo}}(1)).$$

(2) $S_\Lambda \cong H_{\mathrm{quo}}$ as $\mathfrak{h}$-modules, and $M_\Lambda \cong \tilde{H}_{\mathrm{quo}}$ as $\mathfrak{H}$-modules.
Here the $\mathfrak{h}$-module (resp. $\mathfrak{H}$-module) structure of S_Λ (resp. M_Λ) is as in 1.5.12.

1.7.12. The isomorphisms in 1.7.11 (1) induce isomorphisms

$$S_\Lambda \hat{\otimes} W(\bar{\mathbb{F}}_p) \cong H_{\mathrm{quo}}(1)\hat{\otimes} W(\bar{\mathbb{F}}_p), \quad M_\Lambda \hat{\otimes} W(\bar{\mathbb{F}}_p) \cong \tilde{H}_{\mathrm{quo}}(1)\hat{\otimes} W(\bar{\mathbb{F}}_p).$$

In Ohta [38], [40], these isomorphisms are obtained when $\hat{\otimes} W(\bar{\mathbb{F}}_p)$ is replaced by $\hat{\otimes} O$ where O is the valuation ring of a local field over $\mathbb{Q}_p$ which contains all roots of 1, and are called the Λ-adic Eichler-Shimura isomorphisms.

1.7.13. As $\mathfrak{h}$-modules, H_{sub} is free of rank 1, and S_Λ and H_{quo} are dualizing $\mathfrak{h}$-modules. As $\mathfrak{H}$-modules, M_Λ and $\tilde{H}_{\mathrm{quo}}$ are dualizing $\mathfrak{H}$-modules. Here the $\mathfrak{h}$ (resp. $\mathfrak{H}$)-module structure on S_Λ (resp. M_Λ) is as in 1.7.11. (For S_Λ and M_Λ, see Hida [19]. For H_{quo} and $\tilde{H}_{\mathrm{quo}}$, see Hida [18], Mazur-Wiles [33], Ohta [38], [39], Tilouine [56].)

1.7.14. The $Q(\mathfrak{h})$-module $H \otimes_{\mathfrak{h}} Q(\mathfrak{h})$ is free of rank 2, and $H_{\mathrm{sub}} \otimes_{\mathfrak{h}} Q(\mathfrak{h})$ and $H_{\mathrm{quo}} \otimes_{\mathfrak{h}} Q(\mathfrak{h})$ are free $Q(\mathfrak{h})$-modules of rank 1.

Concerning the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $H \otimes_{\mathfrak{h}} Q(\mathfrak{h})$ over $Q(\mathfrak{h})$, we have the following Λ-adic version of 1.2.9 ([19] section 7):

(1) For a prime number ℓ which does not divide Np, the action is unramified at ℓ, and we have

$$\det(1 - Fr_\ell^{-1}u) = 1 - T(\ell)u + \ell\langle\ell\rangle u^2 = 1 - \langle\ell\rangle T^*(\ell)u + \ell\langle\ell\rangle u^2.$$

Here Fr_ℓ is the arithmetic Frobenius of ℓ.

(2) The determinant of $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is $\kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}$. Here $\langle\sigma\rangle$ denotes $\langle a\rangle$ with a the element of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times = \varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times$ such that $\sigma(\zeta_{Np^r}) = \zeta_{Np^r}^a$ for any $r \geq 1$, and κ is the cyclotomic character.

Note also ([38]):

(3) the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\tilde{H}_{\mathrm{quo}}(1)$ is unramified,

(4) The action of an element σ of the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on H_{sub} is given by $\langle\sigma\rangle^{-1}$.

1.7.15. In [38], Ohta uses $X_1'(M)$. Identify $H^1_{\mathrm{\acute{e}t}}(X_1(M))$ and $H^1_{\mathrm{\acute{e}t}}(X_1'(M))$ via the isomoprhism v_M in section 1.4. This isomorphism v_M changes the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as described in 1.4.5 (1). Hence for X_1', 1.7.14 (4) is replaced as follows. As in Ohta [40], the action of an element σ of the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on H_{sub} is trivial. In fact, H_{sub} for X_1' coincides with the fixed part of H under the action of the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

1.7.16. The following is not used in this paper, but put for the comparison. We have the following exact sequences (cf. 1.4.5).

$$0 \to [\langle\sigma\rangle^{-1}\text{-action}] \to [T^*(p)\text{-ordinary part of } H^1_{\text{ét}}(X_1(M))] \to [\text{unramified}](-1) \to 0.$$

$$0 \to [\text{unramified}] \to [T^*(p)\text{-ordinary part of } H^1_{\text{ét}}(X_1'(M))] \to [\langle\sigma\rangle\text{-action}](-1) \to 0.$$

$$0 \to [\text{unramified}] \to [T(p)\text{-ordinary part of } H^1_{\text{ét}}(X_1(M))] \to [\langle\sigma\rangle^{-1}\text{-action}](-1) \to 0.$$

$$0 \to [\langle\sigma\rangle\text{-action}] \to [T(p)\text{-ordinary part of } H^1_{\text{ét}}(X_1'(M))] \to [\text{unramified}](-1) \to 0.$$

Here $[\langle\sigma\rangle^{-1}$-action] (resp. [unramified]) means a $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-module on which an element σ of the inertia subgroup $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts by $\langle\sigma\rangle^{-1}$ (resp. trivially).

As Ohta used $X_1'(M)$, the second exact sequence appeared in his papers [38] and [40]. We use the first exact sequence in this paper.

1.8 $T^*(p)$ and Frobenius

The aim of this section 1.8 is to prove the following proposition.

Proposition 1.8.1. *Assume $p|M$. Then the action of $T^*(p)$ on $H^1_{ét}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)$ coincides with the action of the arithmetic Frobenius Fr_p. That is, the action of $T^*(p)$ on $D(H^1_{ét}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1))$ coincides with the action of φ^{-1} (1.7.4).*

Remark 1.8.2. (1) This proposition is proved in Ohta [41] 3.4.2 (cf. also Mazur-Wiles [32]) except "one Teichmüller component" in the following sense.

Regard $(\mathbb{Z}/p\mathbb{Z})^\times$ as a subgroup of $(\mathbb{Z}/M\mathbb{Z})^\times$ via the unique injective homomorphism such that the composition $(\mathbb{Z}/p\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ is the identity map and such that if we write $M = Np^r$ with $(N,p) = 1$, then the composition $(\mathbb{Z}/p\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$ is trivial. Consider the direct sum decomposition $D(H^1_{\text{ét}}(X_1(M))^{\mathrm{ord}}_{\mathrm{quo}}(1)) = \oplus_{i\in\mathbb{Z}/(p-1)\mathbb{Z}}$ (the i-component) where the i-component is the part on which $\langle a\rangle$ for $a \in (\mathbb{Z}/p\mathbb{Z})^\times \subset (\mathbb{Z}/M\mathbb{Z})^\times$ acts as $\omega(a)^i$ with ω the Teichmüller character. Ohta [41] 3.4.2 shows that on each i-component with $i \in \mathbb{Z}/(p-1)\mathbb{Z} - \{0\}$, $T^*(p)$ and Fr_p coincide.

(2) It seems that this proposition is reduced to the work Saito [48] on the p-adic representation of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated to an eigen cusp form. The following proof does not use [48], but uses q-expansions.

1.8.3. Consider the two homomorphisms

$$T(p), \varphi_q : \mathbb{Z}_p[[q]]dq \mapsto \mathbb{Z}_p[[q]]dq;$$

$$T(p)(\sum_{n=1}^{\infty} a_n q^n d\log(q)) = \sum_{n=1}^{\infty} a_{np} q^n d\log(q),$$

$$\varphi_q(\sum_{n=1}^{\infty} a_n q^n d\log(q)) = \sum_{n=1}^{\infty} a_n q^{np} d\log(q^p) = p \cdot \sum_{n=1}^{\infty} a_n q^{np} d\log(q).$$

We have $T(p) \circ \varphi_q = p$, to which we are going to reduce 1.8.1.

1.8.4. Consider the canonical map of de Rham cohomology groups over $\mathbb{Q}_p$

$$a_{dR}\ :\ H^1_{dR}(X_1(M)_{\mathbb{Q}_p}) \to H^1_{dR}(\mathrm{Spec}(\mathbb{Z}_p[[q]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)) = (\mathbb{Z}_p[[q]]dq)/d\mathbb{Z}_p[[q]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

which is induced from $\mathrm{Spec}(\mathbb{Z}_p[[q]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(\zeta_M)) \to X_1'(M) \overset{w_M}{\to} X_1(M)$ where the first arrow is the q-expansion.

Via this map, $T^*(p)$ on $H^1_{dR}(X_1(M)_{\mathbb{Q}_p})$ is compatible with the usual Hecke operator $T(p) : \mathbb{Z}_p[[q]]dq \to \mathbb{Z}_p[[q]]dq$ (1.8.3).

The restriction of a_{dR} to $S_2(M)_{\mathbb{Q}_p} = \Gamma(X_1(M)_{\mathbb{Q}_p}, \Omega^1) \subset H^1_{dR}(X_1(M)_{\mathbb{Q}_p})$ sends $f \in S_2(M)_{\mathbb{Q}_p}$ to the class of the q-expansion of $w_M(f)$.

1.8.5. Consider the functor D_{pst} of Fontaine ([9]), [10]). We have a homomorphism

$$a_{pst}\ :\ D_{pst}(H^1_{\text{ét}}(X_1(M))_{\mathbb{Q}_p}) \to (\mathbb{Z}_p[[q]]dq)/d\mathbb{Z}_p[[q]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_{p,ur}$$

defined as follows. Here $\mathbb{Q}_{p,ur} \subset \bar{\mathbb{Q}}_p$ is the maximal unramified extension of $\mathbb{Q}_p$. The space $(\mathbb{Z}_p[[q]]dq)/d\mathbb{Z}_p[[q]]$ is regarded as the crystalline cohomology of $\mathbb{F}_p[[q]]$. On the other hand, $X_1(M)_{\mathbb{Q}_p}$ has a model $\mathfrak{X}$ of semi-stable reduction over O_L for some finite extension $L \subset \bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and by Tsuji [57], if L_0 denotes the largest unramified extension of $\mathbb{Q}_p$ contained in L, then $D_{pst}(H^1_{\text{ét}}(X_1(M))_{\mathbb{Q}_p})$ is identified with $H^1_{\log\text{crys}}(C)\otimes_{O_{L_0}}\mathbb{Q}_{p,ur}$ where C is the reduction of $\mathfrak{X}$ and $H^1_{\log\text{crys}}$ is the log crystalline cohomology. The map $\mathrm{Spec}(O_L[[q]]\otimes_{O_L}L(\zeta_M)) \to \mathfrak{X}$ which is compatible with $\mathrm{Spec}(\mathbb{Z}_p[[q]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(\zeta_M)) \to X_1(M)$ induces a homomorphism

$$H^1_{\log\text{crys}}(C) \to (O_{L_0}[[q]]dq)/dO_{L_0}[[q]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

of log crystalline cohomology. Hence we have the above homomorphism a_{pst}.

The homomorphisms a_{dR} and a_{pst} are compatible with the isomorphism

$$H^1_{dR}(X_1(M)_{\mathbb{Q}_p})\otimes_{\mathbb{Q}_p}\bar{\mathbb{Q}}_p = D_{dR}(H^1_{\text{ét}}(X_1(M))_{\mathbb{Q}_p})\otimes_{\mathbb{Q}_p}\bar{\mathbb{Q}}_p \cong D_{pst}(H^1_{\text{ét}}(X_1(M))_{\mathbb{Q}_p})\otimes_{\mathbb{Q}_{p,ur}}\bar{\mathbb{Q}}_p.$$

Via the homomorphism a_{pst}, $T^*(p)$ on $D_{pst}(H^1_{\text{ét}}(X_1(M))_{\mathbb{Q}_p})$ and $T(p)$ on $\mathbb{Z}_p[[q]]dq$ (1.8.3) are compatible. Furthermore, via a_{pst}, φ on $D_{pst}(H^1_{\text{ét}}(X_1(M))_{\mathbb{Q}_p})$ ([8]) and $\varphi_q\otimes Fr_p$ on $\mathbb{Z}_p[[q]]dq\otimes_{\mathbb{Z}_p}\mathbb{Q}_{p,ur}$ (1.8.3) are compatible because the former is compatible with the map φ on $H^1_{\log\text{crys}}(C)$ induced by the p-th power map in characteristic p and the latter is also induced by the p-th power map in characteristic p.

1.8.6. We have an exact sequence

$$0 \to D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub},\mathbb{Q}_p}) \to D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\mathbb{Q}_p}) \to D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo},\mathbb{Q}_p}) \to 0.$$

The φ on $D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo},\mathbb{Q}_p})$ coincides with p times the φ on $D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo},\mathbb{Q}_p}(1))$ and hence coincides with p times the φ on $D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Lemma 1.8.7. *The map a_{pst} kills $D_{pst}(H^1_{\acute{e}t}(X_1(M))^{\text{ord}}_{\text{sub},\mathbb{Q}_p})$.*

Proof. By 1.7.2, the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_M))$ on $H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub}}$ is unramified. Hence the action of φ on $D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub},\mathbb{Q}_p})$ is of slope 0. That is, there is a finitely generated $\mathbb{Z}_p$-submodule L of $D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub},\mathbb{Q}_p})$ which generates $D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{sub},\mathbb{Q}_p})$ over $\mathbb{Q}_p$ such that $\varphi(L) = L$. Since this φ is compatible with φ_q on $\mathbb{Z}_p[[q]]dq$ (1.8.5) and the image of $\varphi_q^n : \mathbb{Z}_p[[q]]dq \to \mathbb{Z}_p[[q]]dq$ is contained in $p^n\mathbb{Z}_p[[q]]dq$ for any $n \geq 0$, we have the result. $\square$

1.8.8. By 1.8.7, a_{pst} induces a homomorphism

$$b : D_{pst}(H^1_{\text{ét}}(X_1(M))_{\text{quo},\mathbb{Q}_p}) \to (\mathbb{Z}_p[[q]]d\log(q))/d\mathbb{Z}_p[[q]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_{p,ur}.$$

The composition of this map with $M_2(M)_{\mathbb{Z}_p} \xrightarrow{\cong} D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1))$ (note $D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1))_{\mathbb{Q}_p} = D_{pst}(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1)_{\mathbb{Q}_p})$ is the map induced by a_{dR}. Hence the image of $D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1))$ under b is contained in $(\mathbb{Z}_p[[q]]dq)/d\mathbb{Z}_p[[q]]$.

Note that for $n \geq 0$, the map $T(p)^n : \mathbb{Z}_p[[q]]dq \to \mathbb{Z}_p[[q]]dq$ induces

$$(\mathbb{Z}_p[[q]]dq)/d\mathbb{Z}_p[[q]] \to (\mathbb{Z}_p[[q]]dq)/p^n d\mathbb{Z}_p[[q]].$$

The composition

$$T(p)^n \circ a_{dR} \circ T^*(p)^{-n} : S_2(M)^{\text{ord}}_{\mathbb{Z}_p} \to (\mathbb{Z}_p[[q]]dq)/p^n d\mathbb{Z}_p[[q]]$$

sends $f \in S_2(M)^{\text{ord}}_{\mathbb{Z}_p}$ to the class of q-expansion of $w_M(f)$. Hence the inverse limit of the last maps for $n \geq 1$ is the map $S_2(M)^{\text{ord}}_{\mathbb{Z}_p} \to \mathbb{Z}_p[[q]]d\log(q)$ which sends $f \in S_2(M)^{\text{ord}}_{\mathbb{Z}_p}$ to the q-expansion of $w_M(f)$, and which is clearly injective. Hence the inverse limit

$$\tilde{b} : D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1)) \to \mathbb{Z}_p[[q]]dq$$

of

$$T(p)^n \circ b \circ T^*(p)^{-n} : D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1)) \to (\mathbb{Z}_p[[q]]dq)/p^n d\mathbb{Z}_p[[q]]$$

is injective. Via $\tilde{b}$, $T^*(p)$ (resp. $p\varphi$) on $D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1))$ is compatible with $T(p)$ (resp. φ_p) on $\mathbb{Z}_p[[q]]dq$. Since $T(p) \circ \varphi_q = p$ on $\mathbb{Z}_p[[q]]d\log(q)$, we have $T^*(p) \circ \varphi = 1$ on $D(H^1_{\text{ét}}(X_1(M))^{\text{ord}}_{\text{quo}}(1))$.

1.9 Eisenstein ideal, Drinfeld-Manin splitting

We review some notions concerning Hecke operators which are necessary for this paper.

1.9.1. (Eisenstein ideal.) Assume $p|M$. Let $N \geq 1$, $(N,p) = 1$, and let $\mathfrak{h} = \varprojlim_r \mathfrak{h}(Np^r)^{\text{ord}}_{\mathbb{Z}_p}$, $\mathfrak{H} = \varprojlim_r \mathfrak{H}(Np^r)^{\text{ord}}_{\mathbb{Z}_p}$ as usual. Let R be one of $\mathfrak{h}(M)_{\mathbb{Z}_p}$, $\mathfrak{H}(M)_{\mathbb{Z}_p}$, $\mathfrak{h}$, $\mathfrak{H}$.

The Eisenstein ideal of R is defined to be the ideal generated by $1 - T^*(\ell) + \ell\langle\ell\rangle^{-1}$ for all prime numbers $\ell \nmid M$ and $1 - T^*(\ell)$ for all prime numbers $\ell|M$.

We will denote the Eisenstein ideal of R by I.

1.9.2. (Eisenstein component.) Let R be as above. Then R is a complete semi-local ring, and hence $R = \prod_m R_m$ where m ranges over all maximal ideals (there are only finitely many m) and R_m is the local ring of R at m which is a complete local ring.

The Eisenstein component R_E of R is defined to be $\prod_m R_m$ where m ranges over all maximal ideals of R which contain the Eisenstein ideal of R.

1.9.3. There is a unique left inverse of the canonical injection $H^1(X_1(M)(\mathbb{C}),\mathbb{Q}) \to H^1(Y_1(M)(\mathbb{C}),\mathbb{Q})$ (resp. $S_2(M)_{\mathbb{Q}} \to M_2(M)_{\mathbb{Q}}$) which is compatible with the actions of $\mathfrak{H}(M)_{\mathbb{Q}}$.

Let $N \geq 1$, $(N,p)=1$, and let $H, \tilde{H}, S_\Lambda$, M_Λ and $\mathfrak{H}$ be as before. Then there is a unique left inverse of the canonical injection $H \otimes_\Lambda Q(\Lambda) \to \tilde{H} \otimes_\Lambda Q(\Lambda)$ (resp. $S_\Lambda \otimes_\Lambda Q(\Lambda) \to M_\Lambda \otimes_\Lambda Q(\Lambda)$) which is compatible with the actions of $\mathfrak{H}$.

These left inverses are called Drinfeld-Manin splittings.

2 Beilinson elements in K_2 of modular curves

Beilinson elements in K_2 of modular curves were defined by Beilinson in [2], and studied by him and then later by [22]. In this section, we review basic definitions (section 2.1), norm relations of Beilinson elements (section 2.2), and the relations of Beilinson elements to values at $s=1$ of complex L-functions of modular forms (sections 2.3 and 2.4). We slightly improve results in [22]: Beilinson elements of the type $\{g_{0,\alpha}, g_{0,\beta}\}$ with two 0 in the left entries, which appear in this section and play important roles in this paper, were not considered in [22]. Beilinson elements in section 2.4 related to the modular symbols $[u:v]_r$ were not considered in [22].

2.1 Beilinson elements on $Y(m,M)$

2.1.1. Siegel units ([25]).

For $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^2 - \{(0,0)\}$ and for an integer c which is prime to 6 and to the orders of α and β, the Siegel unit ${}_cg_{\alpha,\beta}$ is defined. It is an element of $\mathcal{O}(Y(m,M)\otimes\mathbb{Z}[1/mM])^\times$ for integers $m, M \geq 1$, $m+M \geq 5$ such that $m\alpha = 0$ and $M\beta = 0$. It is characterized by its q-expansion as follows. Let

$$\gamma_q(t) := \prod_{n\geq 0}(1-q^n t)\cdot\prod_{n\geq 1}(1-q^n t^{-1}) \in \mathbb{Z}[t,1/t][[q]]^\times,$$

$${}_c\theta(t) := q^{\frac{c^2-1}{12}}(-t)^{\frac{c-c^2}{2}}\gamma_q(t)^{c^2}\gamma_q(t^c)^{-1} \in \mathbb{Z}[t,1/t][[q]][q^{-1}]^\times.$$

Write $\alpha = a/m \bmod \mathbb{Z}$, $\beta = b/M \bmod \mathbb{Z}$ $(a,b\in\mathbb{Z})$. Then ${}_cg_{\alpha,\beta}$ has the q-expansion

$${}_cg_{\alpha,\beta} = {}_c\theta(q^{a/m}\zeta_M^b) \quad in \quad \mathbb{Z}[1/mM,\zeta_M][[q^{1/m}]][q^{-1}]^\times.$$

Here $q^{1/m} = e^{2\pi i\tau/m}$ $(\tau\in\mathcal{H})$. Taking c such that $c\equiv 1 \mod m$, $c \equiv 1 \mod M$ and $c \neq \pm 1$, let

$$g_{\alpha,\beta} = {}_cg_{\alpha,\beta}\otimes(c^2-1)^{-1} \in \mathcal{O}(Y(m,M)\otimes\mathbb{Z}[1/mM])^\times\otimes\mathbb{Q}.$$

Then $g_{\alpha,\beta}$ is independent of the choice of such c.

2.1.2. The following distribution property of Siegel units is often used in section 2.2.

Let $L \geq 1$, and assume $(c,L)=1$. Then we have

$$\prod_{\alpha',\beta'} {}_cg_{\alpha',\beta'} = {}_cg_{\alpha,\beta} \quad \text{in } \mathcal{O}(Y(mL,ML)\otimes\mathbb{Z}[1/mML])^\times.$$

where (α',β') ranges over all elements of $(\mathbb{Q}/\mathbb{Z})^2$ such that $\alpha = L\alpha'$ and $\beta = L\beta'$.

2.1.3. If $m|L$, $M|L$, for $\begin{pmatrix} s & u \\ t & v \end{pmatrix} \in GL(2, \mathbb{Z}/L\mathbb{Z})$,

$$\begin{pmatrix} s & u \\ t & v \end{pmatrix}^* ({}_cg_{\alpha,\beta}) = {}_cg_{\alpha',\beta'} \quad \text{with} \quad (\alpha', \beta') = (\alpha, \beta) \begin{pmatrix} s & u \\ t & v \end{pmatrix}.$$

In particular, since the action of the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on $Y(L)$ is trivial, we have

(1) ${}_cg_{\alpha,\beta} = {}_cg_{-\alpha,-\beta}$.

2.1.4. For a matrix $R = \begin{pmatrix} s & u \\ t & v \end{pmatrix} \in M(2, \mathbb{Z})$ satisfying the condition (i) below, define Beilinson elements ${}_{c,d}z_{m,M}(R) \in K_2(Y(m, M) \otimes \mathbb{Z}[1/mM])$ and $z_{m,M}(R) \in K_2(Y(m, M) \otimes \mathbb{Z}[1/mM]) \otimes \mathbb{Q}$, where $c, d \in \mathbb{Z}$ and $(cd, 6mM) = 1$, as follows.

$${}_{c,d}z_{m,M}(R) = {}_{c,d}z_{m,M} \begin{pmatrix} s & u \\ t & v \end{pmatrix} := \{{}_cg_{s/m,u/M}, {}_dg_{t/m,v/M}\} \in K_2(Y(m, M) \otimes \mathbb{Z}[1/mM]),$$

$$z_{m,M}(R) = \{g_{s/m,u/M}, g_{t/m,v/M}\} \in K_2(Y(m, M) \otimes \mathbb{Z}[1/mM]) \otimes \mathbb{Q}.$$

Here $\{-, -\}$ is the Steinberg symbol.

(i) Neither the images of (s, u) nor (t, v) in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z}$ is $(0, 0)$.

These Beilinson elements depend only on s, t mod m and u, v mod M, and so we often regard them as functions of s, t mod m and u, v mod M.

In the case $m = M$ and $R' \in SL(2, \mathbb{Z})$, by 2.1.3, we have

$$(R')^*(z_{M,M}(R)) = z_{M,M}(RR'), \quad (R')^*({}_{c,d}z_{M,M}(R)) = {}_{c,d}z_{M,M}(RR').$$

2.2 Norm relations

We prove Propositions 2.2.2, 2.2.3, 2.2.4, 2.2.5 on norms of Beilinson elements.

For an integer $m \geq 1$, let prime(m) be the set of all prime divisors of m.

We will often use the following lemma.

Lemma 2.2.1. *Let m, M, L be positive integers. Assume $m|L$, $M|L$. Then we have the following result.*

(1) *If $m|m'|L$, and $M|M'|L$, $G_L(m', M')$ is a normal subgroup of $G_L(m, M)$.*

(2) *Let ℓ be a prime. Then $G_L(m, M)/G_L(m\ell, M\ell)$ is isomorphic to*

$$\begin{cases} M(2, \mathbb{F}_\ell) & \left(\text{by } \begin{pmatrix} 1+mx & my \\ Mz & 1+Mu \end{pmatrix} \mapsto \begin{pmatrix} x & y \\ z & u \end{pmatrix}\right) \quad \text{if } \ell|m \text{ and } \ell|M \\[2ex] \begin{pmatrix} 1 & \mathbb{F}_\ell & \mathbb{F}_\ell \\ 0 & \mathbb{F}_\ell^\times & \mathbb{F}_\ell \\ 0 & 0 & 1 \end{pmatrix} & \left(\text{by } \begin{pmatrix} x & my \\ Mz & 1+Mu \end{pmatrix} \mapsto \begin{pmatrix} 1 & z & u \\ 0 & x & y \\ 0 & 0 & 1 \end{pmatrix}\right) \quad \text{if } \ell \nmid m \text{ but } \ell|M \\[2ex] GL(2, \mathbb{F}_\ell) & (\text{by } \bmod \ell) \quad \text{if } \ell \nmid mM. \end{cases}$$

Proposition 2.2.2. *Let $m, M \geq 1$ and assume $m + M \geq 5$. Assume prime$(m) \subset$ prime(M). Let $L \geq 1$. Let $R \in M(2, \mathbb{Z})$, assume that the condition (i) in 2.1.4 is satisfied, and assume further that the image of R in $M(2, \mathbb{Z}/L\mathbb{Z})$ belongs to $GL(2, \mathbb{Z}/L\mathbb{Z})$. Let $c, d \in \mathbb{Z}$ and assume $(cd, 6mML) = 1$.*

Then the norm map $K_2(Y(mL, ML) \otimes \mathbb{Z}[1/mML]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mML])$ sends ${}_{c,d}z_{mL,ML}(R)$ to

$$\Big(\prod_{\ell|L, \ell\nmid m} P_\ell\Big) \cdot {}_{c,d}z_{m,M}(R),$$

where $\prod_{\ell|L,\ell\nmid m}$ is the product over all prime divisors ℓ of L which do not divide m, and

$$P_\ell = 1 - T^*(\ell)\begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^* + \begin{pmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{pmatrix}^* \cdot \ell \quad \text{if } \ell \nmid M,$$

$$P_\ell = 1 - T^*(\ell)\begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^* \quad \text{if } \ell | M.$$

Proof. Write $R = \begin{pmatrix} s & u \\ t & v \end{pmatrix}$. The proof is essentially the same as that of Proposition 2.4 in [22], where we treated the case $s = 1,\ u = 0, t = 0, v = 1$.

We may assume that L is a prime number ℓ. In the following (1) (resp. (2), resp. (3)), we treat the case $\ell | m$ (resp. $\ell \nmid m$ but $\ell | M$, resp. $\ell \nmid M$).

(1) Assume $\ell | m$. Take an integer $J \geq 1$ such that $m\ell | J$ and $M\ell | J$. Let

$$H = \{h \in G_J(m, M) \mid (\frac{t}{m\ell}, \frac{v}{M\ell})h = (\frac{t}{m\ell}, \frac{v}{M\ell})\}$$

$$= \{h \in G_J(m, M) \mid h \equiv \begin{pmatrix} 1 + xvm & yvm \\ -xtM & 1 - ytM \end{pmatrix} \bmod G_J(m\ell, M\ell) \text{ for some } x, y \in \mathbb{Z}\}.$$

Then the morphism $Y(m\ell, M\ell) \to Y(m, M)$ factors as $Y(m\ell, M\ell) \to H \setminus Y(J) \to Y(m, M)$, and the norm map $K_2(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mM\ell])$ factors as $K_2(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell]) \xrightarrow{N_1} K_2(H \setminus Y(J) \otimes \mathbb{Z}[1/mM\ell]) \xrightarrow{N_2} K_2(Y(m, M) \otimes \mathbb{Z}[1/mM\ell])$, where N_1 and N_2 are norm maps.

Claim 1. N_1 sends ${}_{c,d}z_{mL,ML}(R)$ to $\{{}_cg_{s/m,u/M}, {}_dg_{t/m\ell,v/M\ell}\}$.

We prove Claim 1. Since ${}_dg_{t/m\ell,v/M\ell} \in \mathcal{O}(H \setminus Y(J) \otimes \mathbb{Z}[1/mM\ell])^\times$,

$$N_1({}_{c,d}z_{mL,ML}(R)) = \{N_1({}_cg_{s/m\ell,u/M\ell}),\ {}_dg_{t/m\ell,v/M\ell}\},$$

where N_1 on the right hand side is the norm map $N_1 : \mathcal{O}(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell])^\times \to \mathcal{O}(H \setminus Y(J) \otimes \mathbb{Z}[1/mM\ell])^\times$. Let $\tilde{\mathbb{F}}_\ell$ be a set in $\mathbb{Z}$ of representatives of $\mathbb{F}_\ell$. We have

$$N_1({}_cg_{s/m\ell,u/M\ell}) = \prod_{x,y \in \tilde{\mathbb{F}}_\ell} \begin{pmatrix} 1 + xvm & yvm \\ -xtM & 1 - ytM \end{pmatrix}^* {}_cg_{s/m\ell,u/M\ell}$$

$$= \prod_{x,y \in \tilde{\mathbb{F}}_\ell} {}_cg_{s/m\ell+(sv-tu)x/\ell,u/M\ell+(sv-tu)y/\ell} = {}_cg_{s/m,u/M}$$

by Lemma 2.2.1 and by the distribution property 2.1.2 of Siegel units. This proves Claim 1.

Since ${}_c g_{s/m,u/M} \in \mathcal{O}(Y(m,M) \otimes \mathbb{Z}[1/mM])^\times$,

$$N_2(\{{}_c g_{s/m,u/M}, {}_d g_{t/m\ell,v/M\ell}\}) = \{{}_c g_{s/m,u/M}, N_2({}_d g_{t/m\ell,v/M\ell})\},$$

where N_2 on the right hand side is the norm map $N_2 : \mathcal{O}(H \setminus Y(J) \otimes \mathbb{Z}[1/mM\ell])^\times \to \mathcal{O}(Y(m,M) \otimes \mathbb{Z}[1/mM\ell])^\times$. We have similarly,

$$N_2({}_d g_{t/m\ell,v/M\ell}) = {}_d g_{t/m,v/M}.$$

(2) Assume $\ell \not| m$ but $\ell | M$. Let $J \geq 1$ be as in the proof of (1).

By using Lemma 2.2.1 (2), we see that there exists an element $h \in G_J(m,M)$ such that

$$(\frac{s}{m\ell}, \frac{u}{M\ell})h \text{ or } (\frac{t}{m\ell}, \frac{v}{M\ell})h \in (1/m)\mathbb{Z}/\mathbb{Z} \times (1/M\ell)\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}. \tag{2.1}$$

In what follows, we show the result assuming that we can take $h \in G_J(m,M)$ such that $(t/m\ell, v/M\ell)h \in (1/m)\mathbb{Z}/\mathbb{Z} \times (1/M\ell)/\mathbb{Z}$. The other case can be treated in the same way.

Put

$$h^*({}_{c,d}z_{mL,ML}(R)) = \{{}_c g_{s'/m\ell,u'/M\ell},\ {}_d g_{t\ell'/m,v'/M\ell}\}$$

with some $s', u', v', \ell' \in \mathbb{Z}$ satisfying $s' \equiv s \mod m$, $u' \equiv u \mod M$, $v' \equiv v \mod M$, and $\ell\ell' \equiv 1 \mod m$. By our assumption $(t/m\ell, v/M\ell)h \in (1/m)\mathbb{Z}/\mathbb{Z} \times (1/M\ell)/\mathbb{Z}$, we have $\ell \not| s'v'$. It is clear that $h^*({}_{c,d}z_{mL,ML}(R))$ has the same image as ${}_{c,d}z_{mL,ML}(R)$ under the norm map $K_2(Y(m\ell, M\ell)) \to K_2(Y(m,M))$ in question.

The morphism $Y(m\ell, M\ell) \to Y(m,M)$ factors as $Y(m\ell, M\ell) \to Y(m, M(\ell)) \to Y(m,M)$. Note that $Y(m, M(\ell))$ is the quotient of $Y(m\ell, M\ell)$ by the action of the matrices $\begin{pmatrix} \tilde{x} & my \\ 0 & 1+Mw \end{pmatrix}$ with $x \in \tilde{\mathbb{F}}_\ell^\times$, $y, w \in \tilde{\mathbb{F}}_\ell$, where $\tilde{x}$ denotes an integer whose image in $(\mathbb{Z}/m\ell\mathbb{Z})^\times \cong (\mathbb{Z}/\ell\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^\times$ is $(x, 1)$. The norm map $K_2(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell])$ sends $h^*({}_{c,d}z_{mL,ML}(R))$ to

$$\sum_{x \in \tilde{\mathbb{F}}_\ell^\times, y, w \in \tilde{\mathbb{F}}_\ell} \{{}_c g_{s'\tilde{x}/m\ell,\, u'/M\ell + y/\ell},\ {}_d g_{t\ell'/m,\, v'/M\ell + w/\ell}\} \quad \text{(denote this by } A).$$

As in 1.2.3, let $\pi : Y(m, M(\ell)) \to Y(m,M)$ be the canonical projection and let $\psi_\ell : Y(m, M(\ell)) \to Y(m,M)$ be the unique morphism which is compatible with $\mathcal{H} \to \mathcal{H}$; $\tau \mapsto \ell\tau$ so that $T^*(\ell) = \pi_* \psi_\ell^*$. We have

$$\psi_\ell^*({}_c g_{a/m,b/M}) = \prod_{v \in \tilde{\mathbb{F}}_\ell} {}_c g_{a/m, b/M\ell + v/\ell}$$

for any $a, b \in \mathbb{Z}$. Since

$$\prod_{w', y \in \tilde{\mathbb{F}}_\ell} {}_c g_{s'\tilde{x}/m\ell + w'/\ell, u'/M\ell + y/\ell} = {}_c g_{s/m,u/M}$$

(note that $s'\tilde{x} \equiv s' \equiv s \mod m$ and $u' \equiv u \mod M$),

$$\begin{aligned}\prod_{x\in\tilde{\mathbb{F}}_\ell^\times, y\in\tilde{\mathbb{F}}_\ell} {}_cg_{s'\tilde{x}/m\ell, u'/M\ell+y/\ell} &= (\prod_{x,y\in\tilde{\mathbb{F}}_\ell} {}_cg_{s'/m\ell+x/\ell, u'/M\ell+y/\ell})(\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_cg_{s\ell'/m, u'/M\ell+y/\ell})^{-1}\\ &= ({}_cg_{s/m,u/M})(\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_cg_{s\ell'/m,u'/M\ell+y/\ell})^{-1}.\end{aligned}$$

Hence

$$A = \{{}_cg_{s/m,u/M},\ \prod_{w\in\tilde{\mathbb{F}}_\ell} {}_dg_{t\ell'/m,v'/M\ell+w/\ell}\} - \psi_\ell^*\{{}_cg_{s\ell'/m,u/M},\ {}_dg_{t\ell'/m,v/M}\}.$$

The norm map $\pi_* : K_2(Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mM\ell])$ sends A to

$$\{{}_cg_{s/m,u/M},\ \pi_* \prod_{w\in\tilde{\mathbb{F}}_\ell} {}_dg_{t\ell'/m,v'/M\ell+w/\ell}\} - T^*(\ell)\{{}_cg_{s\ell'/m,u/M},\ {}_dg_{t\ell'/m,v/M}\}.$$

As $Y(m, M)$ is the quotient of $Y(m, M(\ell))$ by the action of $\begin{pmatrix} 1 & 0 \\ w'M & 1 \end{pmatrix}$ $(w' \in \tilde{\mathbb{F}}_\ell)$, it follows

$$\pi_* \prod_{w\in\tilde{\mathbb{F}}_\ell} {}_dg_{t\ell'/m,v'/M\ell+w/\ell} = \prod_{w',w\in\tilde{\mathbb{F}}_\ell} {}_dg_{t\ell'/m+w'/\ell,v'/M\ell+w/\ell} = {}_dg_{t/m,v/M}.$$

Hence we obtain the result.

(3) Assume $\ell \nmid M$. Let $J \geq 1$ be as in the proof of (1).

As in (2), there exists $h \in G_J(m, M)$ such that (2.1) in the proof of (2) is satisfied. As in (2), we show the result assuming that we can take $h \in G_J(m, M)$ such that $(t/m\ell, v/M\ell)h \in (1/m)\mathbb{Z}/\mathbb{Z} \times (1/M\ell)/\mathbb{Z}$. The other case can be treated in the same way.

Take an integer ℓ' satisfying $\ell\ell' \equiv 1 \mod mM$. Put

$$h^*({}_{c,d}z_{mL,ML}(R)) = \{{}_cg_{s'/m\ell,u'/M\ell},\ {}_dg_{t\ell'/m,v'/M\ell}\}$$

for $s', u', v' \in \mathbb{Z}$ satisfying $s' \equiv s \mod m$, $u' \equiv u \mod M$, and $v' \equiv v \mod M$. We also have $l \nmid s'v'$. Clearly $h^*({}_{c,d}z_{mL,ML}(R))$ has the same image as ${}_{c,d}z_{mL.ML}(R)$ under the norm map $K_2(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mM\ell])$.

The morphism $Y(m\ell, M\ell) \to Y(m, M)$ factors as $Y(m\ell, M\ell) \to Y(m, M\ell) \to Y(m, M(\ell)) \to Y(m, M)$. The $Y(m, M\ell)$ is the quotient of $Y(m\ell, M\ell)$ by the action of the matrices $\begin{pmatrix} \tilde{x} & my \\ 0 & 1 \end{pmatrix}$ with $x \in \tilde{\mathbb{F}}_\ell^\times$, $y \in \tilde{\mathbb{F}}_\ell$, where $\tilde{x}$ denotes an integer whose image in $(\mathbb{Z}/m\ell\mathbb{Z})^\times \cong (\mathbb{Z}/\ell\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^\times$ is $(x, 1)$. Since ${}_dg_{t\ell'/m,v'/M\ell} \in \mathcal{O}(Y(m, M\ell) \otimes \mathbb{Z}[1/mM\ell])^\times$, the norm map $N_1 : K_2(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M\ell) \otimes \mathbb{Z}[1/mM\ell])$ sends $h^*({}_{c,d}z_{mL,ML}(R))$ to

$$\begin{aligned}&N_1(h^*({}_{c,d}z_{mL,ML}(R))) = \{N_1({}_cg_{s'/m\ell,u'/M\ell}),\ {}_dg_{t\ell'/m,v'/M\ell}\}\\ &\quad = \{\prod_{x\in\tilde{\mathbb{F}}_\ell^\times, y\in\tilde{\mathbb{F}}_\ell} {}_cg_{s'\tilde{x}/m\ell,\ u'/M\ell+y/\ell},\ {}_dg_{t\ell'/m,\ v'/M\ell}\} = \{\frac{{}_cg_{s/m,u/M}}{\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_cg_{s\ell'/m,\ u'/M\ell+y/\ell}},\ {}_dg_{t\ell'/m,\ v'/M\ell}\}.\end{aligned}$$

Here N_1 on the right hand side of the first equality is the norm map $N_1 : \mathcal{O}(Y(m\ell, M\ell) \otimes \mathbb{Z}[1/mM\ell])^\times \to \mathcal{O}(Y(m, M\ell) \otimes \mathbb{Z}[1/mM\ell])^\times$, and we see $N_1({}_c g_{s'/m\ell, u'/M\ell}) \in \mathcal{O}(Y(m, M(\ell)) \otimes \mathbb{Z}[1_m M\ell])^\times$.

The $Y(m, M(\ell))$ is the quotient of $Y(m, M\ell)$ by the action of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \tilde{w} \end{pmatrix}$ with $\tilde{w} \in \tilde{\mathbb{F}}_\ell^\times$, where $\tilde{w}$ denotes an integer such that whose image in $(\mathbb{Z}/M\ell\mathbb{Z})^\times \cong (\mathbb{Z}/\ell\mathbb{Z})^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$ is $(w, 1)$. The norm map $N_2 : K_2(Y(m, M\ell) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell])$ sends $N_1(h^*({}_{c,d}z_{mL,ML}(R)))$ to

$$\begin{aligned}
N_2(N_1(h^*({}_{c,d}z_{mL,ML}(R)))) &= \{\frac{{}_c g_{s/m,u/M}}{\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_c g_{s\ell'/m,\, u'/M\ell+y/\ell}},\ N_2({}_d g_{t\ell'/m,\, v'/M\ell})\} \\
&= \{\frac{{}_c g_{s/m,u/M}}{\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_c g_{s\ell'/m,\, u'/M\ell+y/\ell}},\ \frac{\prod_{w\in\tilde{\mathbb{F}}_\ell} {}_d g_{t\ell'/m,\, v'/M\ell+w/\ell}}{{}_d g_{t\ell'/m,\, v'\ell'/M}}\} \\
&= \{{}_c g_{s/m,\, u/M},\ \prod_{w\in\tilde{\mathbb{F}}_\ell} {}_d g_{t\ell'/m,\, v'/M\ell+w/\ell}\} + \{\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_c g_{s\ell'/m,\, u'/M\ell+y/\ell},\ {}_d g_{t\ell'/m,\, v'\ell'/M}\} \\
&\quad - \psi_\ell^*\{{}_c g_{s\ell'/m,\, u/M},\ {}_d g_{t\ell'/m,\, v/M}\} - \{{}_c g_{s/m,u/M},\ {}_d g_{t\ell'/m,\, v'\ell'/M}\}.
\end{aligned}$$

Here $\psi_\ell : Y(m, M(\ell)) \to Y(m, M)$ is as in 1.2.3. Let $\pi : Y(m, M(\ell)) \to Y(m, M)$ be the canonical projection. Recall that $\pi_* \psi_\ell^* = T^*(\ell)$. Now the norm map $\pi_* : K_2(Y(m, M(\ell)) \otimes \mathbb{Z}[1/mM\ell]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mM\ell])$ sends $N_2(N_1(h^*({}_{c,d}z_{mL,ML}(R))))$ to

$$\begin{aligned}
&\{{}_c g_{s/m,\, u/M},\ \pi_*(\prod_{w\in\tilde{\mathbb{F}}_\ell} {}_d g_{t\ell'/m,\, v'/M\ell+u/\ell})\} + \{\pi_*(\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_c g_{s\ell'/m,\, u'/M\ell+y/\ell}),\ {}_d g_{t\ell'/m,\, v\ell'/M}\} \\
&\quad - T^*(\ell)\{{}_c g_{s\ell'/m,\, u/M},\ {}_d g_{t\ell'/m,\, v/M}\} - (\ell+1)\{{}_c g_{s/m,u/M},\ {}_d g_{t\ell'/m,\, v\ell'/M}\}.
\end{aligned}$$

We can show

$$\pi_*(\prod_{w\in\tilde{\mathbb{F}}_\ell} {}_d g_{t\ell'/m,\, v'/M\ell+w/\ell}) = {}_d g_{t/m,\, v/M} \cdot ({}_d g_{t\ell'/m,\, v\ell'/M})^\ell,$$

$$\pi_*(\prod_{y\in\tilde{\mathbb{F}}_\ell} {}_c g_{s\ell'/m,\, u'/M\ell+y/\ell}) = {}_c g_{s/m,\, u/M} \cdot ({}_c g_{s\ell'/m,\, u\ell'/M})^\ell,$$

just as in the computation of π_* at the end of the proof of the case (2). Hence we obtain the result. □

Proposition 2.2.3. *Let $m, M \geq 1$ and assume $m + M \geq 5$. Let $L \geq 1$ and assume prime$(mL) \subset$ prime(M). Let $c, d \in \mathbb{Z}$ and assume $(cd, 6mML) = 1$. Let $R \in M(2, \mathbb{Z})$ and assume the image of R in $M(2, \mathbb{Z}/L\mathbb{Z})$ belongs to $GL(2, \mathbb{Z}/L\mathbb{Z})$. Let*

$$g : Y(m, ML) \to Y(m, L)$$

be the unique morphism which is compatible with $\mathcal{H} \to \mathcal{H}$; $\tau \mapsto L\tau$. Over $\mathbb{Z}[1/mML]$, this morphism is described as $(E, e_1, e_2) \mapsto (E', e_1', e_2')$ where E' is the quotient of E by the subgroup generated by Me_2, and e_i' is the image of e_i in E'. Then the norm map $g_* : K_2(Y(m, ML) \otimes \mathbb{Z}[1/mML]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mML])$ associated to g sends ${}_{c,d}z_{m,ML}(R)$ to ${}_{c,d}z_{m,M}(R)$.

Proof. We may and do assume that L is a prime number ℓ.

Then the morphism g factors as

$$Y(m, M\ell) \to Y(m, M(\ell)) \xrightarrow[\cong]{\varphi_\ell} Y(m(\ell), M) \to Y(m, M).$$

Here the arrows $\to$ are the canonical projections and φ_ℓ is the unique isomorphism which is compatible with $\tau \mapsto \ell\tau$ on the upper half plane.

Write $R = \begin{pmatrix} s & u \\ t & v \end{pmatrix}$.

The first arrow in the above factorization of g sends ${}_{c,d}z_{m,ML}(R)$ to

$$\begin{aligned}
&\sum_{x \in \mathbb{F}_\ell} \{{}_c g_{s/m, u/M\ell + ux/\ell},\ {}_d g_{t/m, v/M\ell + vx/\ell}\} \\
&= \sum_{x,y,z,w,w' \in \mathbb{F}_\ell} \{{}_c g_{s/m\ell + y/\ell, u/M\ell^2 + u\tilde{x}/\ell^2 + z/\ell},\ {}_d g_{t/m\ell + w/\ell, v/M\ell^2 + v\tilde{x}/\ell^2 + w'/\ell}\} = \varphi_\ell^*(\eta)
\end{aligned}$$

with

$$\eta = \sum_{x,y,w \in \mathbb{F}_\ell} \{{}_c g_{s/m\ell + y/\ell, u/M\ell + ux/\ell},\ {}_d g_{t/m\ell + w/\ell, v/M\ell + vx/\ell}\}.$$

Here the first $=$ is by the distribution properties of Siegel units. Here $\tilde{x}$ is a lifting of x to $\mathbb{Z}/\ell^2\mathbb{Z}$.

The norm map of $Y(m(\ell), M) \otimes \mathbb{Z}[1/mM\ell] \to Y(m, M) \otimes \mathbb{Z}[1/mM\ell]$ sends η to

$$\begin{aligned}
&\sum_{z \in \mathbb{F}_\ell} \eta \begin{pmatrix} 1 & mz \\ 0 & 1 \end{pmatrix} = \sum_{x,y,z,w \in \mathbb{F}_\ell} \{{}_c g_{s/m\ell + y/\ell, sz/\ell + u/M\ell + ux/\ell},\ {}_d g_{t/m\ell + w/\ell, tz/\ell + v/M\ell + vx/\ell}\} \\
&= \sum_{x,y,z,w \in \mathbb{F}_\ell} \{{}_c g_{s/m\ell + y/\ell, u/M\ell + x/\ell},\ {}_d g_{t/m\ell + w/\ell, v/M\ell + z/\ell}\} = \{{}_c g_{s/m, u/M},\ {}_d g_{t/m, v/M}\}.
\end{aligned}$$

$\square$

Proposition 2.2.4. *Let $m, M \geq 1$ and assume $m + M \geq 5$. Let $L \geq 1$ and assume $\mathrm{prime}(mL) \subset \mathrm{prime}(M)$. Let $c, d \in \mathbb{Z}$ and assume $(cd, 6mML) = 1$. Let $R \in M(2, \mathbb{Z})$ and assume the image of R in $M(2, \mathbb{Z}/L\mathbb{Z})$ belongs to $GL(2, \mathbb{Z}/L\mathbb{Z})$. Let*

$$f : Y(mL, M) \to Y(m, M)$$

be the unique morphism which is compatible with $\mathcal{H} \to \mathcal{H}$; $\tau \mapsto \tau/L$. Over $\mathbb{Z}[1/mML]$, this morphism is described as $(E, e_1, e_2) \mapsto (E', e_1', e_2')$ where E' is the quotient of E by the subgroup generated by Me_1, and e_i' is the the image of e_i in E'. Then the norm map $f_* : K_2(Y(mL, M) \otimes \mathbb{Z}[1/mML]) \to K_2(Y(m, M) \otimes \mathbb{Z}[1/mML])$ associated to f sends ${}_{c,d}z_{mL,M}(R)$ to

$$\Big(\prod_{\ell \in C} \Big(1 - T^*(\ell) \begin{pmatrix} 1/\ell & 0 \\ 0 & 1 \end{pmatrix}^*\Big)\Big)({}_{c,d}z_{m,M}(R)),$$

where C denotes the set of all prime divisors of L which do not divide m.

Proof. Consider the unique morphism $g : Y(mL, ML) \to Y(mL, M)$ which is compatible with $\mathcal{H} \to \mathcal{H}\ ; \tau \mapsto L\tau$ considered in Proposition 2.2.3 (m there is mL here). Then the composition $f \circ g$ coincides with the canonical projection $Y(mL, ML) \to Y(m, M)$. By Proposition 2.2.2 it is sufficient to prove that the norm map g_* associated to g sends ${}_{c,d}z_{mL,ML}(R) \in K_2(Y(mL, ML) \otimes \mathbb{Z}[1/mML])$ to ${}_{c,d}z_{mL,M}(R) \in K_2(Y(mL, M) \otimes \mathbb{Z}[1/mML])$. But this follows from Proposition 2.2.3 which we apply by taking mL as m there.

□

Proposition 2.2.5. *Assume $p|M$. Consider the map*

$$K_2(Y_1(Mp) \otimes \mathbb{Z}[1/M])[\mathbb{Z}/Mp\mathbb{Z}] \to K_2(Y_1(M) \otimes \mathbb{Z}[1/M])[\mathbb{Z}/M\mathbb{Z}],$$

where $K_2(Y_1(Mp) \otimes \mathbb{Z}[1/M]) \to K_2(Y_1(M) \otimes \mathbb{Z}[1/M])$ is the norm map and $\mathbb{Z}/Mp\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$ is given by the natural projection. For a divisor M' of M such that M/M' is prime to p, under this map, we have

$$\sum_{a \in \mathbb{Z}/Mp\mathbb{Z}, (a,p)=1} \{{}_cg_{0,a/Mp}, {}_dg_{0,1/M'p}\} \cdot [a] \mapsto T^*(p) \sum_{a \in \mathbb{Z}/M\mathbb{Z}, (a,p)=1} \{{}_cg_{0,a/M}, {}_dg_{0,1/M'}\} \cdot [a].$$

Proof. For each $a \in \mathbb{Z}/M\mathbb{Z}$ such that $(a, p) = 1$, take a lifting $\tilde{a}$ in $\mathbb{Z}/Mp\mathbb{Z}$ of a. The image of $\sum_{a \in \mathbb{Z}/Mp\mathbb{Z}, (a,p)=1} \{{}_cg_{0,a/Mp}, {}_dg_{0,1/M'p}\}[a]$ in $K_2(Y_1(Mp) \otimes \mathbb{Z}[1/Mp])[\mathbb{Z}/M\mathbb{Z}]$ is

$$\sum_{a \in \mathbb{Z}/M\mathbb{Z}, (a,p)=1} \{\prod_{x \in \tilde{\mathbb{F}}_p} {}_cg_{0,\tilde{a}/Mp+x/p}, {}_dg_{0,1/M'p}\}[a] = \sum_{a \in \mathbb{Z}/M\mathbb{Z}, (a,p)=1} \{\psi_p^*({}_cg_{0,a/M}), {}_dg_{0,1/M'p}\}[a].$$

Since $\psi_p^*({}_cg_{0,a/M}) \in K_1(Y(1, M(p)) \otimes \mathbb{Z}[1/Mp])$, the norm map $K_2(Y_1(Mp) \otimes \mathbb{Z}[1/Mp]) \to K_2(Y_1(1, M(p)) \otimes \mathbb{Z}[1/Mp])$ sends $\{\psi_p^*({}_cg_{0,a/M}), {}_dg_{0,1/M'p}\}$ to $\{\psi_p^*({}_cg_{0,a/M}), \mathcal{N}({}_dg_{0,1/M'p})\}$, where $\mathcal{N}$ is the norm map $\mathcal{O}(Y_1(Mp) \otimes \mathbb{Z}[1/Mp])^\times \to \mathcal{O}(Y(1, M(p)) \otimes \mathbb{Z}[1/Mp])^\times$. We have

$$\mathcal{N}({}_dg_{0,1/M'p}) = \prod_{y \in \tilde{\mathbb{F}}_p} \begin{pmatrix} 1 & 0 \\ 0 & 1 + yM \end{pmatrix}^* {}_dg_{0,1/M'p} = \prod_{y \in \tilde{\mathbb{F}}_p} {}_dg_{0,1/M'p+y/p} = \psi_p^*({}_dg_{0,1/M'}).$$

Hence the norm map $K_2(Y_1(Mp) \otimes \mathbb{Z}[1/Mp]) \to K_2(Y(1, M(p)) \otimes \mathbb{Z}[1/Mp])$ sends $\{\psi_p^*({}_cg_{0,a/M}), {}_dg_{0,1/M'p}\}$ to $\psi_p^*(\{{}_cg_{0,a/M}, {}_dg_{0,1/M'}\})$. Since $T^*(p) = \pi_*\psi_p^*$ where $\pi : Y(1, M(p)) \to Y_1(M)$ is the canonical projection, the norm map $K_2(Y_1(Mp) \otimes \mathbb{Z}[1/Mp]) \to K_2(Y_1(M) \otimes \mathbb{Z}[1/Mp])$ sends $\{\psi_p^*({}_cg_{0,a/M}), {}_dg_{0,1/M'p}\}$ to $T^*(p)\{{}_cg_{0,a/M}, {}_dg_{0,1/M'}\}$. □

Remark 2.2.6. These propositions have the evident versions for $z_{m,M}(R) \in K_2(Y(m, M) \otimes \mathbb{Z}[1/mM]) \otimes \mathbb{Q}$ without c, d, which can be proved in the same way as these propositions.

2.3 Beilinson elements on $Y(m, M)$ and zeta values

We review the relation of Beilinson elements to the values of L-functions of modular forms at $s = 1$ obtained in [22] section 9. The L-functions here are the complex L-functions, but the relation is given p-adically. We will use this relation in section 3.

Concerning the relations of Beilinson elements to these L-functions at $s = 0$ (through the regulator maps), see [2] section 5 (cf. also [22] section 2).

2.3.1. Zeta function $Z_{m,M}(s)$. We consider the following zeta function of $Y(m,M)$ which is an operator-valued function acting on $H^1(Y(m,M)(\mathbb{C}),\mathbb{C})$.

$$Z_{m,M}(s) = \sum_{n\geq 1,(n,m)=1} T^*(n)\begin{pmatrix}1/n & 0\\ 0 & 1\end{pmatrix}^* \cdot n^{-s} = \prod_{\ell \text{ prime}} P_\ell(\ell^{-s})^{-1}$$

where

$$P_\ell(u) = 1 - T^*(\ell)\begin{pmatrix}1/\ell & 0\\ 0 & 1\end{pmatrix}^* \cdot u + \begin{pmatrix}1/\ell & 0\\ 0 & 1/\ell\end{pmatrix}^* \cdot \ell u^2 \quad \text{if } \ell \not| mM,$$

$$P_\ell(u) = 1 - T^*(\ell)\begin{pmatrix}1/\ell & 0\\ 0 & 1\end{pmatrix}^* \cdot u \quad \text{if } \ell \not| m \text{ but } \ell | M,$$

$$P_\ell(u) = 1 \quad \text{if } \ell | m.$$

$Z_{m,M}(s)$ converges when $Re(s) > 2$, and has meromorphic analytic continuation to the whole complex plane $\mathbb{C}$ and is holomorphic on $\mathbb{C}$ except at $s = 2$.

2.3.2. When $m = M$, $Z_{M,M}(s)$ commutes with the action of $GL(2,\mathbb{Z}/M\mathbb{Z})$.

2.3.3. Let $m, M \geq 1, m+M \geq 5$. For $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, let $\{\alpha,\beta\}_{Y(m,M)} \in H_1(X(m,M)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z})$ be the class of the image in $X(m,M)(\mathbb{C})$ of the route on the upper half plane from α to β. Via the canonical isomorphism of Poincaré duality

$$H_1(X(m,M)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}) \cong H^1(Y(m,M)(\mathbb{C}),\mathbb{Z})(1),$$

we regard as

$$\{\alpha,\beta\}_{Y(m,M)} \in H^1(Y(m,M)(\mathbb{C}),\mathbb{Z})(1).$$

Here (1) is the Tate twist $\otimes_{\mathbb{Z}}\mathbb{Z}(1)$ where $\mathbb{Z}(1) = \mathbb{Z}\cdot 2\pi i \subset \mathbb{C}$. Note that the above isomorphism of Poincaré duality preserves the action of the complex conjugation (the complex conjugation acts also on $\mathbb{Z}(1)$ here).

2.3.4. Let $M_2(m,M)_{\mathbb{Q}}$ be the space of all modular forms of weight 2 on $X(m,M)\otimes\mathbb{Q}$. It is the space of differential forms on $X(m,M)\otimes\mathbb{Q}$ having (possibly) log poles at cusps.

We have the period map

$$\text{per} : M_2(m,M)_{\mathbb{Q}} \to H^1(Y(m,M)(\mathbb{C}),\mathbb{C}).$$

2.3.5. For a finite extension L of $\mathbb{Q}_p$ and for a finite dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $\mathrm{Gal}(\bar{L}/L)$ which is a de Rham representation of $\mathrm{Gal}(\bar{L}/L)$, let

$$\exp^* : H^1(L,V) \to D^0_{dR}(V) = D^0_{dR}(L,V)$$

be the dual exponential homomorphism. Here $H^1(L,-)$ denotes the continuous Galois cohomology of $\mathrm{Gal}(\bar{L}/L)$.

In the case $L = \mathbb{Q}_p$ and $V = H^1_{\text{ét}}(Y(m,M))(1)$, $D^0_{dR}(V) = M_2(m,M)_{\mathbb{Q}_p}$. Hence we have the dual exponential map

$$\exp^* : H^1(\mathbb{Q}_p, H^1_{\text{ét}}(Y(m,M))(1)) \to M_2(m,M)_{\mathbb{Q}_p}.$$

2.3.6. We consider the following composition.

$$\varprojlim_n K_2(Y(mp^n, Mp^n)) \xrightarrow{(1)} \varprojlim_n H^2(Y(mp^n, Mp^n), \mathbb{Z}_p(2)) \cong \varprojlim_n H^2(Y(mp^n, Mp^n), \mathbb{Z}_p(1))$$
$$\to H^2(Y(m, M), \mathbb{Z}_p(1)) \xrightarrow{(2)} H^1(\mathbb{Z}[1/mM], H^1_{\text{ét}}(Y(m, M))(1)) \to H^1(\mathbb{Q}_p, H^1_{\text{ét}}(Y(m, M)))$$
$$\xrightarrow{\exp^*} M_2(m, M)_{\mathbb{Q}_p}.$$

Here (1) is the Chern class map, the isomorphism id defined by $(\otimes\zeta_{p^n}^{\otimes -1})_n$ where ζ_{p^n} is the p^n-th root of 1 on $Y(mp^n, Mp^n)$ whose pull back to $\mathcal{H}$ via $\mathcal{H} \to Y_1(mp^n, Mp^n)(\mathbb{C})$ is $\exp(2\pi i/p^n)$, and (2) comes from the spectral sequence

$$H^i(\mathbb{Z}[1/mM], H^j_{\text{ét}}(Y(m, M))(1)) \Rightarrow H^{i+j}(Y(m, M) \otimes \mathbb{Z}[1/mM], \mathbb{Z}_p(1))$$

and the fact $H^2_{\text{ét}}(Y(m, M)) = 0$ (because $Y(m, M)$ is an affine curve).

We will denote this composite map also by $\exp^*$.

2.3.7. (The case $k = 2$ and $r = r' = 1$ of) [22] Theorem 4.6 and Theorem 9.5.

Let $c, d \in \mathbb{Z}$, $(cd, 6mM) = 1$. Let

$$\gamma = (c^2 - c \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^*)(d^2 - d \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^*) \cdot \frac{\{0, \infty\}_{Y(m,M)}}{2\pi i} \in H^1(Y(m, M)(\mathbb{C}), \mathbb{Z}).$$

Then the composite map $\exp^*$ in 2.3.6 sends $({}_{c,d}z_{mp^n,Mp^n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})_{n\ge 1}$ to an element ω of $M_2(m, M)_{\mathbb{Q}}$ (without $\otimes\mathbb{Q}_p$) such that

$$\mathrm{per}(\omega)^+ = Z_{m,M}(1) \cdot \gamma^+$$

in $H^1(Y(m, M)(\mathbb{C}), \mathbb{C})$. Here $(-)^+$ denotes the $\mathbb{C}$-linear map $H^1(Y(m, M)(\mathbb{C}), \mathbb{C}) \to H^1(Y(m, M)(\mathbb{C}), \mathbb{C})$ induced by $(1 + \iota)/2 : H^1(Y(m, M)(\mathbb{C}), \mathbb{Q}) \to H^1(Y(m, M)(\mathbb{C}), \mathbb{Q})$, where ι is the map induced by the complex conjugation $Y(m, M)(\mathbb{C}) \to Y(m, M)(\mathbb{C})$.

2.4 Beilinson elements on $Y_1(M) \otimes \mathbb{Q}(\zeta_m)$ and zeta values

We review modular symbols (Manin symbols) which are used in [51] (cf. [27]).

Let $M \ge 4$.

2.4.1. For $u, v \in \mathbb{Z}/M\mathbb{Z}$ such that $(u, v) = (1)$ as an ideal of $\mathbb{Z}/M\mathbb{Z}$, let

$$[u : v]_{Y_1(M)} \in H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1)$$

be as follows ([51] 3.1). Take liftings $\tilde{u}, \tilde{v} \in \mathbb{Z}$ of u, v and $x, y \in \mathbb{Z}$ such that $x\tilde{u} - y\tilde{v} = 1$. Let

$$[u : v]_{Y_1(M)} = \{-\frac{\tilde{v}}{xM}, -\frac{\tilde{u}}{yM}\}_{Y_1(M)}$$

where $\{-, -\}_{Y_1(M)}$ is as in 2.3.3. It can be shown that this is independent of the choices of $\tilde{u}, \tilde{v}, s, t$.

We have

$$[-u:-v]_{Y_1(M)} = [u:v]_{Y_1(M)}, \quad [v:u]_{Y_1(M)} = -[u:-v]_{Y_1(M)}.$$

In the case an integer $N \geq 1$ which is prime to p is fixed, we denote $[u:v]_{Y_1(Np^r)}$ by $[u:v]_r$ as in [51].

The following facts are known.

(1) The elements $[u:v]_{Y_1(M)}$ generate $H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ as a $\mathbb{Z}$-module.

(2) For $a \in (\mathbb{Z}/M\mathbb{Z})^\times$, the diamond operator

$$\langle a \rangle : H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1) \to H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1)$$

sends $[u:v]_{Y_1(M)}$ to $[au:av]_{Y_1(M)}$.

(3) The complex conjugation $\iota : H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1) \to H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1)$ sends $[u:v]_{Y_1(M)}$ to $-[v:u]_{Y_1(M)}$.

We consider Beilinson elements in $K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m])$ which are closely related to $[u:v]_{Y_1(M)}$.

2.4.2. Let $M \geq 4$, $m \geq 1$.

Let $u, v \in \mathbb{Z}/M\mathbb{Z}$, $(u, v) = (1)$. In the case $m = 1$, we assume $u \neq 0$, $v \neq 0$.

We define

$${}_{c,d}z_{1,M,m}(u, v) \in K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m]), \quad z_{1,M,m}(u, v) \in K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m]) \otimes \mathbb{Q}$$

as follows.

Take liftings $\tilde{u}, \tilde{v} \in \mathbb{Z}$ of u, v and take integers s, t such that $s\tilde{v} - t\tilde{u} = 1$. Let ${}_{c,d}z_{1,M,m}(u, v)$ be the image of ${}_{c,d}z_{m,Mm}\begin{pmatrix} s & \tilde{u} \\ t & \tilde{v} \end{pmatrix} \in K_2(Y(m, Mm) \otimes \mathbb{Z}[1/Mm])$ under the norm map $K_2(Y(m, Mm) \otimes \mathbb{Z}[1/Mm]) \to K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m])$. (The definition of $z_{1,M,m}(u, v)$ without c, d is similar.)

Then this is independent of the choices of $\tilde{u}, \tilde{v}, s, t$. We prove this.

Take also $\tilde{u}', \tilde{v}', s', t'$ and let $R' = \begin{pmatrix} s' & \tilde{u}' \\ t' & \tilde{v}' \end{pmatrix}$. Let

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} := R^{-1}R' \in SL(2, \mathbb{Z}).$$

Then

$$y = \tilde{u}'\tilde{v} - \tilde{u}\tilde{v}' \equiv \tilde{u}\tilde{v} - \tilde{u}\tilde{v} \equiv 0, \quad w = s\tilde{v}' - t\tilde{u}' \equiv s\tilde{v} - t\tilde{u} \equiv 1 \mod M.$$

We have

$${}_{c,d}z_{m,Mm}(R') = \begin{pmatrix} x & y/M \\ zM & w \end{pmatrix}^* {}_{c,d}z_{m,Mm}(R).$$

That is, these Beilinson elements are connected by the automorphism $\begin{pmatrix} x & y/M \\ zM & w \end{pmatrix}$ of $Y(m, Mm) \otimes \mathbb{Z}[1/Mm]$ over $Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m]$. Hence their norms to $Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m]$ coincide.

In particular,

$$ {}_{c,d}z_{1,M,1}(u, v) = \{{}_c g_{0,u/M}, {}_d g_{0,v/M}\}. $$

The (without c, d)-version is defined similarly.

2.4.3. For $a \in (\mathbb{Z}/m\mathbb{Z})^\times$, let σ_a be the element of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ characterized by $\sigma_a(\zeta_m) = \zeta_m^a$.

Proposition 2.4.4. *(Norm relation.) Let $L \geq 1$. Then the norm map $K_2(Y_1(M) \otimes \mathbb{Z}[1/MmL, \zeta_{mL}]) \to K_2(Y_1(M) \otimes \mathbb{Z}[1/MmL, \zeta_m])$ sends ${}_{c,d}z_{1,M,mL}(u, v)$ to*

$$ \prod_{\ell \in C} (1 - \sigma_\ell^{-1} \otimes T^*(\ell)) \cdot {}_{c,d}z_{1,M,m}(u, v), $$

where C denotes the set of all prime numbers which divide L but which do not divide m.

This follows from Proposition 2.2.2.

In the rest of this section 2.4, we consider the relation of these Beilinson elements and the value at $s = 1$ of L-functions of modular forms.

2.4.5. Zeta function $Z_{1,M,m}(s)$.

We consider the following operator-valued zeta function acting on

$$ \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{C}). $$

Here -1 in $\{\pm 1\}$ acts on $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]$ as the complex conjugation in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ and acts on $H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ as the map induced by the complex conjugation $Y_1(M)(\mathbb{C}) \to Y_1(M)(\mathbb{C})$. Let

$$ Z_{1,M,m}(s) = \sum_{(n,m)=1} \sigma_n^{-1} \otimes T^*(n) n^{-s} = \prod_{\ell:\text{prime}} P_\ell(\ell^{-s})^{-1}. $$

Here σ_n is as in 2.4.3, and

$$ P_\ell(u) = 1 - \sigma_\ell^{-1} \otimes T^*(\ell) u + \sigma_\ell^{-2} \otimes \langle \ell \rangle^{-1} \cdot \ell u^2 \quad \text{if } \ell \nmid Mm, $$

$$ P_\ell(u) = 1 - \sigma_\ell^{-1} \otimes T^*(\ell) u \quad \text{if } \ell | M \text{ but } \ell \nmid m, $$

$$ P_\ell(u) = 1 \quad \text{if } \ell | m. $$

This zeta function and the zeta function $Z_{M,m}(s)$ in 2.3.1 are compatible through the projection $Y(m, M) \to Y_1(M) \otimes \mathbb{Q}(\zeta_m)$.

2.4.6. We define an element

$$_{c,d}[u:v]_{1,M,m} \in \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$$

as follows:

$$_{c,d}[u:v]_{1,M,m} = c^2d^2 \otimes [u:v]_{Y_1(M)} - c^2\sigma_d \otimes [u:dv]_{Y_1(M)} - d^2\sigma_c \otimes [cu:v]_{Y_1(M)} + \sigma_{cd} \otimes [cu:dv]_{Y_1(M)}.$$

If $c \equiv d \equiv 1 \bmod M$, then

$$_{c,d}[u:v]_{1,M,m} = (c^2 - \sigma_c)(d^2 - \sigma_d) \otimes [u:v]_{Y_1(M)}.$$

2.4.7. We consider the period map

$$M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_{mp^n}) \to \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(X_1(M)(\mathbb{C}), \mathbb{C})\ ;$$
$$x \otimes y \mapsto \sum_{\sigma \in G} \sigma \otimes \sigma^{-1}(y)\mathrm{per}(x)$$

($G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, where per is the period map $M_2(M)_{\mathbb{Q}} \to H^1(Y_1(M)(\mathbb{C}), \mathbb{C})$).

This period map and the period map for $Y(M,N)$ in 2.3.4 are related as in the following commutative diagram.

$$\begin{array}{ccc} M_2(m, Mm)_{\mathbb{Q}} & \overset{\mathrm{per}}{\to} & H^1(Y(m, Mm)(\mathbb{C}), \mathbb{C})/\iota \\ \downarrow & & \downarrow \\ M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_{mp^n}) & \overset{\mathrm{per}}{\to} & \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{C}). \end{array}$$

Here $(-)/\iota$ denotes $(-)/(1-\iota)(-)$, and the right vertical arrow is induced from the trace map

$$H^1(Y(m, Mm)(\mathbb{C}), \mathbb{Z}) \to H^1((Y_1(M) \otimes \mathbb{Q}(\zeta_m))(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}} H^1(Y_1(M)(\mathbb{C}), \mathbb{Z}).$$

2.4.8. We consider the following composite map, which is defined similarly to 2.3.6.

$$\begin{aligned} &\varprojlim_n K_2(Y_1(M) \otimes \mathbb{Z}[\zeta_{mp^n}]) \to \varprojlim_n H^2(Y_1(M) \otimes \mathbb{Z}[\zeta_{mp^n}], \mathbb{Z}_p(2)) \\ &\quad \to \varprojlim_n H^1(\mathbb{Z}[1/m, \zeta_{mp^n}], H^1_{\text{ét}}(Y_1(M)(2)) \cong \varprojlim_n H^1(\mathbb{Z}[1/m, \zeta_{mp^n}], H^1_{\text{ét}}(Y_1(M)(1)) \\ &\quad \to H^1(\mathbb{Q}_p \otimes \mathbb{Q}(\zeta_m), H^1_{\text{ét}}(Y_1(M)(1))) \overset{\exp^*}{\to} M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p. \end{aligned}$$

We will denote this composite map also by $\exp^*$. This composite map $\exp^*$ is compatible with the composite map $\exp^*$ in 2.3.6 as in the following commutative diagram.

$$\begin{array}{ccc} \varprojlim_n K_2(Y(mp^n, Mmp^n) \otimes \mathbb{Z}[1/Mm]) & \overset{\exp^*}{\to} & M_2(m, Mm)_{\mathbb{Q}_p} \\ \downarrow & & \downarrow \\ \varprojlim_n K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_{mp^n}]) & \overset{\exp^*}{\to} & M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_m) \otimes \mathbb{Q}_p. \end{array}$$

Theorem 2.4.9. *Assume $p|M$, $p|m$. Then the composite map* $\exp^*$ *in 2.4.8 sends $({}_{c,d}z_{1,M,mp^n}(u,v))_{n\geq 0}$ to an element of $M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_m)$ (without $\otimes \mathbb{Q}_p$) whose period image in $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{C})$ coincides with $Z_{1,M,m}(1) \cdot {}_{c,d}[v:u]_{1,M,m}$.*

Proof. We use Proposition 2.2.4 and the zeta value formula at $s=1$ in section 2.3.

Take liftings $\tilde{u},\tilde{v}$ of u,v to $\mathbb{Z}$, respectively, such that $(\tilde{u},\tilde{v})=1$, and take $s,t\in\mathbb{Z}$ such that $s\tilde{v}-t\tilde{u}=1$.

Let $f':Y(Mm,Mm)\to Y(m,Mm)$ be the unique morphism which commutes with $\mathcal{H}\to\mathcal{H}\ ;\ \tau\mapsto\tau/M$, and let $f:Y(Mm,Mm)\to Y_1(M)\otimes\mathbb{Z}[\zeta_m]$ be the composition $Y(Mm,Mm)\xrightarrow{f'}Y(m,Mm)\to Y_1(M)\otimes\mathbb{Z}[\zeta_m]$ where the second arrow is the canonical projection. Let $\gamma\in H^1(Y(Mm,Mm)(\mathbb{C}),\mathbb{Z})$ be as in section 2.3 (we replace M and N in section 2.3 by Mm and Mm, respectively). Consider the commutative diagrams

$$\begin{array}{ccc}
\varprojlim_n K_2(Y(Mmp^n,Mmp^n)) & \xrightarrow{\exp^*} & M_2(Mm,Mm)_{\mathbb{Q}_p}\\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*}\\
\varprojlim_n K_2(Y_1(M)\otimes\mathbb{Z}[\zeta_{mp^n}]) & \xrightarrow{\exp^*} & M_2(M)_{\mathbb{Q}}\otimes\mathbb{Q}(\zeta_m)\otimes\mathbb{Q}_p,
\end{array}$$

$$\begin{array}{ccc}
M_2(Mm,Mm)_{\mathbb{Q}} & \xrightarrow{\mathrm{per}} & H^1(Y(Mm,Mm)(\mathbb{C}),\mathbb{C})/\iota\\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*}\\
M_2(M)_{\mathbb{Q}}\otimes\mathbb{Q}(\zeta_{mp^n}) & \xrightarrow{\mathrm{per}} & \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]\otimes_{\mathbb{Z}[\{\pm1\}]}H^1(Y_1(M)(\mathbb{C}),\mathbb{C}).
\end{array}$$

Here $-/\iota$ denotes $(-)/(1-\iota)(-)$. By 2.3.7, by the upper horizontal rows in these diagrams, we have

$$({}_{c,d}z_{Mmp^n,Mmp^n}\begin{pmatrix}1&0\\0&1\end{pmatrix})_{n\geq0}\mapsto\omega,\quad \omega\mapsto Z_{Mm,Mm}(1)\cdot\gamma$$

for some $\omega\in M_2(Mm,Mm)_{\mathbb{Q}}$, where γ is as in 2.3.7. Applying $f_*\begin{pmatrix}s&u\\t&v\end{pmatrix}^*$ to this, we have that by the lower horizontal rows of these diagrams,

$$\begin{aligned}
&f_*({}_{c,d}z_{Mmp^n,Mmp^n}\begin{pmatrix}s&\tilde{u}\\t&\tilde{v}\end{pmatrix})_{n\geq0}\mapsto f_*\begin{pmatrix}s&\tilde{u}\\t&\tilde{v}\end{pmatrix}^*\omega,\\
&f_*\begin{pmatrix}s&\tilde{u}\\t&\tilde{v}\end{pmatrix}^*\omega\mapsto f_*\begin{pmatrix}s&\tilde{u}\\t&\tilde{v}\end{pmatrix}^*Z_{Mm,Mm}(1)\cdot\gamma\\
&\qquad\qquad=(\prod_{\ell\in C}(1-\sigma_\ell^{-1}\otimes T^*(\ell)\ell^{-1}))\cdot Z_{1,M,m}(1)\cdot f_*\begin{pmatrix}s&\tilde{u}\\t&\tilde{v}\end{pmatrix}^*\gamma.
\end{aligned}$$

Here C denotes the set of prime numbers which divide M but which do not divide m. By Proposition 2.2.4, we have

$$f_*({}_{c,d}z_{Mmp^n,Mmp^n}\begin{pmatrix}s&\tilde{u}\\t&\tilde{v}\end{pmatrix})_{n\geq0}=(\prod_{\ell\in C}(1-\sigma_\ell^{-1}\otimes T^*(\ell)))\cdot({}_{c,d}z_{1,M,mp^n}(u,v))_{n\geq1}.$$

We have $\exp^*\circ(\sigma_\ell^{-1}\otimes T^*(\ell))=(\sigma_\ell^{-1}\otimes T^*(\ell))\ell^{-1}\circ\exp^*$ for the lower horizinral arrow $\exp^*$ of the first diagram.

Claim 1. For $\ell\in C$, the endomorphism $1-\sigma_\ell^{-1}\otimes T^*(\ell)\ell^{-1}$ of $M_2(M)_{\mathbb{Q}}\otimes\mathbb{Q}(\zeta_m)$ and the endomorphism $1-\sigma_\ell^{-1}\otimes T^*(\ell)\ell^{-1}$ of $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})]\otimes_{\mathbb{Z}[\{\pm1\}]}H^1(Y_1(M)(\mathbb{C}),\mathbb{Q})$ are isomorphisms.

This is because the eigen values of $T^*(\ell)$ are algebraic numbers whose complex absolute values are 1 or $\ell^{1/2}$.

By Claim 1, we have that the lower horizontal arrow of the first diagram sends $({}_{c,d}z_{1,M,mp^n}(u,v))_{n\geq 0}$ to an element ω' of $M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_m)$ whose image under the lower horizontal arrow of the second diagram is $Z_{1,M,m}(1) \cdot f_*(\gamma)$. It remains to prove that $f_*(\gamma) = -{}_{c,d}[u:v]_{1,M,m}/(2\pi i)$. We have

$$f_*\begin{pmatrix} s & \tilde{u} \\ t & \tilde{v} \end{pmatrix}^* \{0,\infty\}_{Y(Mm,Mm)} = f_*\begin{pmatrix} \tilde{v} & -\tilde{u} \\ -t & s \end{pmatrix}_* \{0,\infty\}_{Y(Mm,Mm)}$$
$$= f_*\{-\frac{\tilde{u}}{s}, -\frac{\tilde{v}}{t}\}_{Y(Mm,Mm)} = \{-\frac{\tilde{u}}{sM}, -\frac{\tilde{v}}{tM}\}_{Y_1(M)} = [v:u]_{Y_1(M)}.$$

By modifying this computation slightly, we obtain

$$f_*\begin{pmatrix} s & \tilde{u} \\ t & \tilde{v} \end{pmatrix}^* \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^* \{0,\infty\}_{Y(Mm,Mm)} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^* [cu:v]_{Y_1(M)} = \sigma_c \otimes [cu:v]_{Y_1(M)},$$

$$f_*\begin{pmatrix} s & \tilde{u} \\ t & \tilde{v} \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^* \{0,\infty\}_{Y(Mm,Mm)} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}^* [d^{-1}u:v]_{Y_1(M)}$$
$$= \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^* \langle d\rangle^* [d^{-1}u:v]_{Y_1(M)} = \sigma_d \otimes [u:dv]_{Y_1(M)}.$$

These prove $f_*(\gamma) = -{}_{c,d}[u:v]_{1,M,m}/(2\pi i)$.

In the above, we used the facts

$$\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^* = \sigma_c, \quad \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}^* = \langle d\rangle^*$$

for $Y_1(M)$. □

3 Beilinson elements in Galois cohomology

In this section 3, we construct a correspondence

modular symbol $\mapsto$ Beilinson element (in Galois cohomology)

having characterizations by zeta values. In section 3.1, we construct systems of Beilinson elements in the direction of cyclotomic extensions. In section 3.2, we consider the ordinary component of 3.1 and extend it also to the level direction. In section 3.3, for the ordinary component, we construct other systems of Beilinson elements in the direction of level changes of modular curves.

This section 3 is an improvement of Theorem 12.5 of [22] in which we considered the f-components of the above correspondences associated to individual Hecke-eigen cusp forms f. The improvement in this section is in the similar direction as in Ochiai [37] in which individual Hecke-eigen cusp forms are replaced by Hida families of cusp forms.

3.1 Cyclotomic direction

3.1.1. Fix $m, M \geq 1$. Assume $p \not| m$, $p|M$. Let

$$\Lambda(mp^\infty) := \varprojlim_{n\geq 1} \mathbb{Z}_p[(\mathbb{Z}/mp^n\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/m\mathbb{Z})^\times]].$$

Let $\mathbb{Q}(\zeta_{mp^\infty}) = \cup_n \mathbb{Q}(\zeta_{mp^n})$. We have the isomorphism

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q}) \cong \varprojlim_r (\mathbb{Z}/mp^r\mathbb{Z})^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/m\mathbb{Z})^\times,$$
$$\sigma_a \leftrightarrow a, \quad \sigma_a(\zeta_{mp^n}) = \zeta_{mp^n}^a.$$

Via this isomorphism, we identify $\Lambda(mp^\infty)$ with $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q})]]$.

We consider the $\Lambda(mp^\infty)$-modules

$$\mathfrak{Y} = \varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\text{ét}}(X_1(M))), \quad \tilde{\mathfrak{Y}} = \varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\text{ét}}(Y_1(M))).$$

3.1.2. In the case there is no danger of confusion, we will denote the image of ${}_{c,d}z_{1,M,mp^n}(u,v) \in K_2(Y_1(Mp^n) \otimes \mathbb{Z}[1/Mm, \zeta_{mp^n}])$ in $H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\text{ét}}(Y_1(M))(2))$ by the same notation ${}_{c,d}z_{1,M,mp^n}(u,v)$, and the image of $z_{1,M,mp^n}(u,v)$ in $H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\text{ét}}(Y_1(M))(2)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by the same notation $z_{1,M,mp^n}(u,v)$.

Proposition 3.1.3. *(1) $\mathfrak{Y}$ is $\Lambda(mp^\infty)$-torsion free.*

(2) The map $\mathfrak{Y} \to \tilde{\mathfrak{Y}}$ is injective.

(3) $\tilde{\mathfrak{Y}}$ is p-torsion free.

(4) The $\Lambda(mp^\infty)$-torsion of $\tilde{\mathfrak{Y}}$ is killed by $1 - a\sigma_a$ for any $a \in \mathbb{Z}$ such that $(a, mp) = 1$ and $a \equiv 1 \bmod M$.

Proof. (The following proofs of (1) and (3) are essentially the same as the proof of a similar result [22] Theorem 12.4 (2) for each Hecke eigen cusp form which is given in ibid. section 13.)

We prove (1). We prove first that $\mathfrak{Y}$ is p-torsion free. The proof of (3) is similar. Let $U = H^1_{\text{ét}}(X_1(M))$. The exact sequence $0 \to U \xrightarrow{p} U \to U/pU \to 0$ induces an exact sequence $A \to B \xrightarrow{p} B$ where $A = \varprojlim_n H^0(\mathbb{Z}[1/Mm, \zeta_{mp^n}], U/pU)$ and $B = \varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], U)$. By the fact U/pU is finite, we have easily $A = 0$.

Hence for the proof of (1), it is sufficient to prove that $\mathfrak{Y} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is $\Lambda(mp^\infty)$-torsion free.

Recall that we have a direct decomposition $H^1_{\text{ét}}(X_1(M)) = \oplus_i V(f_i)$ which is compatible with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and with the action of the subring $\mathfrak{h}(M)'_{\mathbb{Z}_p}$ of $\mathfrak{h}(M)_{\mathbb{Z}_p}$ generated by $T^*(n)$ for all n such that $(n, M) = 1$. Here for each i, f_i is a non-zero cusp form of weight 2 which is an eigen form for any $T^*(n)$ such that $(n, M) = 1$, the action of $\mathfrak{h}(M)'_{\mathbb{Z}_p}$ on $V(f_i)$ factors through a quotient L_i of $\mathfrak{h}(M)'_{\mathbb{Z}_p}$ such that L_i is a field of finite degree over $\mathbb{Q}_p$, $V(f_i)$ is two dimensional over L_i, and the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $V(f_i)$ over L_i, called the Galois representation associated to f_i, is irreducible. This shows that there is no $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbb{Q}_p$-subspace of $H^1_{\text{ét}}(X_1(M))$ on which the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is abelian.

Let $\Lambda(mp^\infty)^\sharp$ be $\Lambda(mp^\infty)$ with the following action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. If $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ with the image σ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q})$, τ acts on $\Lambda(mp^\infty)^\sharp$ as the multiplication by $\sigma^{-1} \in \Lambda(mp^\infty)$. Then we have $\mathfrak{Y} = H^1(\mathbb{Z}[1/Mm], U)$ where $U = H^1_{\text{ét}}(X_1(M)) \otimes_{\mathbb{Z}_p} \Lambda(mp^\infty)^\sharp$ with the diagonal action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let f be a non-zero-divisor of $\Lambda(mp^\infty)$. The exact sequence $0 \to U \xrightarrow{f} U \to U/fU \to 0$ induces an exact sequence $A \to B \xrightarrow{f} B$, where $A = H^0(\mathbb{Z}[1/Mm], U/fU)$ and $B = H^1(\mathbb{Z}[1/Mm], U)$. Hence it is sufficient to prove $H^0(\mathbb{Z}[1/Mm], U/fU) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$. We have $H^0(\mathbb{Z}[1/Mm], U/fU) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{Hom}_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}(V, H^1_{\text{ét}}(X_1(M)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ where $V = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda(mp^\infty)^\sharp/f\,\Lambda(mp^\infty)^\sharp, \mathbb{Q}_p)$. Since the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on V is abelian and $H^1_{\text{ét}}(X_1(M)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has no non-zero $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbb{Q}_p$-subspace on which the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is abelian, we have $H^0(\mathbb{Z}[1/Mm], U/fU) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$.

(2) follows from the fact $\mathfrak{Y}$ is p-torsion free and the Drinfeld-Manin splitting 1.9.3.

We prove (4). We prove the equivalent statement that the $\Lambda(mp^\infty)$-torsion of the Tate twist $\mathfrak{Y}(1)$ is killed by $1 - \sigma_a$ for any $a \in \mathbb{Z}$ such that $(a, mp) = 1$ and $a \equiv 1 \bmod M$. Let $U = (H^1_{\text{ét}}(Y_1(M))/H^1_{\text{ét}}(X_1(M)))(1)$. By (1) and (2), it is sufficient to prove that the $\Lambda(mp^\infty)$-torsion of $\varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], U)$ is killed by $1 - \sigma_a$ for any $a \in \mathbb{Z}$ such that $(a, mp) = 1$ and $a \equiv 1 \bmod M$. As a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over $\mathbb{Z}_p$, U is embedded in the free $\mathbb{Z}_p$-module whose base is the set of all cusps of $X_1(M) \otimes \bar{\mathbb{Q}}$. By the same method as in the proof of (1), we have that $\varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], U)$ is p-torsion free. Since the residue fields of all cusps of $Y_1(M)$ are subfields of $\mathbb{Q}(\zeta_M)$ (1.3.3), the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_M))$ on U is trivial. Since the kernel of $\varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], U) = H^1(\mathbb{Z}[1/Mm], U \otimes_{\mathbb{Z}_p} \Lambda(mp^\infty)^\sharp) \to H^1(\mathbb{Z}[1/Mm, \zeta_M], U \otimes_{\mathbb{Z}_p} \Lambda(mp^\infty)^\sharp) = U \otimes_{\mathbb{Z}_p} H^1(\mathbb{Z}[1/Mm, \zeta_M], \Lambda(mp^\infty)^\sharp)$ is killed by the non-zero integer $[\mathbb{Q}(\zeta_M) : \mathbb{Q}]$, and hence is zero, it is sufficient to prove that the $\Lambda(mp^\infty)$-torsion of $H^1(\mathbb{Z}[1/Mm, \zeta_M], \Lambda(mp^\infty)^\sharp)$ is killed by $1 - \sigma_a$. Let f be a non-zero-divisor of $\Lambda(mp^\infty)$. By using the exact sequence $0 \to \Lambda(mp^\infty)^\sharp \xrightarrow{f} \Lambda(mp^\infty)^\sharp \to \Lambda(mp^\infty)^\sharp/f\,\Lambda(mp^\infty)^\sharp \to 0$, we are reduced to proving that $H^0(\mathbb{Z}[1/Mm, \zeta_M], \Lambda(mp^\infty)^\sharp/f\,\Lambda(mp^\infty)^\sharp)$ is killed by $1 - \sigma_a$. Since $a \equiv 1 \bmod M$, there is $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_M))$ whose image in $\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^\infty})/\mathbb{Q})$ is σ_a. Since the multiplication by $\sigma_a \in \Lambda(mp^\infty)$ on $\Lambda(mp^\infty)^\sharp/f\,\Lambda(mp^\infty)^\sharp$ coincides with the action of τ^{-1}, it acts trivially on $H^0(\mathbb{Z}[1/Mm, \zeta_M], \Lambda(mp^\infty)^\sharp/f\,\Lambda(mp^\infty)^\sharp)$. Hence $1 - \sigma_a$ kills $H^0(\mathbb{Z}[1/Mm, \zeta_M], \Lambda(mp^\infty)^\sharp/f\,\Lambda(mp^\infty)^\sharp)$. □

Lemma 3.1.4. *The kernel of*

$$\exp^* : \tilde{\mathfrak{Y}}(1) \to \varprojlim_n M_2(M) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{mp^n}) \otimes \mathbb{Q}_p$$

coincides with the $\Lambda(mp^\infty)$*-torsion of* $\tilde{\mathfrak{Y}}(1)$.

Proof. By Drinfeld-Manin splitting (1.9.3), we have

$$H^1_{\text{ét}}(Y_1(M)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^1_{\text{ét}}(X_1(M)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \oplus H^1_{\text{ét}}(Y_1(M))/H^1_{\text{ét}}(X_1(M)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

as a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Furthermore, $H^1_{\text{ét}}(X_1(M)) = \oplus_i V(f_i)$ as in the proof of (1) of 3.1.3. The map $\exp^* : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \tilde{\mathfrak{Y}}(1) \to \varprojlim_n M_2(M) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{mp^n}) \otimes \mathbb{Q}_p$ is the direct sum of the maps

$$\exp^* : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], T_i(1)) \to \varprojlim_n D^0_{dR}(V(f_i)(1)) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{mp^n}),$$

where T_i is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbb{Z}_p$-lattice in $V(f_i)$, and the map

$$\exp^* : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], S) \to \varprojlim_n D^0_{dR}((H^1_{\text{ét}}(Y_1(M))/H^1_{\text{ét}}(X_1(M)))(1)) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{mp^n}),$$

where S is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbb{Z}_p$-lattice in $(H^1_{\text{ét}}(Y_1(M))/H^1_{\text{ét}}(X_1(M)))(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The former map is injective by [22]. By the fact the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_M))$ on S is trivial, the kernel of the latter map coincides with the $\Lambda(mp^\infty)$-torsion part. □

Theorem 3.1.5. *Let the notation be as in 3.1.1. Then, there exists a unique $\Lambda(mp^\infty)$-homomorphism*

$$\begin{aligned} z_{1,M,mp^\infty} &: \Lambda(mp^\infty) \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{Z}) \\ &\longrightarrow \tilde{\mathfrak{Y}} \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty)) = (\varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\acute{e}t}(Y_1(M))(1))) \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty)) \end{aligned}$$

characterized by the following (i) and (ii).

Here -1 in $\{\pm 1\}$ acts on $\Lambda(mp^\infty)$ as σ_{-1} and acts on $H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ by the complex conjugation on $Y_1(M)(\mathbb{C})$.

(i) For any $\gamma \in \Lambda(mp^\infty) \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ and for any c, d such that $(cd, 6Mm) = 1$ and $c \equiv d \equiv 1 \bmod M$, $(\sigma_c - c)(\sigma_d - d) z_{1,M,mp^\infty}(\gamma)$ belongs to the image of $\tilde{\mathfrak{Y}}$ in $\tilde{\mathfrak{Y}} \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty))$.

(ii) Consider the dual exponential map

$$\exp^*_{mp^n} : H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\acute{e}t}(Y_1(M))(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_{mp^n}) \otimes \mathbb{Q}_p$$

and the period map

$$\begin{aligned} per_{mp^n} : M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_{mp^n}) &\to \mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{C}) \; ; \\ x \otimes y &\mapsto \sum_{\sigma \in G} \sigma \otimes \sigma^{-1}(y) per(x) \end{aligned}$$

where $G = \mathrm{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q})$ and the last per is the period map $M_2(M)_{\mathbb{Q}} \to H^1(Y_1(M)(\mathbb{C}), \mathbb{C})$.

For $\gamma \in H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ and for $n \geq 1$, the image of $z_{1,M,mp^\infty}(\gamma) := z_{1,M,mp^\infty}(1 \otimes \gamma)$ in $M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_{mp^n}) \otimes \mathbb{Q}_p$ under $\exp^*$ is an element of $M_2(M)_{\mathbb{Q}} \otimes \mathbb{Q}(\zeta_{mp^n})$ whose image in $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q})] \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{C})$ under per_{mp^n} coincides with

$$Z_{1,M,mp^n}(1) \cdot \gamma.$$

Here $Z_{1,M,mp^n}(s)$ is the zeta function in section 2.4.

Here in (ii), $\exp^*$ is applied to $z_{1,M,mp^\infty}(\gamma)$ as $(c-\sigma_c)^{-1}(d-\sigma_d)^{-1} \circ \exp^* \circ (c-\sigma_c)(d-\sigma_d)$ ($c, d \in \mathbb{Z}$, $(cd, 6Mm) = 1$, $c \equiv d \equiv 1 \bmod M$, and $c, d \notin \{\pm 1\}$).

Proof. The relation of Beilinson elements and zeta values considered in section 2 plays an essential role in the proof.

The uniqueness follows from Lemma 3.1.4 and the injectivity of the period maps per_{mp^n}.

We prove the existence of z_{1,M,mp^∞}. Let $\gamma \in H^1(Y_1(M))(\mathbb{C}), \mathbb{Z})$. Write $\gamma = \sum_i a_i[u_i : v_i]$ with $a_i \in \mathfrak{H}(M)_{\mathbb{Z}}$ and $u_i, v_i \in \mathbb{Z}/M\mathbb{Z}$ such that $(u_i, v_i) = (1)$ as an ideal of $\mathbb{Z}/M\mathbb{Z}$. Take $c, d \in \mathbb{Z} - \{\pm 1\}$ such that $(cd, 6Mm) = 1$. Then $c - \sigma_c$ and $d - \sigma_d$ are non-zero-divisors in $\Lambda(mp^\infty)$. Let

$$z_{1,M,mp^\infty}(\gamma) = (c^2 - c\sigma_c)^{-1}(d^2 - d\sigma_d)^{-1} \sum_i a_i \mathrm{tw}_{-1}({}_{c,d}z_{1,M,mp^\infty}(u_i, v_i)) \in \tilde{\mathfrak{Y}}(1) \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty)).$$

Here tw_{-1} is the isomorphism $\mathfrak{Y}(2) \xrightarrow{\cong} \mathfrak{Y}(1)$; $x \mapsto x \otimes (\zeta_{p^n})_n^{-1}$ where $\zeta_{p^n} = \exp(2\pi i/p^n) \in \bar{\mathbb{Q}} \subset \mathbb{C}$. We show that $z_{1,M,mp^\infty}(\gamma)$ is independent of the choices of c, d and the presentation of γ as above. Assume we have other c', d' and another presentation $\gamma = \sum_i a'_i[u'_i : v'_i]$ with $a'_i \in \mathfrak{H}(M)_{\mathbb{Z}}$. Then by Theorem 2.4.9, $\mathrm{per} \circ \exp^*$ sends both $A = \sum_i (c^2 - c\sigma_c)^{-1}(d^2 - d\sigma_d)^{-1} \cdot a_i \cdot \mathrm{tw}_{-1}({}_{c,d}z_{1,M,mp^\infty}(u_i, v_i))$ and $B = \sum_i ((c')^2 - c'\sigma_{c'})^{-1}((d')^2 - (d')\sigma_{d'})^{-1} \cdot a'_i \cdot \mathrm{tw}_{-1}({}_{c',d'}z_{1,M,mp^\infty}(u'_i, v'_i))$ to $Z_{1,M,mp^n}(1) \cdot \gamma$. Since the period maps per_{mp^n} $(n \geq 1)$ are injective, the images of A and B in $\varprojlim_n M_2(M)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{mp^n}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ coincide. By Lemma 3.1.4, we have $A = B$.

We prove that $z_{1,M,m}(\iota\gamma) = \sigma_{-1}(z_{1,M,m}(\gamma))$ ($\gamma \in H^1_{\text{ét}}$ and ι is the complex conjugation). It is sufficient to prove $\sigma_{-1\,c,d}z_{1,M,m}(u, v) = {}_{c,d}z_{1,M,m}(u, -v)$. Take liftings $\tilde{u}, \tilde{v}$ of u, v to $\mathbb{Z}$ and take integers s, t such that $s\tilde{v} - t\tilde{u} = 1$. Then $\sigma_{-1}({}_{c,d}z_{1,M,m}(u, v))$ is the image under $K_2(Y(m, Mm) \otimes \mathbb{Z}[1/Mm]) \to K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m])$ of

$$\begin{aligned}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^* \{{}_c g_{s/m, \tilde{u}/mM}, {}_d g_{t/m, \tilde{v}/Mm}\} &= \{{}_c g_{-s/m, \tilde{u}/mM}, {}_d g_{-t/m, \tilde{v}/Mm}\} \\ &= \{{}_c g_{-s/m, \tilde{u}/mM}, {}_d g_{t/m, -\tilde{v}/Mm}\}.\end{aligned}$$

(The second $=$ here is by ${}_d g_{\alpha,\beta} = {}_d g_{-\alpha,-\beta}$ (2.1.3 (1)).) The image of the last element in $K_2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m])$ is ${}_{c,d}z_{1,M,m}(u, -v)$ since $(-s)(-\tilde{v}) - t\tilde{u} = 1$.

This completes the proof of the theorem. □

Remark 3.1.6. For each Hecke-eigen cusp form f, the "f-component " version of this theorem was obtained in [22] Theorem 12.5.

The next Propositions 3.1.7, 3.1.8 follow from the proof of the above theorem.

Proposition 3.1.7. *This homomorphism z_{1,M,mp^∞} commutes with the action of $\mathfrak{H}(M)_{\mathbb{Z}_p}$.*

Proposition 3.1.8. *Denote the homomorphism*

$$\begin{aligned} z_{1,M,mp^\infty} : \Lambda(mp^\infty) \otimes_{\mathbb{Z}[\{\pm 1\}]} H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1) \\ \longrightarrow \tilde{\mathfrak{Y}}(2) \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty)) \\ = (\varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\acute{e}t}(Y_1(M))(2))) \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty)) \end{aligned}$$

induced by z_{1,M,mp^∞} by the same later z_{1,M,mp^∞}.

(i) This map sends ${}_{c,d}[u:v]_{1,M,mp^\infty}$ to $(-{}_{c,d}z_{1,M,mp^n}(u,v))_n$.

(ii) Let $\gamma=[u:v]_{Y_1(M)}$. Then the image of $z_{1,M,mp^\infty}(1\otimes\gamma)$ in $H^1(\mathbb{Z}[1/Mm,\zeta_{mp^n}],H^1_{ét}(Y_1(M))(2))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ coincides with the image of $-z_{1,M,mp^n}(u:v)$ $(n\ge 1)$.

Proposition 3.1.9. *Assume $M|M'$, $m|m'$, prime(M) = prime(M'). Consider the norm map*

$$\begin{aligned} \mathit{Norm}\ :\ &(\varprojlim_n H^1(\mathbb{Z}[1/Mm',\zeta_{m'p^n}],H^1_{ét}(Y_1(M'))(1)))\otimes_{\Lambda(m'p^\infty)}Q(\Lambda(m'p^\infty))\\ &\longrightarrow (\varprojlim_n H^1(\mathbb{Z}[1/Mm',\zeta_{mp^n}],H^1_{ét}(Y_1(M))(1)))\otimes_{\Lambda(mp^\infty)}Q(\Lambda(mp^\infty)). \end{aligned}$$

Then for $\gamma\in H^1(Y_1(M')(\mathbb{C}),\mathbb{Z})$, we have

$$\mathit{Norm}(z_{1,M',m'p^\infty}(\gamma))=(\prod_\ell P_\ell(\ell^{-1}))\cdot z_{1,M,mp^\infty}(\bar\gamma),$$

where ℓ ranges over all prime numbers which divide m' but which do not divide m, $P_\ell(u)$ is as in 2.4.5, and $\bar\gamma$ is the image of γ in $H^1(Y_1(M)(\mathbb{C}),\mathbb{Z})$.

Proof. Since prime(M) = prime(M'), $Z_{1,M'}(s)$ and $Z_{1,M}(s)$ are compatible with the trace map $H^1(Y_1(M')(\mathbb{C}),\mathbb{C})\to H^1(Y_1(M)(\mathbb{C}),\mathbb{C})$. By taking $\mathrm{p}er\circ\exp^*$, Proposition 3.1.9 is seen from the relation Theorem 2.4.9 of Beilinson elements to the values of these zeta functions at $s=1$. □

3.2 Cyclotomic and level directions for the ordinary part

3.2.1. We consider the ordinary component of section 3.1, and consider the inverse limits for varying levels. Fix $N,m\ge 1$ which are prime to p.

As in section 1.5, let

$$H=\varprojlim_r H^1_{\mathrm{et}}(X_1(Np^r)\otimes\bar{\mathbb{Q}},\mathbb{Z}_p)^{\mathrm{ord}}\subset\tilde H=\varprojlim_r H^1_{\mathrm{et}}(Y_1(Np^r)\otimes\bar{\mathbb{Q}},\mathbb{Z}_p)^{\mathrm{ord}}.$$

In this section 3.2, let

$$\begin{aligned} \mathbf{\Lambda}&:=\varprojlim_{n,r}\mathbb{Z}_p[(\mathbb{Z}/mp^n\mathbb{Z})^\times\times(\mathbb{Z}/Np^r\mathbb{Z})^\times]\\ &\supset\Lambda:=\varprojlim_r\mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]=\mathbb{Z}_p[[\mathbb{Z}_p^\times\times(\mathbb{Z}/N\mathbb{Z})^\times]]. \end{aligned}$$

In $\mathbf{\Lambda}$, we identify the left $(\mathbb{Z}/mp^n\mathbb{Z})^\times$ with $\mathrm{Gal}(\mathbb{Q}(\zeta_{mp^n})/\mathbb{Q})$ by $a\mapsto\sigma_a$ (3.1.1), and the right $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ with the group of diamond operators. Then $\varprojlim_n H^1(\mathbb{Z}[1/mNp,\zeta_{mp^n}],\tilde H)$ is regarded as a $\mathbf{\Lambda}$-module.

We consider the $\mathbf{\Lambda}$-modules

$$\mathfrak{Z}=\varprojlim_n H^1(\mathbb{Z}[1/Nmp,\zeta_{mp^n}],H)\subset\tilde{\mathfrak{Z}}=\varprojlim_n H^1(\mathbb{Z}[1/Nmp,\zeta_{mp^n}],\tilde H).$$

3.2.2. Take $c \in \mathbb{Z}$ such that $c \equiv 1 \bmod p$, $c \not\equiv 1 \bmod p^2$, $(c, m) = 1$, $c \equiv 1 \bmod N$, take $d \in \mathbb{Z}$ having the same properties as c, and let

$$\lambda = (c - \sigma_c)(d - \sigma_d\langle d\rangle) \in \mathbf{\Lambda}.$$

Then λ is a non-zero-divisor of $\mathbf{\Lambda}$. The ideal of $\mathbf{\Lambda}$ generated by λ is independent of the choices of c, d, and hence the $\mathbf{\Lambda}$-submodule $\mathbf{\Lambda}\lambda^{-1}$ of $Q(\mathbf{\Lambda})$ which appears below is also independent of the choices of c, d.

Theorem 3.2.3. *Let the notation be as in 3.2.1. Let $\mathfrak{T}$ be the set of all elements of $\tilde{\mathfrak{Z}}$ which are killed by $1 - a\sigma_a$ for any integer a such that $(a, mp) = 1$ and $a \equiv 1 \bmod N$.*

Then there exists a unique $\mathbf{\Lambda}$-homomorphism

$$z_{1,Np^\infty,mp^\infty} : \mathbf{\Lambda} \otimes_{\Lambda[\{\pm 1\}]} \tilde{H} \longrightarrow (\tilde{\mathfrak{Z}}/\mathfrak{T})(1) \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}\lambda^{-1}$$

which induces the ordinary component of z_{1,Np^r,mp^∞} (3.1.5) for any $r \geq 1$.

Here -1 in $\{\pm 1\}$ acts on $\mathbf{\Lambda}$ as σ_{-1} in the Galois group and acts on $\tilde{H}$ by the complex conjugation.

Note that $\mathfrak{T}(1)$ is killed by λ.

For the proof of Theorem 3.2.3, we use the following propositions 3.2.4 and 3.2.6.

Proposition 3.2.4. *The action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ *on* $(H^1_{ét}(Y_1(Np^r))^{\mathrm{ord}}/H^1_{ét}(X_1(Np^r))^{\mathrm{ord}})(1)$ *is trivial.*

Proof. For $i \in \mathbb{Z}$ such that $0 \leq i \leq r$, let $S_i \subset H^1_{\text{ét}}(Y_1(Np^r))$ be the intersection of the kernels of the boundary maps $H^1_{\text{ét}}(Y_1(Np^r)) \to \mathbb{Z}_p(-1)$ associated to cusps of $X_1(Np^r)\otimes\bar{\mathbb{Q}}$ defined by $(c, d) \in P_{Np^r}$ (section 1.3) such that $c = \tilde{c} \bmod Np^r$ for some $\tilde{c} \in \mathbb{Z}$ satisfying $\mathrm{ord}_p(\tilde{c}) \leq i$. We have $S_0 \supset S_1 \supset \cdots \supset S_r = H^1_{\text{ét}}(X_1(Np^r))$.

By 1.3.3, the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ on $(H^1_{\text{ét}}(Y_1(Np^r))/S_0)(1)$ is trivial. Hence, for the proof of proposition, it is sufficient to prove that $H^1_{\text{ét}}(Y_1(Np^r))^{\mathrm{ord}} \cap S_0 = H^1_{\text{ét}}(X_1(Np^r))^{\mathrm{ord}}$. By using 1.3.5 (2) and (3) applied to $\ell = p$, we can prove the following (1) and (2).

(1) For $0 \leq i < r - 1$, we have $T^*(p)S_i \subset S_{i+1}$.

(2) $T^*(p)S_{r-1} \subset pS_{r-1} + S_r$.

By (1) and (2), the action of $T^*(p)$ on S_0/S_r is topologically nilpotent. Hence we have $H^1_{\text{ét}}(Y_1(Np^r))^{\mathrm{ord}} \cap S_0 = S_0^{\mathrm{ord}} = S_r^{\mathrm{ord}} = H^1_{\text{ét}}(X_1(Np^r))^{\mathrm{ord}}$. □

Lemma 3.2.5. *As an $\mathfrak{H}$-module, $\tilde{H}$ is generated by the elements $([p^{r-1}u : v]_r)_{r\geq 1} \in \tilde{H}$ ([51], Lemma 3.2), where v ranges over all integers which are prime to p and u ranges over integers such that $(u, v, N) = 1$.*

Proof. By the case $k = 2$ of Proposition 1.5.8, we have an isomorphism $\tilde{H}\otimes_\Lambda \mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \cong H^1_{\text{ét}}(Y_1(Np))^{\mathrm{ord}}$. By this isomorphism, the elements $([p^{r-1}u : v]_r)_{r\geq 1}$ are sent to $[u : v]_1$ which generate $H^1_{\text{ét}}(Y_1(Np))^{\mathrm{ord}}$. Hence we are done by Nakayama's lemma. □

Proposition 3.2.6. *Let* $A = \varprojlim_n H^1(\mathbb{Z}[1/Mm, \zeta_{mp^n}], H^1_{\acute{e}t}(Y_1(Np^r))^{\mathrm{ord}})$.

(1) For $\gamma \in H^1_{\acute{e}t}(Y_1(Np^r))^{\mathrm{ord}}$, $\lambda \cdot z_{1,Np^r,mp^\infty}(\gamma)$ *belongs to the image of* A *in* $A \otimes_{\Lambda(mp^\infty)} Q(\Lambda(mp^\infty))$.

(2) The $\Lambda(mp^\infty)$*-torsion part of* A *is killed by* $1 - a\sigma_a$ *for any integer* a *such that* $(a, mp) = 1$ *and* $a \equiv 1 \bmod N$.

Proof. (1) follows from 3.2.5 by the fact ${}_{c,d}[p^{r-1}u : v]_r = (c - \sigma_c)(d - \sigma_d\langle d\rangle)[p^{r-1}u : v]_r$ if $c \equiv d \equiv 1 \mod Np$.

By using 3.2.4, (2) can be proved in the same way as in the proof of Proposition 3.1.3 (4). □

3.2.7. Proof of the Theorem 3.2.3. This is reduced to the finite levels $M = Np^r$ by Theorem 3.1.5 and Proposition 3.2.6.

3.3 Elements $z^\sharp_{Np^\infty}(\gamma)$ in the level direction for the ordinary part

3.3.1. Fix $N \geq 1$ which is prime to p.

In this section 3.3, let

$$\Lambda = \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times]],$$

and regard it as the ring of diamond operators on $X_1(Np^r)$ $(r \geq 1)$.

In this section 3.3, we consider Beilinson elements in $H^1(\mathbb{Z}[1/Np], \tilde{H}(2)) \otimes_\Lambda Q(\Lambda)$.

Proposition 3.3.2. *(1)* $1 - T^*(p) \in \mathfrak{h}$ *is a non-zero-divisor.*

(2) The actions of $1 - T^*(p)$ *on* H, H_{sub}, H_{quo} *are injective.*

To prove this proposition, since H_{sub} is a free $\mathfrak{h}$-module and H_{quo} is a faithful $\mathfrak{h}$-module, it is sufficient to prove that the action of $1 - T^*(p)$ on H_{quo} is injective. By Proposition 1.8.1, we are reduced to the following proposition.

Proposition 3.3.3. $1 - Fr_p$ *is injective on* $H_{\mathrm{quo}}(1)$. *Equivalently,* $1 - \varphi$ *on* $D(H_{\mathrm{quo}}(1))$ *is injective.*

Proof. Take $k \geq 2$. Let $r \geq 1$. By Saito [48], all eigen values of the operator φ on $D_{pst}(V_k(X_1(Np^r))_{\mathbb{Q}_p})$ in $\bar{\mathbb{Q}}_p$ are algebraic numbers whose all conjugates over $\mathbb{Q}$ in $\mathbb{C}$ have absolute values in $\{p^{(k-2)/2}, p^{(k-1)/2}, p^{k/2}\}$. By Proposition 1.5.8, the quotient $D(H_{\mathrm{quo}}(1)) \otimes_\Lambda \mathbb{Q}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ of $D(H_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $\Lambda \to \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ is given through tw_{2-k} as in 1.5.8, is a quotient of $D_{pst}(V_k(X_1(Np^r))_{\mathbb{Q}_p}(k-1)$ as a space with an operator φ. Hence on this quotient, all eigen values of φ in $\bar{\mathbb{Q}}_p$ are algebraic numbers whose all complex conjugates over $\mathbb{Q}$ in $\mathbb{C}$ have absolute values in $\{p^{-k/2}, p^{(1-k)/2}, p^{(2-k)/2}\}$. Take $k \geq 3$. Then since these absolute values are not 1, $1-\varphi$ is injective on this quotient of $D(H_{\mathrm{quo}}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Hence $1 - \varphi$ is injective on $D(H_{\mathrm{quo}}(1))$. □

3.3.4. By 3.3.2 (1) and by 1.5.4, the kernel of $1 - T^*(p)$ on $H^1(\mathbb{Z}[1/Np], H(2))$ is contained in the Λ-torsion part.

Lemma 3.3.5. *(1)* $H^0(\mathbb{Z}[1/Np], \tilde{H}(2)) = H^0(\mathbb{Q}_p, \tilde{H}(2)) = 0$.

(2) $H^0(\mathbb{Z}[1/Np], (\tilde{H}/H)(2)) = H^0(\mathbb{Q}_p, (\tilde{H}/H)(2)) = 0$.

(3) The canonical maps $H^1(\mathbb{Z}[1/Np], H(2)) \to H^1(\mathbb{Z}[1/Np], \tilde{H}(2))$ *and* $H^1(\mathbb{Q}_p, H(2)) \to H^1(\mathbb{Q}_p, \tilde{H}(2))$ *are injective.*

Proof. (1) and (2). These are reduced to $H^0(\mathbb{Q}_p, H_{\rm sub}(2)) = 0$, $H^0(\mathbb{Q}_p, \tilde{H}_{\rm quo}(2)) = 0$, and $H^0(\mathbb{Q}_p, (\tilde{H}/H)(2)) = 0$, which can be seen easily.

(3) follows from (2). □

Concerning the Λ-torsion of $H^1(\mathbb{Z}[1/Np], \tilde{H}(2))$ and that of $H^1(\mathbb{Q}_p, \tilde{H}(2))$, we have

Proposition 3.3.6. *(1) The canonical map from the* Λ*-torsion part of* $H^1(\mathbb{Z}[1/Np], \tilde{H}(2))$ *to the* Λ*-torsion part of* $H^1(\mathbb{Q}_p, \tilde{H}(2))$ *is injective.*

(2) On the Λ*-torsion part of* $H^1(\mathbb{Q}_p, \tilde{H}(2))$ *(and hence on the* Λ*-torsion part of* $H^1(\mathbb{Z}[1/Np], \tilde{H}(2))$ *by (1)), the diamond operator* $\langle c\rangle$ *for* $c = (a, 1) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ *acts as* a^2.

Proof. We prove (1). Let x be a non-zero-divisor of Λ, and let $E = \tilde{H}(2)/x\tilde{H}(2)$. Since $\tilde{H}$ is $\mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$-flat (1.5.4), $x : \tilde{H}(2) \to \tilde{H}(2)$ is injective. Hence by (1), the kernel of the action of x on $H^1(\mathbb{Z}[1/Np], \tilde{H}(2))$ (resp. $H^1(\mathbb{Q}_p, \tilde{H}(2))$) is isomorphic to $H^0(\mathbb{Z}[1/Np], E)$ (resp. $H^0(\mathbb{Q}_p, E)$). Thus we are reduced to the evident fact that the map $H^0(\mathbb{Z}[1/Np], E) \to H^0(\mathbb{Q}_p, E)$ is injective.

We prove (2). Let L be the maximal unramified extension of $\mathbb{Q}_p$. We prove first that $H^1(\mathbb{Q}_p, \tilde{H}_{\rm quo}(2))$ is Λ-torsion free. By the exact sequence

$$0 \to H^1(\mathrm{Gal}(L/\mathbb{Q}_p), H^0(L, \tilde{H}_{\rm quo}(2))) \to H^1(\mathbb{Q}_p, \tilde{H}_{\rm quo}(2)) \to H^1(L, \tilde{H}_{\rm quo}(2))$$

and by $H^0(L, \tilde{H}_{\rm quo}(2)) = \tilde{H}_{\rm quo}(1)\hat{\otimes} H^0(L, \mathbb{Z}_p(1)) = 0$ ($\hat{\otimes}$ is the topological tensor product defined as in 1.7.3), it is sufficient to prove that $H^1(L, \tilde{H}_{\rm quo}(2))$ is Λ-torsion free. But $H^1(L, \tilde{H}_{\rm quo}(2)) = \tilde{H}_{\rm quo}(1)\hat{\otimes} H^1(L, \mathbb{Z}_p(1))$, and this is Λ-torsion free.

We next prove that on the Λ-torsion part of $H^1(\mathbb{Q}_p, H_{\rm sub}(2))$, $\langle c\rangle$ ($c = (a, 1) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$) acts as a^2. Let $R = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ (we denote the group element of R corresponding to $a \in \mathbb{Z}_p^\times$ by $[a]$) and let $R^\sharp$ be R with the following action of $\mathrm{Gal}(\bar{L}/L)$: $\sigma \in \mathrm{Gal}(\bar{L}/L)$ acts on $R^\sharp$ by $[\kappa(\sigma)]$, where κ denotes the cyclotomic character. Let $R \to \Lambda$ be a ring homomorphism $[a] \mapsto \langle (a, 1)\rangle$. Since $\sigma \in \mathrm{Gal}(\bar{L}/L)$ acts on $H_{\rm sub}$ as $\langle (\kappa(\sigma), 1)\rangle$ (1.7.14 (4)) and since $H_{\rm sub}$ is a finitely generated projective R-module, we have $H^i(L, H_{\rm sub}(2)) \cong H_{\rm sub} \otimes_R H^i(L, R^\sharp(2))$ for any $i \in \mathbb{Z}$, where R acts on $H_{\rm sub}$ via $R \to \Lambda$. By the exact sequence

$$0 \to H^1(\mathrm{Gal}(L/\mathbb{Q}_p), H^0(L, H_{\rm sub}(2))) \to H^1(\mathbb{Q}_p, H_{\rm sub}(2)) \to H^1(L, H_{\rm sub}(2))$$

and by $H^0(L, H_{\rm sub}(2)) = H_{\rm sub} \otimes_R H^0(L, R^\sharp(2)) = 0$, it is sufficient to prove that on the Λ-torsion part of $H^1(L, H_{\rm sub}(2))$, $\langle c\rangle$ acts by a^2. We have

$$H^1(L, H_{\rm sub}(2)) = H_{\rm sub} \otimes_R H^1(L, R^\sharp(2)) \cong H_{\rm sub} \otimes_R \varprojlim_n H^1(L(\zeta_{p^n}), \mathbb{Z}_p(2)),$$

and the action of $\langle c\rangle$ on these modules coincide with the action of $1\otimes\sigma_a$ on the last module, where $\sigma_a\in\mathrm{Gal}(L(\zeta_{p^\infty})/L)$, $\sigma_a(\zeta_{p^n})=\zeta_{p^n}^a$. Note $\varprojlim_n H^1(L(\zeta_{p^n}),\mathbb{Z}_p(2))\cong\varprojlim_n(L(\zeta_{p^n})^\times)^\wedge(1)$, where $(-)^\wedge$ denotes the p-adic completion. By local Iwasawa theory,

$$(\text{the } R\text{-torsion of } \varprojlim_n(L(\zeta_{p^n})^\times)^\wedge(1))\cong\mathbb{Z}_p(2)$$

on which the action of σ_a coincides with the action of a^2. This proves (3). □

3.3.7. Let $H^1_{\text{ét}}(Y_1(Np^r))_0$ be the part of $H^1_{\text{ét}}(Y_1(Np^r))$ generated by $[u:v]_r$ $(u,v\in\mathbb{Z}/Np^r\mathbb{Z},\ (u,v)=(1))$ satisfying $u\neq0$, $v\neq0$. It coincides with the part of $H^1_{\text{ét}}(Y_1(Np^r))$ consisting of all elements whose boundaries at 0-cusps are zero.

All Hecke operators $T^*(n)$ preserve this part (this is seen by 1.3.5). In particular, the ordinary part $H^1(Y_1(Np^r))_0^{\text{ord}}$ of $H^1(Y_1(Np^r))_0$ is defined. Note that $T^*(p)$ on $H^1_{\text{ét}}(Y_1(Np^r))$ commutes with the identity maps on 0-cusps via the boundary map (this is seen from 1.3.5).

For $r\geq1$, we have a commutative diagram

$$\begin{array}{ccc}
H^1_{\text{ét}}(Y_1(Np^{r+1}))(1) & \xrightarrow{\partial} & \mathbb{Z}_p[\{\text{0-cusps}\}]=\mathbb{Z}_p[(\mathbb{Z}/Np^{r+1}\mathbb{Z})^\times/\{\pm1\}] \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(Y_1(Np^{r}))(1) & \xrightarrow{\partial} & \mathbb{Z}_p[\{\text{0-cusps}\}]=\mathbb{Z}_p[(\mathbb{Z}/Np^{r}\mathbb{Z})^\times/\{\pm1\}],
\end{array}$$

where the 0-cusp associated to $\infty_M(a,0)$ $(a\in(\mathbb{Z}/M\mathbb{Z})^\times)$ for $M=Np^r,Np^{r+1}$ corresponds to the class of the group element $[a]$ of $\mathbb{Z}_p[(\mathbb{Z}/M\mathbb{Z})^\times]$. Hence we can take

$$\tilde{H}_0:=\varprojlim_r H^1_{\text{ét}}(Y_1(Np^r))_0^{\text{ord}}.$$

Lemma 3.3.8. *As a Λ-module, $\tilde{H}_0(1)$ is generated by the elements $([p^{r-1}u:v]_r^{\text{ord}})_{r\geq1}$, where u and v range over integers such that $(u,v,N)=1$, $p\nmid v$, and $u\not\equiv0 \bmod Np$.*

Proof. This is a variant of Lemma 3.2.5, and is proved in the same way. □

Theorem 3.3.9. *There is a unique $\mathfrak{H}$-homomorphism*

$$z^\sharp_{Np^\infty}:\tilde{H}_0(1)\to H^1(\mathbb{Z}[1/Np],\tilde{H}(2))\otimes_\Lambda Q(\Lambda)$$

having the following properties (i)–(iii).

(i) For any element x in the image of $z^\sharp_{Np^\infty}$ and for any integer c such that $c\equiv 1 \bmod Np$, $p\cdot(c^2-\langle c\rangle)x$ belongs to the image of the canonical map $H^1(\mathbb{Z}[1/Np],\tilde{H}(2))\to H^1(\mathbb{Z}[1/Np],\tilde{H}(2))\otimes_\Lambda Q(\Lambda)$.

That is, the above map is in fact a map into

$$H^1(\mathbb{Z}[1/Np],\tilde{H}(2))\otimes_\Lambda\Lambda\,\mu^{-1},$$

where

$$\mu=p((1+Np)^2-\langle1+Np\rangle)\in\Lambda\,.$$

(ii) For any $r \geq 1$, in $H^1(\mathbb{Z}[1/Np], H^1_{ét}(Y_1(Np^r))(2)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, the image $(1 - T^*(p)) z^{\sharp}_{Np^\infty}(\gamma)$ coincides with the image of $z_{1,Np^r,p^\infty}(\gamma)$.

(iii) For each $r \geq 1$, the following diagram is commutative.

$$\begin{array}{ccc} \tilde{H}_0(1) & \xrightarrow{z^{\sharp}_{Np^\infty}} & H^1(\mathbb{Z}[1/Np], \tilde{H}(2)) \otimes_\Lambda \Lambda\, \mu^{-1} \\ \downarrow & & \downarrow \\ \oplus_{(a,b)\in P_{Np^r}} \mathbb{Z}_p & \xrightarrow{f} & \oplus_{(a,b)\in P_{Np^r}} H^1(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \end{array}$$

Here P_{Np^r} is as in 1.3.1, f is the map explained in 3.3.10 below, and the vertical arrows are induced from

$$\tilde{H}(1) \to H^1_{ét}(Y(Np^r))(1) \to H^1(\mathbb{Z}[1/Np, \zeta_{Np^r}][[q^{1/Np^r}]][q^{-1}], \mathbb{Z}_p(1)) \to \mathbb{Z}_p$$

associated to $\infty_{Np^r}(a,b)$. The last arrow here is the boundary map which sends the Kummer class of q^{1/Np^r} to 1, and f is as below.

3.3.10. The map f in the diagram in the above theorem is as follows.

For integers x, B such that $B \geq 1$, let $\zeta_{\equiv x \bmod B}(s)$ be the partial Riemann zeta function which is given as $\sum_{n\geq 1, n\equiv x \mod B} n^{-s}$ when $Re(s) > 1$ and which is extended to $\mathbb{C}$ as a meromorphic function holomorphic except at $s = 1$. Note that $\zeta_{\equiv x \bmod B}(r) \in \mathbb{Q}$ for any integer $r \leq 0$.

Then f is the homomorphism which sends 1 at $\infty_{Np^r}(a,b)$ ($a \in \mathbb{Z}/Np^r\mathbb{Z}$, $b \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$) to

$$\sum_{x,y} (\{1 - \zeta^y_{Np^r/A}\} \otimes \zeta_{\equiv x \bmod A}(-1) \text{ at } \infty_{Np^r}(ax, by))$$

where A is the positive divisor of Np^r such that a is Np^r/A times a unit of $\mathbb{Z}/Np^r\mathbb{Z}$, x ranges over all elements of $\mathbb{Z}/A\mathbb{Z}$, and y ranges over all invertible elements of $(\mathbb{Z}/Np^r\mathbb{Z})/(ax)$.

Lemma 3.3.11. *Let $u, v \in \mathbb{Z}/Np^r\mathbb{Z}$, $(u,v) = (1)$. Let R (resp. S) be the positive divisor of Np^r such that $(u) = (R)$ (resp. $(v) = (S)$) in $\mathbb{Z}/Np^r\mathbb{Z}$. Then, the boundary of $[u : v]_r$ is given by*

$$\infty_{Np^r}(\frac{Np^r}{R} \cdot v', u') - \infty_{Np^r}(\frac{Np^r}{S} \cdot u'', v''),$$

where u', v', u'', v'' denote elements of $\mathbb{Z}/Np^r\mathbb{Z}$ such that $(u/R)u' \equiv 1 \bmod Np^r/R$, $vv' \equiv 1 \bmod R$, $uu'' \equiv 1 \bmod S$, $(v/S)v'' \equiv 1 \bmod Np^r/S$.

Lemma 3.3.12. *At the cusp of $Y_1(Np^r) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{Np^r})$ associated to $\infty_{Np^r}(a,b)$, the boundary of $z_{1,Np^r,1}(u,v)$ coincides with $\{1 - \zeta^{bu}_{Np^r}\} \otimes \zeta_{\equiv av \bmod Np^r}(-1)$ if $au = 0$, with $\{1 - \zeta^{bv}_{Np^r}\} \otimes \zeta_{\equiv au \bmod Np^r}(-1)$ if $av = 0$, and with 0 otherwise.*

Proof. Take $c, d \in \mathbb{Z}$ such that $bc - ad = 1$. Then the boundary in problem coincides with the boundary at ∞ of

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix}^* z_{1,Np^r,1}(u,v) = \{g_{au/Np^r, bu/Np^r}, g_{av/Np^r, bv/Np^r}\}.$$

The lemma follows from this by explicit computation of this boundary. $\square$

3.3.13. Proof of Theorem 3.3.9.

The uniqueness follows from the exact sequence

$$0 \to H^1(\mathbb{Z}[1/Np], H(2)) \to H^1(\mathbb{Z}[1/Np], \tilde{H}(2)) \to H^1(\mathbb{Z}[1/Np], (\tilde{H}/H)(2))$$

and the fact that the kernel of $1-T^*(p)$ on $H^1(\mathbb{Z}[1/Np], H(2))$ is contained in the Λ-torsion of $H^1(\mathbb{Z}[1/Np], H(2))$.

We prove the existence of the homomorphism. By Lemmas 3.3.11, 3.3.12, we are reduced to showing the following. For any $\gamma \in \tilde{H}_0(1)$, we can find $x \in H^1(\mathbb{Z}[1/Np], \tilde{H}(2)) \otimes_\Lambda \Lambda\,\mu^{-1}$ such that for any $r \geq 1$, if x_r and y_r denote the images of x and $z_{1,Np^r,p^\infty}(\gamma)$ in $H^1(\mathbb{Z}[1/Np], H^1_{\text{ét}}(Y(Np^r))(2)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, respectively, then

$$y_r = (1 - T^*(p))x_r.$$

By Lemma 3.3.8, it is sufficient to check this for $\gamma = ([p^{r-1}u : v]_r)_r$ where $(u, v, N) = 1$, $(v, p) = 1$, and $u \not\equiv 0 \bmod Np$. Taking $c = d = 1 + Np$, x_r in this case is given by

$$x_r = ({}_{c,d}z_{1,Np^r,1}(p^{r-1}u, v) \otimes (c^2 - 1)^{-1}(d^2 - \langle d \rangle)^{-1})_r.$$

Proposition 3.3.14. *The map $z^\sharp_{Np^\infty}$ of Theorem 3.3.9 sends $H(1)$ into $H^1(\mathbb{Z}[1/Np], H(2)) \otimes_\Lambda \Lambda\,\mu^{-1}$.*

This follows from (iii) in Theorem 3.3.9.

From the proof of Theorem 3.3.9, we obtain the following proposition.

Proposition 3.3.15. *The map $z^\sharp_{Np^\infty}$ of Theorem 3.3.9 is compatible with a homomorphism*

$$H^1_{\acute{e}t}(Y_1(Np^r))_0 \to H^1(\mathbb{Z}[1/Np], H^1_{\acute{e}t}(Y_1(Np^r))(2)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

which sends $[u : v]_r$ to $-z_{1,Np^r,1}(u, v)$ and sends ${}_{c,d}[u : v]_{1,Np^r}$ to $-{}_{c,d}z_{1,Np^r,1}(u, v)$.

Remark 3.3.16. It may be possible that the map

modular symbol $\mapsto$ Beilinson element

in this section can be obtained also from the work of Goncharov [14] on motivic cohomology. In this section, we proved that this map commutes with the actions of Hecke operators by using the relations between Beilinson elements and zeta values. (This commutativity will be the key for the proof of Conjecture 5.8 in [51] given in section 5.2.) It may be possible that the commutativity is proved also by a motivic method of [14].

4 p-adic L-functions in two variables

There are two kinds of p-adic L-functions in two variables for modular forms. One is the L-function in two variables of Mazur-Kitagawa ([30], [24]). The other is related to Beilinson elements ([12], [37]) and has the shape of a product of two Λ-adic Eisenstein series ([12], [43]). Both play important roles in this paper. We review and study these two, and describe (Theorem 4.4.3) the relation between them.

4.1 Classical p-adic zeta functions

4.1.1. We review the classical theory of p-adic zeta functions. See [59] for example.

Let $N \geq 1$ be an integer which is prime to p, and let $\Lambda = \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times] = \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times]]$.

Then for an integer $i \leq 0$, there is a unique element

$$^i\xi = {}^i\xi_{Np^\infty} \ \ (\text{resp. } {}^i\xi^* = {}^i\xi^*_{Np^\infty}) \ \in Q(\Lambda)$$

satisfying the following condition. Let $r \geq 0$ and let $\psi : (\mathbb{Z}/Np^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be a homomorphism. Then ${}^i\xi$ (resp. ${}^i\xi^*$) belongs to the local ring $\Lambda_\mathfrak{p}$ of Λ at the prime ideal $\mathfrak{p} = \mathrm{Ker}\,(\psi : \Lambda \to \bar{\mathbb{Q}}_p)$ and the image of ${}^i\xi$ (resp. ${}^i\xi^*$) under $\psi : \Lambda_\mathfrak{p} \to \bar{\mathbb{Q}}_p$ coincides with the complex zeta value $L_{(Np)}(i, \psi^{-1})$ (resp. $L_{(Np)}(i, \psi)$). Here $L_{(Np)}(s, -)$ denotes the Dirichlet L-function without Euler factors at the prime divisors of Np.

Furthermore, for $i \leq 0$, ${}^i\xi$ coincides with the i-th Tate twist of ${}^0\xi$ and ${}^i\xi^*$ coincides with the transpose of ${}^i\xi$. That is, ${}^i\xi$ coincides with the image of ${}^0\xi$ under the ring isomorphism $\Lambda \xrightarrow{\cong} \Lambda$; $[(a, b)] \mapsto a^i[(a, b)]$ $(a \in \mathbb{Z}_p^\times,\ b \in (\mathbb{Z}/N\mathbb{Z})^\times)$, and ${}^i\xi^*$ coincides with the image of ${}^i\xi$ under the ring isomorphism $\Lambda \xrightarrow{\cong} \Lambda$; $[c] \mapsto [c^{-1}]$ $(c \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times)$.

For any integer i, let ${}^i\xi \in Q(\Lambda)$ be the i-th Tate twist of ${}^0\xi$, and let ${}^i\xi^* \in Q(\Lambda)$ be the transpose of ${}^i\xi$.

4.1.2. The ξ in Introduction of this paper is as follows. Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ and Λ^- be as in Introduction. Then $\Lambda^- \xrightarrow{\cong} \prod_i \Lambda_{(i)}$ where i ranges over all odd elements of $\mathbb{Z}/(p-1)\mathbb{Z}$ and $\Lambda_{(i)} = \Lambda\,/([\omega(a)] - \omega(a)^i\ ;\ a \in (\mathbb{Z}/p\mathbb{Z})^\times)$. Here ω is the Teichmüller lifting. The image of ${}^0\xi \in Q(\Lambda)$ in $Q(\Lambda_{(i)})$ belongs to $\Lambda_{(i)}$ unless $i \neq 1$, and the image of ${}^0\xi$ in $Q(\Lambda_{(1)})$ has the shape f/g where f is an invertible element of $\Lambda_{(1)}$ and g is a generator of the kernel of the ring homomorphism $\mathrm{Ker}\,(\Lambda_{(1)} \to \mathbb{Z}_p)$ induced from $\Lambda \to \mathbb{Z}_p$; $[a] \mapsto a$ $(a \in \mathbb{Z}_p^\times)$.

In Introduction, ξ denotes an element of Λ^- whose image in $\Lambda_{(i)}$ for an odd element i of $\mathbb{Z}/(p-1)\mathbb{Z}$ is the image of ${}^0\xi$ if $i \neq 1$, and is an invertible element of $\Lambda_{(1)}$ if $i = 1$. (We can choose this invertible element freely.)

4.1.3. In sections 6–11, ξ will denote the image of ${}^{-1}\xi$ in a certain quotient of $Q(\Lambda)$ (see 6.1.6).

4.2 Coleman power series

4.2.1. Let T be a pro-p abelian group endowed with a continuous unramified action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $D(T)$ be as in 1.7.4.

We discuss the homomorphism of Iwasawa-Coates-Wiles-Coleman

$$\mathrm{Col} : \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), T(1)) \to S^{-1}(D(T)[[\mathbb{Z}_p^\times]])$$

for a certain multiplicative subset S of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ consisting of non-zero-divisors, and its variant

$$\mathrm{Col}^\flat : H^1(\mathbb{Q}_p, T(1)) \to D(T).$$

4.2.2. First we consider $\mathrm{Col}^\flat$. Let L be the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{ur}} \subset \bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. The valuation ring O_L of L is $W(\bar{\mathbb{F}}_p)$. We have the exact sequence $0 \to H^1(\mathbb{F}_p, H^0(L, T(1)) \to H^1(\mathbb{Q}_p, T(1)) \to H^0(\mathbb{F}_p, H^1(L, T(1)) \to 0$, and $H^0(L, T(1)) = T\hat{\otimes}H^0(L, \mathbb{Z}_p(1)) = 0$, $H^1(L, T(1)) = T\hat{\otimes}H^1(L, \mathbb{Z}_p(1)) \cong T\hat{\otimes}L^\times$. Hence $H^1(\mathbb{Q}_p, T(1)) \cong (T\hat{\otimes}L^\times)^{Fr_p=1}$. We have $L^\times = p^{\mathbb{Z}} \oplus O_L^\times$. The homomorphism

$$l_\varphi : (O_L)^\times \to O_L \ ; \ x \mapsto p^{-1}\log(x^p/Fr_p(x))$$

induces an isomorphism $\varprojlim_n O_L^\times/(O_L^\times)^{p^n} \cong O_L$. Thus we have an isomorphism

$$H^1(\mathbb{Q}_p, T(1)) = (T\hat{\otimes}(\mathbb{Z} \oplus O_L))^{Fr_p=1} \cong T^{Fr_p=1} \oplus D(T).$$

We define $\mathrm{Col}^\flat : H^1(\mathbb{Q}_p, T(1)) \to D(T)$ to be the second component of this isomorphism.

Thus $\mathrm{Col}^\flat$ is surjective and the kernel is isomorphic to $T^{Fr_p=1}$.

4.2.3. Let $\varphi : D(T) \to D(T)$ be as in 1.7.4.

We have

$$D(T)/(1-\varphi)D(T) = D(T/(1-Fr_p)T) = T/(1-Fr_p)T.$$

Proposition 4.2.4. *The composition*

$$H^1(\mathbb{Q}_p, T(1)) \xrightarrow{\mathrm{Col}^\flat} D(T) \to D(T)/(1-\varphi)D(T) = T/(1-Fr_p)T$$

coincides with the composition

$$\cup(1-p^{-1})\log(\kappa) : H^1(\mathbb{Q}_p, T(1)) \to H^2(\mathbb{Q}_p, T(1)) = T/(1-Fr_p)T.$$

Here κ is the cyclotomic character $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$, $(1-p^{-1})\log(\kappa)$ is the composition of κ and $(1-p^{-1})\log : \mathbb{Z}_p^\times \to \mathbb{Z}_p$, and $\cup$ denotes the cup product.

Proof. Replacing T by $T/(1-Fr_p)T$, we are reduced to the case the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on T is trivial. Hence we are reduced to the case $T = \mathbb{Z}_p$ with the trivial action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. In this case, it is sufficient to prove that $l_\varphi : \mathbb{Q}_p^\times \to \mathbb{Z}_p$ (this sends p to 0 and coincides with $(1-p^{-1})\log$ on $\mathbb{Z}_p^\times$) is induced by $\cup(1-p^{-1})\log(\kappa) : H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \to H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) = \mathbb{Z}_p$. This is checked easily. $\square$

4.2.5. Next we consider Col.

Let

$$P = \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), T(1)),$$

and let $U \subset P$ be the kernel of

$$P \to \varprojlim_n H^1(L(\zeta_{p^n}), T(1)) \to T\hat{\otimes} \varprojlim_n L(\zeta_{p^n})^\times/p^n \to T$$

where the last arrow is induced by the valuation $L(\zeta_{p^n})^\times \to \mathbb{Z}$.

We have a canonical homomorphism

$$\mathrm{Col} : U \to D(T)[[\mathbb{Z}_p^\times]]$$

defined as follows. It is the composition

$$U \to (T\hat{\otimes}(\varprojlim_n O_L[\zeta_{p^n}]^\times)^{Fr_p=1} \to (T\hat{\otimes}O_L[[\mathbb{Z}_p^\times]])^{Fr_p=1} = D(T)[[\mathbb{Z}_p^\times]]$$

where the second arrow is induced from the usual homomorphism of Iwasawa-Coates-Wiles-Coleman

$$(1) \qquad \varprojlim_n O_L[\zeta_{p^n}]^\times \to O_L[[\mathbb{Z}_p^\times]]$$

with respect to $(\zeta_{p^n})_{n\geq 1}$. We recall the last map (1). Let $u = (u_n)_{n\geq 1} \in \varprojlim_n O_L[\zeta_{p^n}]^\times$. Then there is a unique element of $g(t) \in O_L[[t-1]]^\times$ (called the Coleman power series associated to u) such that $Fr_p^{-n}(g)(\zeta_{p^n}) = u_n$ for all $n \geq 1$. Here $Fr_p^{-n}(g)$ is defined by applying Fr_p^{-n} to the coefficients of g. The image $\mu \in O[[\mathbb{Z}_p^\times]]$ of u is characterized by the property $\mu t = p^{-1} l_\varphi(g(t))$ where $l_\varphi(g(t)) = \log(g(t)^p / Fr_p(g)(t^p))$. Here μt is defined by the natural $O_L[[\mathbb{Z}_p^\times]]$-module structure of $O_L[[t]]$ for which the group element $[a]$ of $O_L[[\mathbb{Z}_p^\times]]$ for $a \in \mathbb{Z}_p^\times$ sends $f(t) \in O_L[[t]]$ to $f(t^a)$.

We extend $\mathrm{Col} : U \to D(T)[[\mathbb{Z}_p^\times]]$ to

$$\mathrm{Col} : P \to S^{-1}(D(T)[[\mathbb{Z}_p^\times]])$$

where S is the multiplicative subset of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ consisting of all elements of the form us^n where $u \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]^\times$, $n \geq 0$, and s is an element of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ such that the kernel of the augmentation map $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p$; $[a] \mapsto 1$ is generated by s as an ideal of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. The extended map is defined by

$$x \mapsto s^{-1}\mathrm{Col}(sx)$$

for such s.

Note that $D(T)[[\mathbb{Z}_p^\times]] \to S^{-1}(D(T)[[\mathbb{Z}_p^\times]])$ is injective. This is because $(S^{-1}\mathbb{Z}_p[[\mathbb{Z}_p^\times]])/\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ has no p-torsion and hence is flat over $\mathbb{Z}_p$.

4.2.6. In the case $T^{Fr_p=1} = 0$, we have

$$\mathrm{Col} : P \longrightarrow D(T)[[\mathbb{Z}_p^\times]]$$

without S^{-1}, since $P = U$ in this case.

Proposition 4.2.7. *Assume $T^{Fr_p=1} = 0$.*

(1) $\mathrm{Col}^\flat$ is an isomorphism $H^1(\mathbb{Q}_p, T(1)) \xrightarrow{\cong} D(T)$.

(2) We have an exact sequence

$$0 \to \varprojlim_n H^1(\mathbb{Q}(\zeta_{p^n}), T(1)) \xrightarrow{\mathrm{Col}} D(T)[[\mathbb{Z}_p^\times]] \xrightarrow{b} D(T)/(1-\varphi)D(T) \to 0,$$

where $b(x \otimes [a]) = ax$ for $x \in D(T)$ and $a \in \mathbb{Z}_p^\times$.

Proof. (1) is clear from 4.2.2.

As is well known in classical local Iwasawa theory, the homomorphism (1) in 4.2.5 induces an exact sequence

$$0 \to \mathbb{Z}_p(1) \to V \to O_L[[\mathbb{Z}_p^\times]] \to 0$$

where $V := Ker(\varprojlim_n H^1(L(\zeta_{p^n}), \mathbb{Z}_p(1)) \to \mathbb{Z}_p)$ and the map $\mathbb{Z}_p(1) \to V$ comes from the evident embedding of $\mathbb{Z}/p^n\mathbb{Z}(1)$ into $\mathbb{Z}_p[\zeta_{p^n}]^\times$. By the snake lemma for the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & T(1) & \to & T\hat{\otimes}V & \to & T\hat{\otimes}O_L[[\mathbb{Z}_p^\times]] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & T(1) & \to & T\hat{\otimes}V & \to & T\hat{\otimes}O_L[[\mathbb{Z}_p^\times]] & \to & 0 \end{array}$$

where the vertical arrows are $1 - Fr_p$, we have an exact sequence $0 \to U \to D(T)[[\mathbb{Z}_p^\times]] \to T/(1-Fr_p)T$. As is easily seen, the last arrow sends $x \otimes [a]$ to ax if we identify $T/(1-Fr_p)T$ by $D(T)/(1-\varphi)D(T)$. It remains to prove that this last arrow is surjective. By replacing T by $T/(1-Fr_p)T$, this is reduced to the case $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on T trivially, and hence to the case where T is $\mathbb{Z}_p$ with the trivial action of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. In the last case, the surjectivity is known in classical Iwasawa theory (cf. [5]). □

Lemma 4.2.8. *The image of the map*

$$(1-\varphi)\mathrm{Col} : P \to S^{-1}(D(T)[[\mathbb{Z}_p^\times]])$$

is contained in $D(T)[[\mathbb{Z}_p^\times]]$.

Proof. $\varprojlim_n L(\zeta_{p^n})^\times$ is generated by $\varprojlim_n O_L[\zeta_{p^n}]^\times$ and the class of $(1-\zeta_{p^n})_n$. The last element is killed by $1-\varphi$. □

Proposition 4.2.9. *We have a commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{(1-\varphi^{-1})\mathrm{Col}} & D(T)[[\mathbb{Z}_p^\times]] \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}_p, T(1)) & \xrightarrow{\mathrm{Col}^\flat} & D(T). \end{array}$$

Here the vertical arrows are the natural projections.

Proof. Let $u \in U$ and let $g \in T\hat{\otimes}O_L[[t-1]]^\times$ be the element such that $u_n = ((1 \otimes Fr^{-n})(g))(\zeta_{p^n})$ in $H^1(L(\zeta_{p^n}), T(1)) = T\hat{\otimes}L(\zeta_{p^n})^\times$ for any $n \geq 1$. Then if u_0 denotes the image of u in $T\hat{\otimes}(O_L)^\times$ under the canonical projection, we have $u_0 = g(1)(1 \otimes Fr_p^{-1})(g)(1)^{-1}$ where we denote the group law of the right hand side multiplicatively. Hence

$$\mathrm{Col}^\flat(u_0) = (1 \otimes l_\varphi)(u_0) = (1 - 1 \otimes Fr_p^{-1})((1 \otimes l_\varphi)(g)(1)) = (1-\varphi^{-1})\mathrm{Col}(u)_0$$

where $\mathrm{Col}(u)_0$ denotes the image of $\mathrm{Col}(u)$ under the projection $D(T)[[\mathbb{Z}_p^\times]] \to D(T)$.

The proposition follows from this and the fact that $T \otimes (1-\zeta_{p^n})_n$ is killed by $1 - 1 \otimes Fr_p^{-1}$ and projects to $T \otimes p$ which is killed by $1 \otimes l_\varphi$. □

4.2.10. Relation of Col and $\exp^*$ ([21] Chap. II, section 2, [44].) Assume that T is finitely generated as a $\mathbb{Z}_p$-module. Let $x \in \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), T(1))$, let S_1 be the image of S under $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \xrightarrow{\cong} \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$; $[a] \mapsto a[a]$ $(a \in \mathbb{Z}_p^\times)$, and let $\mu \in S_1^{-1}(D(T)[[\mathbb{Z}_p^\times]])$ be the image of $Col(x) \in S^{-1}(D(T)[[\mathbb{Z}_p^\times]])$ under the isomorphism $S^{-1}(D(T)[[\mathbb{Z}_p^\times]]) \to S_1^{-1}(D(T)[[\mathbb{Z}_p^\times]])$; $z \otimes [a] \mapsto az \otimes [a]$ $(z \in D(T),\ a \in \mathbb{Z}_p^\times)$. Let $\psi : \mathbb{Z}_p^\times \to \bar{\mathbb{Z}}_p^\times$ be a continuous homomorphism with finite image, and let $\mu(\psi) \in D(T)\hat{\otimes}\bar{\mathbb{Z}}_p$ be the image of μ by ψ. Let n be the smallest integer ≥ 0 such that ψ factors through $(\mathbb{Z}/p^n\mathbb{Z})^\times$, let x_n be the image of x in $H^1(\mathbb{Q}_p(\zeta_{p^n}), T)$ under the composition $\varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), T(1)) \cong \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), T) \to H^1(\mathbb{Q}_p(\zeta_{p^n}), T)$, and let $\exp^*(x_n) \in D(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n})$ be the image of x_n under the dual exponential map

$$\exp^* : H^1(\mathbb{Q}_p(\zeta_{p^n}), T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to D_{dR}(\mathbb{Q}_p(\zeta_{p^n}), T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = D(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n}).$$

(1) Assume $n \geq 1$. Then

$$\mu(\psi^{-1}) = G(\psi, \zeta_{p^n})^{-1} \cdot p^n \cdot (\varphi^n \otimes 1) \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(a)\sigma_a(\exp^*(x_n)).$$

(2) Assume $n = 0$ (so ψ is trivial). Then

$$(1 - p^{-1}\varphi^{-1})\mu(\psi) = (1 - \varphi)\exp^*(x_0).$$

4.3 p-adic L-function $\mathcal{M}$ in two variables

4.3.1. From now, in this section 4, fix an integer $N \geq 1$ which is prime to p, and let

$$\Lambda = \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times].$$

As in section 1.5, let

$$\tilde{H} = \varprojlim_r H^1_{\mathrm{et}}(Y_1(Np^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)^{\mathrm{ord}} \supset H = \varprojlim_r H^1_{\mathrm{et}}(X_1(Np^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)^{\mathrm{ord}},$$

M_Λ the space of ordinary Λ-adic modular forms, and $S_\Lambda \subset M_\Lambda$ the space of ordinary Λ-adic cusp forms. Recall that the action of $\mathfrak{H}$ on M_Λ and the action of $\mathfrak{h}$ on S_Λ are that $T^*(n)$ acts as the usual $T(n)$.

4.3.2. Consider the maps Col and Col$^\flat$ for the unramified representations $T = \tilde{H}_{\mathrm{quo}}(1)$ or $T = H_{\mathrm{quo}}(1)$ of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

We have

$$\mathrm{Col} : \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), \tilde{H}_{\mathrm{quo}}(2)) \to S^{-1}(D(\tilde{H}_{\mathrm{quo}}(1))[[\mathbb{Z}_p^\times]]) = S^{-1}(M_\Lambda[[\mathbb{Z}_p^\times]])$$

(here $S \subset \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is as in section 4.2). This induces

$$\mathrm{Col} : \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), H_{\mathrm{quo}}(2)) \to D(H_{\mathrm{quo}}(1))[[\mathbb{Z}_p^\times]] = S_\Lambda[[\mathbb{Z}_p^\times]]$$

without S^{-1}, by the fact $H_{\mathrm{quo}}(1)^{Fr_p=1}=0$ (Proposition 3.3.3) and 4.2.6.

We have

$$\mathrm{Col}^{\flat}: H^1(\mathbb{Q}_p, \tilde{H}_{\mathrm{quo}}(2)) \to D(\tilde{H}_{\mathrm{quo}}(1)) = M_\Lambda$$

which induces

$$\mathrm{Col}^{\flat}: H^1(\mathbb{Q}_p, H_{\mathrm{quo}}(2)) \to D(H_{\mathrm{quo}}(1)) = S_\Lambda.$$

The last map is an isomorphisms by $H_{\mathrm{quo}}(1)^{Fr_p=1}=0$ (Proposition 3.3.3).

4.3.3. We define a map

$${}^0\mathcal{M}: \tilde{H} \to M_\Lambda[[\mathbb{Z}_p^\times]] \otimes_{\Lambda[[\mathbb{Z}_p^\times]]} Q(\Lambda[[\mathbb{Z}_p^\times]]) = M_\Lambda \otimes_\Lambda Q(\Lambda[[\mathbb{Z}_p^\times]]).$$

by

$${}^0\mathcal{M}(\gamma) = \mathrm{Col}(z_{1,Np^\infty,p^\infty}(\gamma)) \quad (\gamma \in \tilde{H}).$$

Here $z_{1,Np^\infty,p^\infty}(\gamma)$ is defined as in Theorem 3.2.3.

For $s \in \mathbb{Z}$, let ${}^s\mathcal{M}(\gamma)$ be the element of $M_\Lambda \otimes_\Lambda Q(\Lambda[[\mathbb{Z}_p^\times]])$ obtained from ${}^0\mathcal{M}(\gamma)$ by applying the automorphism of $M_\Lambda \otimes_\Lambda Q(\Lambda[[\mathbb{Z}_p^\times]])$ induced by the automorphism of $\Lambda[[\mathbb{Z}_p^\times]]$ over Λ which sends $[a]$ ($a \in \mathbb{Z}_p^\times$) to $a^s[a]$. We denote ${}^1\mathcal{M}$ simply by $\mathcal{M}$. We will mainly use ${}^0\mathcal{M}$ and $\mathcal{M} = {}^1\mathcal{M}$.

The correspondence $\gamma \mapsto \mathcal{M}(\gamma)$ is characterized by zeta values as in (2) in the following Proposition.

Proposition 4.3.4. *The map $\mathcal{M}$ has the following properties.*

(1) For any $\gamma \in \tilde{H}$ and for any b, c, d such that $(b,p)=1$ and $(cd, 6Nmp)=1$ and $c \equiv d \equiv 1 \bmod Np$, $(1-b\sigma_b)(c-\sigma_c)(d-\sigma_d\langle d\rangle)\mathcal{M}(\gamma)$ belongs to the image of $E := M_\Lambda \otimes_\Lambda \Lambda[[\mathbb{Z}_p^\times]]$ in $E \otimes_{\Lambda[[\mathbb{Z}_p^\times]]} Q(\Lambda[[\mathbb{Z}_p^\times]])$. (Here $\langle a\rangle$ acts on M_Λ as the usual $\langle a\rangle^{-1}$).

(2) Let $r \geq 1$, let $\gamma \in H$, and assume that the image of γ in $H^1_{ét}(Y_1(Np^r))_{\mathbb{Q}_p}$ belongs to $H^1(Y_1(Np^r)(\mathbb{C}), \mathbb{Q})$. Let $\psi: \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}^\times$ be a continuous homomorphism of finite order and of conductor p^n with $n > 0$. Then the image of $\mathcal{M}(\gamma)(\psi) \in (\varprojlim_j M_2(Np^j)_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p$ in $M_2(Np^r)_{\bar{\mathbb{Q}}_p}$ belongs to $M_2(Np^r)_{\bar{\mathbb{Q}}}$, and the period map

$$M_2(Np^r)_{\bar{\mathbb{Q}}} \to H^1(Y_1(Np^r)(\mathbb{C}), \mathbb{Q})^{\pm} \otimes_{\mathbb{Q}} \mathbb{C} \quad (\pm = \psi(-1))$$

sends this image of $\mathcal{M}(\gamma)(\psi)$ in $M_2(Np^r)_{\bar{\mathbb{Q}}}$ to

$$p^n G(\psi, \zeta_{p^n})^{-1} T^*(p)^{-n} Z_{1,Np^r}(\psi, 1) \cdot \gamma^{\pm}.$$

Here

$$Z_{1,Np^r}(\psi, s) = \sum_m \psi(m) T^*(m) m^{-s}$$

where m ranges over all integers ≥ 1 which are prime to p, and $\pm$ denotes the $\pm$-component for the action of complex conjugation.

(3) The map $\gamma \mapsto \mathcal{M}(\gamma)$ commutes with the action of the Hecke algebra $\mathfrak{H}$. (Recall that $T^*(n) \in \mathfrak{H}$ acts on M_Λ by the usual action of $T(n)$.)

Proof. This is reduced to the properties of Beilinson elements. $\square$

4.3.5. In Proposition 4.3.6 below, we describe $\mathcal{M}(\{0,\infty\})$. Here we give some preliminaries to state the result.

We prepare notation for Eisenstein series of weight 1. For $x, y \in \mathbb{Q}/\mathbb{Z}$, the Eisenstein series $E^{(1)}_{x,y}$ of weight 1, which is a modular form on the curve $X(M)_{\mathbb{Q}}$ for $M \geq 3$ such that $Mx = My = 0$, is defined as in [22] section 4. It is characterized by the q-expansion as follows. For $x \in \mathbb{Q}/\mathbb{Z}$, we define

$$\zeta(x,s) = \sum_{m\in\mathbb{Q}, m>0,\ m \bmod \mathbb{Z}=x} m^{-s}, \qquad \zeta^*(x,s) = \sum_{m=1}^{\infty} \exp(2\pi i x m)\cdot m^{-s}.$$

Then $E^{(1)}_{x,y} = \sum_{m\in\mathbb{Q},m\geq 0} a_m q^m$ where a_m with $m > 0$ are given by

$$\sum_{m\in\mathbb{Q},m>0} a_m m^{-s} = \zeta(x,s)\zeta^*(y,s) - \zeta(-x,s)\zeta^*(-y,s),$$

and a_0 is as follows: $a_0 = \zeta(x,0)$ if $x \neq 0$, and $a_0 = (1/2)(\zeta^*(y,0) - \zeta^*(-y,0))$ if $x = 0$.

For $R \in GL(2, \mathbb{Z}/M\mathbb{Z})$, we have

$$(1) \quad R^* E^{(1)}_{x,y} = E^{(1)}_{x',y'} \quad \text{where } (x',y') = (x,y)R.$$

For any integer $m \geq 1$, we have the distribution property

$$(2) \qquad E^{(1)}_{x,y} = m^{-1} \sum_{mx'=x, my'=y} E^{(1)}_{x',y'}.$$

For $a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, we denote the group element of Λ corresponding to a by $[a]$. For $a \in \mathbb{Z}_p^\times$, we denote the group element of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ corresponding to a by σ_a (we will identify this σ_a with $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$). In $\Lambda[[\mathbb{Z}_p^\times]]$ which contains both rings Λ and $\mathbb{Z}_[[\mathbb{Z}_p^\times]]$ in the evident manner, the image of $[a] \in \Lambda$ ($a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$) is denoted by $[a]$, and the image of $\sigma_a \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ ($a \in \mathbb{Z}_p^\times$) is denoted by σ_a.

Proposition 4.3.6. *(1) As an element of* $\varprojlim_{r,n} M_2(Np^r)_{\mathbb{Q}_p}[(\mathbb{Z}/p^n\mathbb{Z})^\times]$, *we have*

$$\mathcal{M}(\{0,\infty\}) =$$

$$\textit{the ordinary component of } (N^{-1}p^{-n-r}\cdot(\sum_{x\in\mathbb{Z}/p^n\mathbb{Z}, b\in(\mathbb{Z}/p^n\mathbb{Z})^\times} E^{(1)}_{x/p^n,b/p^n}\sigma_b)\cdot E^{(1)}_{0,1/Np^r})_{n\geq 1, r\geq 1}.$$

(2) As an element of $M_\Lambda[[\mathbb{Z}_p^\times]] \otimes_{\Lambda[[\mathbb{Z}_p^\times]]} Q(\Lambda[[\mathbb{Z}_p^\times]]$, *for* $s \in \mathbb{Z}$, *we have*

$${}^s\mathcal{M}(\{0,\infty\}) = \textit{ the ordinary part of } AB, \quad \textit{where}$$

where A and B are (Λ-adic) Eisenstein series (of weight s and $2-s$, respectively) defined by

$$A = A_0 + 2 \sum_{i,j\geq 1, (i,p)=1} \sigma_i i^{s-1} q^{Nij},$$

$$B = B_0 + 2 \sum_{i,j \geq 1, (i,Np)=1} \sigma_i[i]\sigma_i^{-1} i^{1-s} q^{ij},$$

whose constant terms A_0 and B_0 are given, respectively, as follows. A_0 is the image of the p-adic zeta function ${}^{1-s}\xi^*_{p^\infty}$ under $[a] \mapsto \sigma_a$, and B_0 is the image of the p-adic zeta function ${}^{s-1}\xi^*_{Np^\infty}$ under $[a] \mapsto [a]\sigma_a^{-1}$.

This (2) is obtained in Fukaya [12]. The fact the system in (1) belongs to the inverse limit is deduced from the distribution property (2) in 4.3.5. The fact the system in (1) corresponds to the Λ-adic modular form in (2) is proved by using (1) in 4.3.5 and using the q-expansion of $E^{(1)}_{x,y}$ in 4.3.5.

The product AB of (2) appears also in the work Panchiskin [43] to obtain p-adic L-function in two variables (relation with Beilinson elements are not considered in [43]).

4.3.7. Let $(-)_{s=0} : D(H_{\mathrm{quo}}(1)) \otimes_\Lambda \Lambda[[\mathbb{Z}_p^\times]] \to D(H_{\mathrm{quo}}(1))$ be the map induced by $\Lambda[[\mathbb{Z}_p^\times]] \to \Lambda\,; \sigma_c \mapsto c^{-1}$. For $\gamma \in \tilde{H}$, let

$$\mathcal{M}_{s=0}(\gamma) := ({}^0\mathcal{M}(\gamma))_{s=0} \in M_\Lambda \otimes_\Lambda Q(\Lambda).$$

We will see in the next section (4.4.3) that, for $\gamma \in H$, $\mathcal{M}(\gamma) \in S_\Lambda[[\mathbb{Z}_p^\times]]$ and hence $\mathcal{M}_{s=0}(\gamma) \in S_\Lambda$ (we do not need the localization $\otimes Q(-)$).

Proposition 4.3.8. *For $\gamma \in H$, we have*

$$\mathcal{M}_{s=0}(\gamma) = \mathrm{Col}^\flat(z^\sharp_{Np^\infty}(\gamma))$$

in $D(H_{\mathrm{quo}}(1))$.

Proof. Let $z_\gamma \in H^1(\mathbb{Q}_p, \tilde{H}_{\mathrm{quo}}(2)) \otimes_\Lambda \Lambda\, \mu^{-1}$ be the image of $z_{1,Np^\infty,p^\infty}(\gamma)$ under the canonical projection. (Here μ is as in 3.3.9.) We have by Propositions 1.8.1 and 4.2.9

$$(1 - T^*(p))\mathcal{M}_{s=0}(\gamma) = (1 - \varphi^{-1})\mathcal{M}_{s=0}(\gamma) = (1 - \varphi^{-1})(\mathrm{Col}(z_{1,Np^\infty,p^\infty}(\gamma))_{s=0})$$

$$= \mathrm{Col}^\flat(z_\gamma) = \mathrm{Col}^\flat((1 - T^*(p))z^\sharp_{Np^\infty}(\gamma)) = (1 - T^*(p))\mathrm{Col}^\flat(z^\sharp_{Np^\infty}(\gamma)).$$

Use the injectivity of $1 - T^*(p)$ on H_{quo} considered in section 3.3. $\square$

4.4 p-adic L-function $\mathcal{L}$ of Mazur-Kitagawa in two variables

We review the definition of the p-adic L-function of Mazur-Kitagawa in two variables and prove a relation (Theorem 4.4.3) to the above p-adic L-function $\mathcal{M}$ in two variables.

4.4.1. By [51] Lemma 3.1, the ordinary component of $[u : v]_r \in H^1_{\text{ét}}(Y_1(Np^r))$ ($u, v \in \mathbb{Z}/Np^r\mathbb{Z}$, $(u,v) = 1$) such that $u \bmod p^r \neq 0$, $v \bmod p^r \neq 0$ belongs to the image of the canonical injection $H^1_{\text{ét}}(X_1(Np^r))^{\mathrm{ord}} \to H^1_{\text{ét}}(Y_1(Np^r))^{\mathrm{ord}}$.

4.4.2. Fix $N \geq 1$ which is prime to p. Mazur-Kitagawa p-adic L-function $\mathcal{L}$ in two variables is defined as

$$\mathcal{L} = (\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} T^*(p)^{-r}\{\infty, a/p^r\}_{Y_1(Np^r)} \otimes [a])_{r\geq 1} = (\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} T^*(p)^{-r}[Na:1]_r \otimes [a])_{r \geq 1}$$
$$\in \tilde{H}[[\mathbb{Z}_p^\times]] = \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}[(\mathbb{Z}/p^r\mathbb{Z})^\times].$$

In fact it belongs to $H[[\mathbb{Z}_p^\times]] \subset \tilde{H}[[\mathbb{Z}_p^\times]]$ by 4.4.1.

(Here we consider the (-1)-Tate twist of the homology.)

Relations between the p-adic L functions $\mathcal{M}$ and $\mathcal{L}$ in two variables were studied by Ochiai [37]. In Theorem 4.4.3 below, we give a new presentation of the relation by using the pairing $((-,-))_\Lambda$ defined in section 1.6.

Theorem 4.4.3. *For $\gamma \in H$, $\mathcal{M}(\gamma)$ belongs to $S_\Lambda[[\mathbb{Z}_p^\times]]$, and we have*

$$((\mathcal{L}, \gamma))_\Lambda = \sigma_{-1} \cdot \mathcal{M}(\gamma)$$

in $S_\Lambda[[\mathbb{Z}_p^\times]]$.

Proof. It is sufficient to prove that for any non-trivial continuous homomorphism $\psi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ with finite image, we have

$$(1) \quad ((\mathcal{L}(\psi), \gamma)) = \psi(-1)\mathcal{M}(\gamma)(\psi) \;\text{ in }\; S_\Lambda \otimes_{\mathbb{Z}_p} \bar{\mathbb{Q}}_p.$$

Let p^c be the conductor of ψ. We work on $X_1(Np^r)$ $(r \geq 1)$. Let $L(\psi) \in H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} \bar{\mathbb{Q}}$ be the image of $\sum_{a \in (\mathbb{Z}/p^c\mathbb{Z})^\times}\{\infty, a/p^c\}_{Y_1(Np^r)} \otimes \psi(a) \in H^1(Y_1(Np^r)(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} \bar{\mathbb{Q}}$ under the Drinfeld-Manin splitting $H^1(Y_1(Np^r)(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} \bar{\mathbb{Q}} \to H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} \bar{\mathbb{Q}}$ (1.9.3). Consider the pairing $(-,-) : H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Q}) \times H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}$ of Poincaré duality. Let $f = \sum_{m=1}^\infty a_m(f)q^m \in S_2(Np^r)_\mathbb{C}$ and consider $\text{per}(f) \in H^1(X_1(Np^r)(\mathbb{C}), \mathbb{C})$.

Claim 1. $(L(\psi), \text{per}(f)) = G(\psi, \zeta_{p^c})L(f, \psi^{-1}, 1)$. Here $L(f, \psi^{-1}, s) = \sum_m a_m(f)\psi^{-1}(m)m^{-s}$ where m ranges over all integers ≥ 1 which are prime to p.

Proof of Claim 1. This is standard.

Claim 2. $\sum_{n=1}^\infty (L(\psi), T(n)\text{per}(f))q^n = G(\psi, \zeta_{p^c})Z_T(\psi^{-1}, 1)f$. Here $Z_T(\psi^{-1}, s) = \sum_m T(m)\psi(m)m^{-s}$ where m ranges over all integers ≥ 1 which are prime to p.

Proof of Claim 2. The right hand side of Claim 1 is $G(\psi, \zeta_{p^c})a_1(Z_T(\psi^{-1}, 1)f)$. Replacing f by $T(n)f$, we have

$$(L(\psi), T(n)\text{per}(f)) = (L(\psi), \text{per}(T(n)f)) = G(\psi, \zeta_{p^c})a_1(Z_T(\psi^{-1}, 1)T(n)f)$$
$$= G(\psi, \zeta_{p^c})a_n(Z_T(\psi^{-1}, 1)f).$$

This proves Claim 2.

Claim 3. For any $\gamma \in H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Q})$, the period map $M_2(Np^r)_{\bar{\mathbb{Q}}} \to H^1(X_1(Np^r)(\mathbb{C}), \mathbb{Q})^\pm \otimes_\mathbb{Q} \mathbb{C}$ sends $\sum_{n=1}^\infty (L(\psi), T(n)\gamma)q^n$ to $G(\psi, \zeta_{p^c})Z_T(\psi^{-1}, 1)\gamma^\pm$. Here $\pm = \psi(-1)$.

Proof of Claim 3. Since the map $\gamma \mapsto (L(\psi), \gamma)$ factors through $\gamma \mapsto \gamma^{\pm}$, this follows from Claim 2.

Now the formula (1) at the beginning of this proof of Theorem 4.4.3 is deduced from Claim 3 and (2) of Proposition 4.3.4. ($\psi(-1)$ appears because $G(\psi, \zeta_{p^c})G(\psi^{-1}, \zeta_{p^c})p^{-c} = \psi(-1)$). □

Remark 4.4.4. The p-adic L-function $\mathcal{L}$ in two variables of Mazur-Kitagawa still works for modular forms which need not be ordinary. But it loses the integrality, for $T^*(p)^{-r}$ which appears in its definition is not integral any more.

However, the p-adic L-function $\mathcal{M}$ in two variables works integrally even for modular forms which need not be ordinary, as is shown in a forthcoming paper of T. Fukaya. It will be shown that we can construct $\mathcal{M} : \varprojlim_r H^1_{\text{ét}}(X_1(Np^r)) \to (\Lambda\text{-adic cusp forms})[[\mathbb{Z}_p^{\times}]]$, $\mathcal{M} : \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r)) \to (\Lambda\text{-adic modular forms})[[\mathbb{Z}_p^{\times}]]$ without taking the ordinary components.

4.5 Relations with p-adic L-functions of cusp forms

We compare the relation of $\mathcal{L}$ to the p-adic L-function of each cusp form with that of $\mathcal{M}$.

4.5.1. We review the p-adic L-function of a cusp form ([1], [31], [58]).

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(Np^r)_{\mathbb{C}}$ ($r \geq 1$) be a normalized Hecke-eigen p-stabilized newform of weight $k \geq 2$, of level Np^r, and of character $\epsilon : (\mathbb{Z}/Np^r\mathbb{Z})^{\times} \to \bar{\mathbb{Q}}^{\times}$. So $T(n)f = a_n f$ ($a_n \in \bar{\mathbb{Q}}$) for $n \geq 1$, $\langle c \rangle f = \epsilon(c) f$ for any $c \in (\mathbb{Z}/Np^r\mathbb{Z})^{\times}$, and $a_1 = 1$. Assume f is $T(p)$-ordinary, that is, a_p is a p-adic unit in $\bar{\mathbb{Q}}_p$. We review the p-adic L function of f.

We have $f \in S_k(Np^r)_{\bar{\mathbb{Q}}}$. Let

$$F = \mathbb{Q}(a_n \; ; \; n \geq 1) \subset \bar{\mathbb{Q}}, \quad L = \mathbb{Q}_p(a_n \; ; \; n \geq 1) \subset \bar{\mathbb{Q}}_p.$$

Then F is a finite extension of $\mathbb{Q}$ and L is a finite extension of $\mathbb{Q}_p$.

Consider the canonical perfect paring

$$(\; , \;) : V_k(X_1(Np^r))_{\mathbb{Q}}(k-1) \times V_k(X_1(Np^r))_{\mathbb{Q}} \to \mathbb{Q}.$$

This pairing commutes with the action of the complex conjugation. We denote the induced pairing $V_k(X_1(Np^r))_{\mathbb{C}} \times V_k(X_1(Np^r))_{\mathbb{C}} \to \mathbb{C}$ also by $(\; , \;)$.

We say that a pair $\Omega = (\Omega^+, \Omega^-)$ of non-zero complex numbers is a period of f if $\Omega^{\pm} = (2\pi i)^{2-k}(\delta^{\pm}, \mathrm{per}(f))$ for the above pairing $(-,-)$ and for some $\delta^{\pm} \in V_k(X_1(Np^r))_{\mathbb{Q}}(k-1)^{\pm}$. (The last $(-)^{\pm}$ denotes the part on which the complex conjugation acts by ± 1). Periods of f fulfill an $F^{\times} \times F^{\times}$-orbit in $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$.

Let $\Omega = (\Omega^+, \Omega^-)$ be a period of f. Then the p-adic L-function $L_{p,\Omega}(f)$ of f with respect to Ω is the unique element of $O_L[[\mathbb{Z}_p^{\times}]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ having the following relations (i) and (ii) with the complex L-functions

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad L(f, \psi, s) = \sum_{n=1}^{\infty} a_n \psi(n) n^{-s}.$$

(Here $\psi(n)$ means 0 if n is divisible by p and ψ is non-trivial.) Let $r \in \mathbb{Z}$, $1 \leq r \leq k-1$. For a homomorphism $\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$, let $L_{p,\Omega}(f,\psi,r) \in \bar{\mathbb{Q}}_p$ be the image of $L_{p,\Omega}$ under the ring homomorphism $\psi\kappa^{r-1} : O_L[[\mathbb{Z}_p^\times]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \bar{\mathbb{Q}}_p$ which sends the group element $[a]$ for $a \in \mathbb{Z}_p^\times$ to $a^{r-1}\psi(a)$. In the case ψ is trivial, we denote $L_{p,\Omega}(f,\psi,r)$ by $L_{p,\Omega}(f,r)$.

Let $\alpha := a_p$.

(i) Let $n \geq 1$ and let $\psi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be a homomorphism which does not factor through $(\mathbb{Z}/p^{n-1}\mathbb{Z})^\times$. Then

$$L_{p,\Omega}(f,\psi,r) = (r-1)! \cdot p^{n(r-1)}\alpha^{-n} \cdot G(\psi,\zeta_{p^n}) \cdot (-2\pi i)^{1-r} \cdot \frac{1}{\Omega^\pm} \cdot L(f,\psi,r)$$

(both sides belong to $\bar{\mathbb{Q}}$). Here $\pm = (-1)^{r-1}\psi(-1)$, and $G(\psi,\zeta_{p^n})$ is the Gauss sum $\sum_{b\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \psi(b)\zeta_{p^n}^b$.

(ii)

$$L_{p,\Omega}(f,r) = (r-1)! \cdot (1-p^{r-1}\alpha^{-1})(1-\epsilon(p)p^{k-r-1}\alpha^{-1}) \cdot (-2\pi i)^{1-r} \cdot \frac{1}{\Omega^\pm} \cdot L(f,r)$$

(both sides belong to $\bar{\mathbb{Q}}$). Here $\pm = (-1)^{r-1}$.

4.5.2. Define the F-vector space $V(f)_F$ (resp. $S(f)_F$) to be the quotient of $V_k(X_1(Np^r))_F$ (resp. $S_k(Np^r)_F$) divided by the F-subspace generated by the images of the operators $T(n) - a_n$ ($n \geq 1$) on $V_k(X_1(Np^r))_F$ (resp. $S_k(Np^r)_F$). Then $V(f)_F$ is a two dimensional F-vector space and $S(f)_F$ is a one dimensional F-vector space. The natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $V_k(X_1(Np^r))_{\mathbb{Q}}$ induces an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $V(f)_F$. Let $V(f)_F^+$ (resp. $V(f)_F^-$) be the part of $V(f)_F$ on which the complex conjugation acts as 1 (resp. -1). Then $\dim_F(V(f)_F^+) = \dim_F(V(f)_F^-) = 1$.

The period map in 1.5.6 induces the period map

$$\mathrm{per} : S(f)_{\mathbb{C}} \to V(f)_{\mathbb{C}}$$

which is injective, where $S(f)_{\mathbb{C}} = S(f)_F \otimes_F \mathbb{C}$, $V(f)_{\mathbb{C}} = V(f)_F \otimes_F \mathbb{C}$.

4.5.3. Let f be as in 4.5.2. Let

$$f^* = \sum_n \bar{a}_n q^n$$

where $\bar{a}_n$ denotes the complex conjugate of a_n. Then $f^* \in S_k(Np^r)_{\mathbb{C}}$ and f^* is a normalized Hecke-eigen p-stabilized new form. So $T(n)f^* = \bar{a}_n f^*$ for $n \geq 1$. We have also $T^*(n)f^* = a_n f^*$ for $n \geq 1$. Hence f^* is $T^*(p)$-ordinary. We have $F = \mathbb{Q}(\bar{a}_n\ ;\ n \geq 1)$.

The F-vector space $V(f^*)_F$ (resp. $S(f^*)_F$) coincides with the quotient of $V_k(X_1(M))_F$ (resp. $S_k(M)_F$) by the F-subspace generated by the images of $T^*(n) - a_n$ ($n \geq 1$). The map $(-,\mathrm{per}(f)) : V_k(X_1(Np^r))_F(k-1) \to \mathbb{C}$ factors through $V(f^*)_F(k-1)$.

4.5.4. Take non-zero elements $\delta^\pm$ of $V(f^*)(k-1)_F^\pm$, let $\Omega^\pm = (2\pi i)^{2-k}(\delta^\pm, \mathrm{per}(f))$, and let

$$L_p(f) = L_{p,\Omega}(f)^+\delta^+ + L_{p,\Omega}(f)^-\delta^- \in V(f^*)_F(k-1) \otimes_{O_F} O_L[[\mathbb{Z}_p^\times]].$$

Then $L_p(f)$ is independent of the choice of δ.

Proposition 4.5.5. *([30], [24].) Let $|_{f^*}$ be the projection from $\varprojlim_r V_k(X_1(Np^r))_{\mathbb{Z}_p}$ to $V(f^*)_L$. We have*

$$\pi_k(\mathcal{L})|_{f^*} = L_p(f)$$

in $V(f^*)_F \otimes_{O_F} O_L[[\mathbb{Z}_p^\times]]$.

4.5.6. We define $L_{p,\gamma}(f) \in S(f^*)_F \otimes_{O_F} O_L[[\mathbb{Z}_p^\times]]$ for $\gamma \in V(f^*)_L$. In the case $\gamma \in V(f^*)_F$ and $\gamma^\pm \neq 0$, it is defined as follows. Take a non-zero element ω of $S(f^*)_F$ and write $\mathrm{per}(\omega) = \Omega^+\gamma^a + \Omega^-\gamma^{-a}$ where $a \in \{\pm\}$ is the sign of $(-1)^k$. (We can take for example, $\omega = w_{Np^r}(f)$.) Then $\Omega = (\Omega^+, \Omega^-)$ is a period of f. Define $L_{p,\gamma}(f) = L_{p.\Omega}(f)\omega$. Then this is independent of the choice of ω. For general $\gamma \in V(f)_L$, writing $\gamma = \sum_i a_i\gamma_i$ with $a_i \in L$ and $\gamma_i \in V(f^*)_F$ such that $\gamma_i^\pm \neq 0$, define $L_{p,\gamma}(f) = \sum_i a_i L_{p,\gamma_i}(f)$. This is independent of the choice of such presentation of γ.

Lemma 4.5.7. $((L_p(f), \gamma))_{r,k} = \sigma_{-1} \cdot L_{p,\gamma}(f)$ *in $S_F(f^*) \otimes_{O_F} O_L[[\mathbb{Z}_p^\times]]$. Here the pairing $((-,-))_{k,r}$ is as in 1.6.8.*

Proof. This can be deduced from the relations of both sides to complex L-values. □

Proposition 4.5.8. *Let $|_{f^*}$ be the projection from $S_{k,\Lambda}$ to $S(f^*)_L$. We have*

$$\pi_k(\mathcal{M}(\gamma))|_{f^*} = L_{p,\gamma}(f)$$

in $S(f^*)_F \otimes_{O_F} O_L[[\mathbb{Z}_p^\times]]$.

Proof. This follows from Theorem 4.4.3, Lemma 4.5.7 and Proposition 1.6.10. □

5 The map ϖ

In this section, we define the map ϖ from the modular symbol side to the ideal class group side. We prove Conjecture 5.8 in [51] of Sharifi.

We illustrate the outline of section 5. In section 3, we constructed the correspondence

(1) modular symbol $\mapsto$ Beilinson element.

In section 5.1, we will construct the correspondence

(2) Beilinson element $\mapsto$ cyclotomic symbol

by taking the value of a Beilinson element at the ∞-cusp. Here cyclotomic symbol means the cup product of two cyclotomic units in Galois cohomology. Composing these (1) and (2), we obtain in section 5.2 and section 5.3 the correspondence

(3) modular symbol $\mapsto$ cyclotomic symbol.

The correspondence (3) is constructed by Sharifi [51], Proposition 5.7 (see 5.2.1–5.2.3 of this paper), but the relation of his correspondence with Hecke operators is stated by him as Conjecture 5.8 in [51] which we review in 5.2.2. By using our construction of the correspondence (3) outlined above, we can prove his conjecture by the fact that the map (1) respects Hecke operators (Theorem 3.3.9) and the map (2) also respects Hecke operators (Theorem 5.1.9 (2)). We point our that in our proof of Theorem 3.3.9 and hence in our proof of this conjecture, the relation of Beilinson elements to zeta values play an essential role.

5.1 From Beilinson elements to cyclotomic symbols

5.1.1. Let $M \geq 1$ and assume $p|M$. For $a, b \in \mathbb{Z}/M\mathbb{Z}$ such that $(a, b) = (1)$, we define in 5.1.3 below a homomorphism

$$H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2)) \to H^2(\mathbb{Z}[1/M, \zeta_M], \mathbb{Z}_p(2))$$

called the evaluation at $\infty_M(a, b)$. The restriction of this homomorphism to $H^2(X_1(M) \otimes \mathbb{Z}[1/M])$ is just the pull back by the morphism $\mathrm{Spec}(\mathbb{Z}[1/M, \zeta_M], \mathbb{Z}_p(2)) \to X_1(M) \otimes \mathbb{Z}[1/M]$ induced by $\infty_M(a, b)$ (1.3.4).

5.1.2. Let R be a Noetherian regular ring. We extend the homomorphism of K-groups

$$K_i(R[[T]]) \to K_i(R)\ ;\ T \mapsto 0$$

to a homomorphism

$$K_i(R[[T]][T^{-1}]) \to K_i(R)$$

as follows. By the localization theory in K-theory, we have an isomorphism

$$K_i(R[[T]]) \oplus K_{i-1}(R) \xrightarrow{\cong} K_i(R[[T]][T^{-1}])\ ;\ (x, y) \mapsto x + \{y, T\}.$$

Using this isomorphism, we define the above homomorphism as the composition

$$K_i(R[[T]][T^{-1}]) \cong K_i(R[[T]]) \oplus K_{i-1}(R) \to K_i(R[[T]]) \to K_i(R)$$

where the former arrow is the first projection and the latter arrow is $T \mapsto 0$.

We have a similar homomorphism

$$H^i(R[[T]][T^{-1}], \mathbb{Z}/p^n\mathbb{Z}(r)) \to H^i(R, \mathbb{Z}/p^n\mathbb{Z}(r))$$

for the étale cohomology group, for a prime number p which is invertible in R and for $i, r \in \mathbb{Z}$. This is obtained similarly from the isomorphism

$$H^i(R[[T]], \mathbb{Z}/p^n\mathbb{Z}(r)) \oplus H^{i-1}(R, \mathbb{Z}/p^n\mathbb{Z}(r-1)) \xrightarrow{\cong} H^i(R[[T]][T^{-1}], \mathbb{Z}/p^n\mathbb{Z}(r))\ ;\ (x, y) \mapsto x + \{y, T\}$$

where $\{-, T\}$ is the cup product with the Kummer class of T in $H^1(R[[T]][T^{-1}], \mathbb{Z}/p^n\mathbb{Z}(1))$.

In what follows, we use the case $i = 2$ of these homomorphisms

$$K_2(R[[T]][T^{-1}]) \to K_2(R), \quad H^2(R[[T]][T^{-1}], \mathbb{Z}/p^n\mathbb{Z}(2)) \to H^2(R, \mathbb{Z}/p^n\mathbb{Z}(2)).$$

The following diagram is commutative.

$$\begin{array}{ccc} K_2(R[[T]][T^{-1}]) & \to & K_2(R) \\ \downarrow & & \downarrow \\ H^2(R[[T]][T^{-1}], \mathbb{Z}/p^n\mathbb{Z}(2)) & \to & H^2(R, \mathbb{Z}/p^n\mathbb{Z}(2)) \end{array}$$

Here the vertical arrows are the Chern class maps.

5.1.3. The value at $\infty_M(a,b)$. Let $M \geq 1$ and assume $p|M$. Let $a, b \in \mathbb{Z}/M\mathbb{Z}$ and assume $(a,b) = (1)$. We have a commutative diagram

$$\begin{array}{ccc} K_2(Y_1(M) \otimes \mathbb{Z}[1/M]) & \overset{\infty_M(a,b)}{\longrightarrow} & K_2(\mathbb{Z}[1/M, \zeta_M]) \\ \downarrow & & \downarrow \\ H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2)) & \overset{\infty_M(a,b)}{\longrightarrow} & H^2(\mathbb{Z}[1/M, \zeta_M], \mathbb{Z}_p(2)) \end{array}$$

in which the horizontal arrows are the compositions

$$K_2(Y_1(M)) \to K_2(\mathbb{Z}[1/M, \zeta_M][[q^{1/M}]][q^{-1}]) \to K_2(\mathbb{Z}[1/M, \zeta_M]),$$

$$H^2(Y_1(M), \mathbb{Z}_p(2)) \to H^2(\mathbb{Z}[1/M, \zeta_M][[q^{1/M}]][q^{-1}], \mathbb{Z}_p(2)) \to H^2(\mathbb{Z}[1/M, \zeta_M], \mathbb{Z}_p(2)),$$

respectively. Here the first arrows are the pull back by $\infty_M(a,b)$ and second arrows are the maps in 5.1.2 for the case $R = \mathbb{Z}[1/M, \zeta_M]$ and $T = q^{1/M}$. We denote the horizontal arrows by $x \mapsto x(\infty_M(a,b))$ and call them the value at $\infty_M(a,b)$.

The value at $\infty_M(0,1)$ is called the value at ∞. We use mainly the value at ∞. But to understand the relation of the value at ∞ with dual Hecke operators, we have to consider the values at other $\infty_M(a,b)$.

Remark. The value at $\infty_M(a,b)$ depends on the pair (a,b), not only on the cusp determined by $\infty_M(a,b)$. That is, even if $b \equiv b' \bmod a$, the value at $\infty_M(a,b)$ and the value at $\infty_M(a,b')$ can differ.

Proposition 5.1.4. *(1) For $a, b \in \mathbb{Z}/M\mathbb{Z}$ such that $(a,b) = (1)$, and for $c \in (\mathbb{Z}/M\mathbb{Z})^\times$, and for any element x of $K_2(Y_1(M) \otimes \mathbb{Z}[1/M])$ or of $H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2))$, we have*

$$(\langle c \rangle x)(\infty_M(a,b)) = x(\infty_M(ca, cb)).$$

(2) For $c \in (\mathbb{Z}/M\mathbb{Z})^\times$ and for any element x of $K_2(Y_1(M) \otimes \mathbb{Z}[1/M])$ or of $H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2))$, we have

$$(\langle c \rangle x)(\infty) = \sigma_c(x(\infty)).$$

(Here $\sigma_c \in \mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ is as in 2.4.3.)

Proof. The composition

$$\mathrm{Spec}(\mathbb{Z}[1/M, \zeta_M][[q^{1/M}][q^{-1}]) \overset{\infty_M(a,b)}{\longrightarrow} Y_1(M) \otimes \mathbb{Z}[1/M] \overset{\langle c \rangle}{\longrightarrow} Y_1(M) \otimes \mathbb{Z}[1/M]$$

coincides with $\infty_M(ca, cb)$. Hence we have (1). (2) follows from (1) and

$$x(\infty_M(0,c)) = \sigma_c(x(\infty)).$$

□

The following 5.1.5 (resp. 5.1.6 (1), resp. 5.1.6 (2)) is obtained from the diagram (1) (resp. (3), resp. (2)) in 1.3.5.

Proposition 5.1.5. *Let ℓ be a prime number which does not divide M. Then for any $x \in H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2))$, we have*

$$((1 - T^*(\ell) + \langle \ell \rangle^{-1} \ell)\, x)(\infty) = 0.$$

Lemma 5.1.6. *Let ℓ be a prime divisor of M. Let $x \in H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2))$. Let $(a, b) \in (\mathbb{Z}/M\mathbb{Z})^2$ and assume $(a, b) = (1)$. Let $R \geq 1$ be the positive divisor of M such that $(a) = (R)$ as an ideal of $\mathbb{Z}/M\mathbb{Z}$.*

(1) Assume $\ell | R$. Then

$$(T^*(\ell)x)(\infty_M(a, b)) = \sum_{a' \in \mathbb{Z}/M\mathbb{Z}, \ell a' = a} x(\infty_M(a', b))$$

where a' ranges over all elements of $\mathbb{Z}/M\mathbb{Z}$ such that $\ell a' = a$

(2) Assume $\ell \not| R$. Assume $b = \ell b'$ for some $b' \in \mathbb{Z}/M\mathbb{Z}$. Then we have

$$(T^*(\ell)x)(\infty_M(a, b)) = \ell \cdot x(\infty_M(a, b')).$$

Proposition 5.1.7. *Let $x \in H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2))$ and assume that $x(\infty_M(a, b)) + x(\infty_M(a, -b)) = 0$ for any $a, b \in \mathbb{Z}/M\mathbb{Z}$ such that $(a, b) = (1)$ and such that $a \neq 0$.*

(1) We have $y(\infty) = 0$ for any $y \in Ix \subset H^2(Y_1(M) \otimes \mathbb{Z}[1/M], \mathbb{Z}_p(2))$ where I is the Eisenstein ideal of $\mathfrak{H}(M)_{\mathbb{Z}_p}$.

(2) Let x^{ord} be the ordinary component of x. Then $x^{\mathrm{ord}}(\infty) = x(\infty)$.

Proof. (1) follows from 5.1.5 and 5.1.6. In fact, what we obtain is $y(\infty_M(0, 1)) + y(\infty_M(0, -1)) = 0$. But $y(\infty_M(0, -1)) = (\langle -1 \rangle y)(\infty) = y(\infty)$ for $\langle -1 \rangle$ acts trivially on $Y_1(M)$.

(2) By (1), we have $(T^*(p)^n)x)(\infty) = x(\infty)$ for any $n \geq 0$ since $T^*(p)^n \equiv 1 \bmod I$. Hence $x^{\mathrm{ord}}(\infty) = (\lim_n T^*(p)^{n!}x)(\infty) = x(\infty)$. □

Proposition 5.1.8. *The following diagram is commutative.*

$$\begin{array}{ccc}
H^2(Y_1(Np^{r+1}) \otimes \mathbb{Z}[1/Np], \mathbb{Z}_p(2)) & \xrightarrow{\infty \circ T^*(p)} & H^2(\mathbb{Z}[1/Np, \zeta_{Np^{r+1}}], \mathbb{Z}_p(2)) \\
\downarrow & & \downarrow \\
H^2(Y_1(Np^r) \otimes \mathbb{Z}[1/Np], \mathbb{Z}_p(2)) & \xrightarrow{\infty} & H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))
\end{array}$$

Here the vertical arrows are the trace maps, the lower horizontal arrow is $x \mapsto x(\infty)$, and the upper horizontal arrow is $x \mapsto (T^*(p)x)(\infty)$.

Proof. We have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Spec}(\mathbb{Z}[1/Np, \zeta_{Np^{r+1}}][[q]][q^{-1}]) & \to & Y_1(Np^{r+1}) & \leftarrow & Y(1, Np^{r+1}(p)) & \xrightarrow{\psi_p} & Y_1(Np^{r+1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\mathbb{Z}[1/Np, \zeta_{Np^r}][[q]][q^{-1}]) & \to & Y(1, Np^r(p)) & = & Y(1, Np^r(p)) & \xrightarrow{\psi_p} & Y_1(Np^r)
\end{array}$$

in which the upper middle arrow is the canonical projection, the left upper horizontal arrow is induced by $\infty_{Np^{r+1}}(0, 1)$, the left lower horizontal arrow is the unique morphism which makes the left diagram commutative (for the moduli interpretation of $Y(1, Np^r(p))$ in 1.2.3, it is defined by the triple (E_q, ζ_{Np^r}, C) where C is the subgroup of E_q generated by the section $\zeta_{Np^{r+1}}$ of E_q), and the squares except the middle one are cartesian. Let $x \in H^2(Y_1(Np^{r+1}), \mathbb{Z}_p(2))$ and let y be the trace of x in $H^2(Y_1(Np^r), \mathbb{Z}_p(2))$. Let $A \in H^2(\mathbb{Z}[1/Np, \zeta_{Np^{r+1}}][[q]][q^{-1}], \mathbb{Z}_p(2))$ be the pull back of $T^*(p)x$ under $\infty_{Np^{r+1}}(0, 1)$, and let the element B (resp. C) of $H^2(\mathbb{Z}[1/Np,\zeta_{Np^r}][[q]][q^{-1}],\mathbb{Z}_p(2))$ be the pull back of $\psi_p^*(y)$ (resp. y) under the left lower horizontal arrow (resp. under $\infty_{Np^r}(0,1)$). Write by Tr the trace maps $H^2(\mathbb{Z}[1/Np,\zeta_{Np^{r+1}}][[q]][q^{-1}],\mathbb{Z}_p(2)) \to H^2(\mathbb{Z}[1/Np,\zeta_{Np^r}][[q]][q^{-1}],\mathbb{Z}_p(2))$ and $H^i(\mathbb{Z}[1/Np,\zeta_{Np^{r+1}}],\mathbb{Z}_p(i)) \to H^i(\mathbb{Z}[1/Np,\zeta_{Np^r}],\mathbb{Z}_p(i))$ $(i=1,2)$. Then the above commutative diagram shows that $\mathrm{Tr}(A)=B$. Write $A=a_2+\{a_1,q\}$ with $a_i \in H^i(\mathbb{Z}[1/Np,\zeta_{Np^{r+1}}],\mathbb{Z}_p(i))$, and write $B=b_2+\{b_1,q\}$, $C=c_2+\{c_1,q\}$ $(b_i,c_i \in H^i(\mathbb{Z}[1/Np,\zeta_{Np^r}],\mathbb{Z}_p(i))$. Then by $\mathrm{Tr}(A)=B$, we have $\mathrm{Tr}(a_i)=b_i$. On the other hand, the composition $\mathrm{Spec}(\mathbb{Z}[1/Np,\zeta_{Np^r}][[q]][q^{-1}]) \to Y(1,Np^r(p)) \to Y_1(Np^r)$ of the left lower horizontal arrow and the canonical projection is the morphism induced by $\infty_{Np^r}(0,1)$. This shows that $b_2=c_2$ and $b_1=pc_1$. Since $(T^*(p)x)(\infty)=a_2$ and $y(\infty)=c_2$, we have $\mathrm{Tr}((T^*(p)x)(\infty))=y(\infty)$. □

Let $M \geq 1$ and assume $p|M$.

Theorem 5.1.9. *(1) The value of* ${}_{c,d}z_{1,M,1}(u,v)$ *at* ∞ *is*

$$\{(-\zeta_M^u)^{(c-c^2)/2}\frac{(1-\zeta_M^u)^{c^2}}{1-\zeta_M^{cu}},\ (-\zeta_M^v)^{(d-d^2)/2}\frac{(1-\zeta_M^v)^{d^2}}{1-\zeta_M^{dv}}\}.$$

(2) The value of any element of $I \cdot {}_{c,d}z_{1,M,1}(u,v)$ *at* ∞ *is zero.*

(3) The value of the ordinary component ${}_{c,d}z_{1,M,1}(u,v)^{\mathrm{ord}}$ *of* ${}_{c,d}z_{1,M,1}(u,v) \in H^1(\mathbb{Z}[1/Np],H^1_{\acute{e}t}(Y_1(Np^r))(2))$ *at* ∞ *coincides with the value of* ${}_{c,d}z_{1,M,1}(u,v)$ *(described in (1)).*

Proof. By the q-expansion of a Siegel unit introduced in 2.1.1, we have the following multiplicative congruence for any $r,s \in \mathbb{Z}$ such that $(r,s) \not\equiv (0,0) \bmod M$:

$${}_cg_{r/M,s/M} \equiv q^{i/M}U(r,s) \mod 1+q^{1/M}\mathbb{Z}[1/M,\zeta_M][[q^{1/M}]][q^{-1}]$$

where $i \in \mathbb{Z}$ and

$$U(r,s)=(-\zeta_M^s)^{(c-c^2)/2}(1-\zeta_M^s)^{c^2}/(1-\zeta_M^{cs}) \text{ in the case } M|r,$$
$$U(r,s)=(-\zeta_M^s)^{t(r/M,c)} \text{ unless } M|r$$

for some integer $t(r/M,c)$ which depends only on $r/M \in \mathbb{Q}/\mathbb{Z}$ and $c \in \mathbb{Z}$. (1) follows from the case $M|r$ of this.

In the case M does not divide r, $U(r,-s)=U(r,s)^{-1}$. Hence by (1) of Proposition 5.1.7 (resp. (2) of Proposition 5.1.7), (2) (resp. (3)) is reduced to the following lemma. □

Lemma 5.1.10. *Unless* $a=0$*, we have*

$${}_{c,d}z_{1,M,1}(u,v)(\infty_M(a,b)) + {}_{c,d}z_{1,M,1}(u,v)(\infty_M(a,-b))=0.$$

Proof. This follows from the above computation on Siegel units by the following argument.

Let $a,b \in \mathbb{Z}$ and assume $(a,b)=1$. Take $x,y \in \mathbb{Z}$ such that $\begin{pmatrix} x & y \\ a & b \end{pmatrix} \in SL(2,\mathbb{Z})$. Then the pull back of $\{{}_cg_{0,u/M},{}_dg_{0,v/M}\}$ in $K_2(\mathbb{Z}[1/M,\zeta_M][[q^{1/M}]][q^{-1}])$ under $\infty_M(a,b)$ coincides with the usual q-expansion of

$$\{{}_cg_{0,u/M},{}_dg_{0,v/M}\}\begin{pmatrix} x & y \\ a & b \end{pmatrix} = \{{}_cg_{au/M,bu/M},{}_dg_{av/M,bv/M}\}.$$

□

5.2 From modular symbols to cyclotomic symbols

Let $N \geq 1$ and assume N is prime to p. Let

$$\Lambda = \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times].$$

We will apply section 5.1 by taking $M = Np^r$ with $r \geq 1$.

5.2.1. We introduce Conjecture 5.8 of Sharifi [51]. Recall that $[u:v]_r$ ($u, v \in \mathbb{Z}/Np^r\mathbb{Z}$, $(u,v,Np)=1$, $u \neq 0$, $v \neq 0$) generate $H^1_{\text{ét}}(Y_1(Np^r))_0(1)$. Here $H^1_{\text{ét}}(Y_1(Np^r))_0$ is as in 3.3.7. By Sharifi [51], Proposition 5.7, we have a homomorphism

$$\varpi_r : H^1_{\text{ét}}(Y_1(Np^r))_0(1) \to H^2(\mathbb{Z}[\frac{1}{Np}, \zeta_{Np^r}], \mathbb{Z}_p(2))$$

which sends $[u:v]_r$ ($u, v \in \mathbb{Z}/Np^r\mathbb{Z}$, $(u,v,Np)=1$, $u \neq 0$, $v \neq 0$) to

$$\{1 - \zeta^u_{Np^r},\ 1 - \zeta^v_{Np^r}\}^+$$

and which factors through the projection

$$H^1_{\text{ét}}(Y_1(Np^r))_0(1) \to H^1_{\text{ét}}(Y_1(Np^r))_0(1)^+ = H^1_{\text{ét}}(Y_1(Np^r))_0^-(1).$$

This homomorphism commutes with the actions of $\mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ where the group element $[a]$ of $\mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ for $a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ acts on $H^1_{\text{ét}}(Y_1(Np^r))_0(1)$ as the diamond operator $\langle a \rangle$ and acts on $H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))$ as $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{Np^r})/\mathbb{Q})$ (2.4.3).

5.2.2. [51], **Conjecture** 5.8. The restriction of ϖ_r to $H^1_{\text{ét}}(X_1(Np^r))(1)$ kills $IH^1_{\text{ét}}(X_1(Np^r))(1)$. Here I is the Eisenstein ideal of $\mathfrak{h}(Np^r)_{\mathbb{Z}_p}$.

The following Theorem 5.2.3 (1) implies that this conjecture is true.

Theorem 5.2.3. *(1) The homomorphism ϖ_r kills $IH^1_{\acute{e}t}(Y_1(Np^r))_0(1)$. Here I is the Eisenstein ideal of $\mathfrak{H}(Np^r)_{\mathbb{Z}_p}$.*

(2) The following diagram is commutative for any $r \geq 1$.

$$\begin{array}{ccc} H^1_{\acute{e}t}(Y_1(Np^{r+1}))_0(1) & \overset{\varpi_{r+1}}{\to} & H^2(\mathbb{Z}[1/Np, \zeta_{Np^{r+1}}], \mathbb{Z}_p(2))^+ \\ \downarrow & & \downarrow \\ H^1_{\acute{e}t}(Y_1(Np^r))_0(1) & \overset{\varpi_r}{\to} & H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))^+. \end{array}$$

Here the vertical arrows are the trace maps. Consequently, we have a Λ-homomorphism

$$\varpi : \tilde{H}_0(1)/I\tilde{H}_0(1) \to \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))^+$$

which sends $(x_r)_r$ to $(\varpi_r(x_r))_r$. (Here Λ acts on the left hand side via diamond operators and on the right as the completed group ring of the Galois group.)

We prove this theorem in this section 5.2.

Lemma 5.2.4. *(1) For* $x \in H^1_{\acute{e}t}(Y_1(Np^r))$*, the images of* x *and* x^{ord} *in* $H^1_{\acute{e}t}(Y_1(Np^r))/IH^1_{\acute{e}t}(Y_1(Np^r))$ *coincide.*

(2) We have an isomorphism

$$H^1_{\acute{e}t}(Y_1(Np^r))^{\mathrm{ord}}/IH^1_{\acute{e}t}(Y_1(Np^r))^{\mathrm{ord}} \xrightarrow{\cong} H^1_{\acute{e}t}(Y_1(Np^r))/IH^1_{\acute{e}t}(Y_1(Np^r)).$$

Proof. (1) is easily seen. (2) follows from (1) (the inverse map is given by $x \mapsto x^{\mathrm{ord}}$). □

Lemma 5.2.5. *Let* $M \geq 1$ *and assume* $p|M$*. Then we have a canonical isomorphism* $H^2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m], \mathbb{Z}_p(2))^{\mathrm{ord}} \cong H^1(\mathbb{Z}[1/Mm, \zeta_m], H^1_{\acute{e}t}(Y_1(M))(2))^{\mathrm{ord}}$*.*

Proof. By the spectral sequence

$$E_2^{i,j} = H^i(\mathbb{Z}[1/Mm, \zeta_m], H^j_{\text{ét}}(Y_1(M))(2)) \Rightarrow H^{i+j}(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m], \mathbb{Z}_p(2))$$

and the fact $H^j_{\text{ét}}(Y_1(M)) = 0$ if $j \geq 2$ for $Y_1(M) \otimes \bar{\mathbb{Q}}$ is an affine curve, we have an exact sequence

$$0 \to H^2(\mathbb{Z}[1/Mm, \zeta_m], \mathbb{Z}_p(2)) \to H^2(Y_1(M) \otimes \mathbb{Z}[1/Mm, \zeta_m], \mathbb{Z}_p(2))$$
$$\to H^1(\mathbb{Z}[1/Mm, \zeta_m], H^1_{\text{ét}}(Y_1(M))(2)) \to 0.$$

Since $T^*(p)$ acts on the part $H^2(\mathbb{Z}[1/Mm, \zeta_m], \mathbb{Z}_p(2))$ as the multiplication by p, the ordinary component of this part vanishes. □

5.2.6. Let

$$\mathcal{S} := \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))^+.$$

Here $(+)$ denotes the $(+)$-part for the complex conjugation. We regard $\mathcal{S}$ as a Λ-module where Λ is identified with $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q})]]$.

Then $\mathcal{S}$ is a free $\mathbb{Z}_p$-module of finite rank by Ferrero-Washington [7].

5.2.7. Let $\mu = p((1 + Np)^2 - \langle 1 + Np \rangle) \in \Lambda$ as in 3.3.9. Consider the composition of Λ-homomorphisms

$$\tilde{H}_0(1) \to H^1(\mathbb{Z}[1/Np], \tilde{H}(2)) \otimes_\Lambda \Lambda\,\mu^{-1}$$
$$= \varprojlim_r H^2(Y_1(Np^r), \mathbb{Z}_p(2))^{\mathrm{ord}} \otimes_\Lambda \Lambda\,\mu^{-1} \to \mathcal{S} \otimes_\Lambda \Lambda\,\mu^{-1}$$

where:

we regard Λ as the ring of diamond operators acting on $\tilde{H}$, except that the last Λ denotes the cyclotomic Iwasawa algebra of $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q})$,

the first arrow is by Theorem 3.3.9,

the $=$ is by Lemma 5.2.5,

the last arrow is $(x_r)_r \mapsto ((T^*(p)^r x_r)(\infty))_r$ (Proposition 5.1.8).

By Theorem 5.1.9 (2) (3), if $(x_r)_r \in \tilde{H}_0(1)$ is sent to $(y_r)_r \in \varprojlim_r H^2(Y_1(Np^r), \mathbb{Z}_p(2))^{\mathrm{ord}} \otimes_\Lambda \Lambda\,\mu^{-1}$, then the above composite map sends $(x_r)_r$ to $(y_r(\infty))_r$.

Lemma 5.2.8. *(1) The map* $\mu : \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is bijective.*

(2)The canonical homomorphism $\mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathcal{S} \otimes_\Lambda \Lambda\,\mu^{-1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is an isomorphism.*

Proof. It is sufficient to prove $\Lambda/(\mu) \otimes_\Lambda \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$. We have

$$\Lambda/(\mu) \otimes_\Lambda \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^2(\mathbb{Z}[1/Np, \zeta_{Np}], \mathbb{Q}_p)^+.$$

The vanishing of the last cohomology group is a consequence of Leopoldt conjecture for abelian fields over $\mathbb{Q}$ proved by Brumer [3] which we apply to the field $\mathbb{Q}(\zeta_{Np})$. □

5.2.9. By composing the map $\tilde{H}_0(1) \to \mathcal{S} \otimes_\Lambda \Lambda\,\mu^{-1}$ in 5.2.7 and the map $\mathcal{S} \otimes_\Lambda \Lambda\,\mu^{-1} \to \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ given by 5.2.8, we obtain a homomorphism

$$\tilde{H}_0(1) \to \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proposition 5.2.10. *(1) The homomorphism in 5.2.9 kills $I \cdot \tilde{H}_0(1)$, where I is the Eisenstein ideal of $\mathfrak{H}$.*

(2) The image of the homomorphism in 5.2.9 is contained in $\mathcal{S}$.

Proof. (1) follows from Theorem 5.1.9 (2), (3). We prove (2).

Claim 1. Let $u, v \in \mathbb{Z}$, $(u, v, N) = 1$, $(v, p) = 1$, $u \not\equiv 0 \bmod Np$. Then for any $s \geq 1$, the homomorphism in 5.2.9 sends $([p^{r-s}u : v]_r)_{r \geq s} \in \varprojlim_{r \geq s} H^1(Y_1(Np^r)(\mathbb{C}), \mathbb{Z})(1)$ to $(\{1 - \zeta_{Np^s}^u, 1 - \zeta_{Np^r}^v\}^+)_{r \geq s} \in \mathcal{S}$ ([51], Lemma 3.2).

Proof. Take $c, d \in \mathbb{Z}$ such that $(cd, 6Np) = 1$ and $c \equiv 1, d \equiv 1 \bmod Np^s$. Then

$${}_{c,d}[p^{r-s}u : v]_r = (c^2 - 1)(d^2 - \langle d \rangle)[p^{r-s}u : v]_r.$$

This goes to $-{}_{c,d}z_{1,Np^r,1}(p^{r-s}u, v)$ and goes to $(c^2-1)(d^2-\sigma_d)(\{1-\zeta_{Np^s}^u, 1-\zeta_{Np^r}^v\}^+)_{r \geq s}$. Hence if x denotes the image of $([p^{r-s}u : v]_r)_r$ in $\mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $(c^2-1)(d^2-\sigma_d)x = (c^2-1)(d^2-\sigma_d)(\{1-\zeta_{Np^s}^u, 1-\zeta_{Np^r}^v\}^+)_{r \geq s}$. By the above 5.2.7 and 5.2.8, we have $x = (\{1-\zeta_{Np^s}^u, 1-\zeta_{Np^r}^v\}^+)_{r \geq s}$.

By Claim 1 and Lemma 3.3.8, the image of $\tilde{H}_0(1) \to \mathcal{S} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is contained in $\mathcal{S}$. □

5.2.11. By Proposition 5.2.10, we obtained a homomorphism $\tilde{H}_0(1) \to \mathcal{S}$. By the case $k = 2$ of Proposition 1.5.8, for each $r \geq 1$, this induces a homomorphism

$$H^1_{\text{ét}}(Y_1(Np^r))_0(1) \to H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))^+. \tag{5.1}$$

Claim 1 in the proof of Proposition 5.2.10 tells that this map sends $[u : v]_r$ to $\{1-\zeta_{Np^r}^u, 1-\zeta_{Np^r}^v\}^+$. Since $[u : v]_r$ generate $H^1_{\text{ét}}(Y_1(Np^r))_0$ (by 2.4.1), this shows that the map (5.1) coincides with the map ϖ_r of Sharifi in 5.2.1.

It kills $I \cdot H^1_{\text{ét}}(Y_1(Np^r))_0^{\text{ord}}(1)$ by Proposition 5.2.10 (1).

Thus Theorem 5.2.3 is proved.

Proposition 5.2.12. *The map $\tilde{H}_0(1) \to \mathcal{S}$ sends Mazur-Kitagawa L-function $\mathcal{L} \in H[[\mathbb{Z}_p^\times]]$ to*

$$\mathcal{C} := \Big(\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \{1 - \zeta_{p^r}^a, 1 - \zeta_{Np^r}\} \cdot [a]\Big)_r. \in \mathcal{S}[[\mathbb{Z}_p^\times]]$$

Furthermore, for any divisor $M \geq 1$ of N, it sends the modified Mazur-Kitagawa L-function

$$\mathcal{L}^{\star}_{N,M} := (\sum_{1\leq j<Np^r,(j,M)=1} T^*(p)^{-r}[j:M]_r \cdot [j]_r)_r$$
$$\in \tilde{H}_0[[\mathbb{Z}_p \times \mathbb{Z}/N\mathbb{Z}]]/\tilde{H}_0$$

($\tilde{H}_0 \to \tilde{H}_0[[\mathbb{Z}_p \times \mathbb{Z}/N\mathbb{Z}]]$ is $x \mapsto x \cdot [0]$) in Sharifi [51] section 6.1, to

$$\mathcal{C}^{\star}_{N,M} := (\sum_{1\leq j<Np^r,(j,M)=1} \{1-\zeta^j_{Np^r}, 1-\zeta^M_{Np^r}\} \cdot [j])_r] \in \mathcal{S}[[\mathbb{Z}_p \times \mathbb{Z}/N\mathbb{Z}]]/\mathcal{S}$$

Proof. This is seen easily. □

5.2.13. Recall that $\tilde{H}_0 \to \mathcal{S}$ is the composition $\tilde{H}_0 \to H^1(\mathbb{Z}[1/Np], \tilde{H}(2)) \otimes_\Lambda \Lambda\,\mu^{-1} \to \mathcal{S}$. The first arrow (the homomorphism $z^\sharp_{Np^\infty}$) sends $\mathcal{L} \in H[[\mathbb{Z}_p^\times]]$ to

$$(\sum_{a\in(\mathbb{Z}/p^r\mathbb{Z})^\times} T^*(p)^{-r} z_{1,Np^r,1}(Na,1)^{\mathrm{ord}} \cdot [a])_r \in H^1(\mathbb{Z}[1/Np], \tilde{H}(2))[[\mathbb{Z}_p^\times]] \otimes_\Lambda \Lambda\,\mu^{-1}$$

and the second arrow sends the last element to $\mathcal{C}$. The fact $(\sum_{a\in(\mathbb{Z}/p^r\mathbb{Z})^\times} T^*(p)^{-r} z_{1,Np^r,1}(Na,1)^{\mathrm{ord}} \cdot [a])_r$ belongs to the inverse limit also follows from Proposition 2.2.5.

5.3 From modular symbols to tame symbols

5.3.1. We will show (Theorem 5.3.3) that in the correspondence

$$\text{modular symbol} \mapsto \text{cyclotomic symbol}$$

in Theorem 5.2.3, the following (1) determines the following (2):

(1) The boundary of the modular symbol at cusps.

(2) The boundary (= the tame symbol) of the corresponding cyclotomic symbol at prime divisors ℓ of N.

It is interesting that the geometric boundary in (1) determines the arithmetic boundary in (2).

Let $N \geq 1$, $r \geq 1$, and assume N is prime to p.

5.3.2. Let ℓ be a prime divisor of N, let λ be a place of $\mathbb{Q}(\zeta_{Np^r})$ lying over ℓ, and let $\kappa(\lambda)$ be the residue field of λ.

We define a homomorphism

$$\bar{\varpi} : \oplus_{(a,b)\in P_{Np^r}} \mathbb{Z}_p \to H^2(\mathbb{Q}(\zeta_{N\ell})_\lambda, \mathbb{Z}_p(2)) \cong H^1(\kappa(\lambda), \mathbb{Z}_p(1)) \cong \kappa(\lambda)^\times \otimes \mathbb{Z}_p,$$

where P_{Np^r} is as in 1.3.1. For $(a,b) \in P_{Np^r}$, the image of $1 \in \mathbb{Z}_p$ at (a,b) under $\bar{\varpi}$ is as follows. It is defined to be zero unless $a = a'\ell^s$ for some $a' \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Here $0 \leq s \leq e$ where e is the ℓ-adic order of N. In the last case, it is defined to be the image of $(1-\zeta^{1/a'\ell^e}_{N'p^r})^{\ell^{e-s}} \in \kappa(\lambda)^\times$ where N' is the prime to ℓ-part of N and $\zeta^{1/a'\ell^e}_{N'p^r}$ is the unique $N'p^r$-th root of 1 whose $a'\ell^e$-th power coincides with $\zeta_{N'p^r}$.

Theorem 5.3.3. *Let ℓ be a prime divisor of N, let $r \geq 1$, and let λ be a place of the field $\mathbb{Q}(\zeta_{Np^r})$ lying over ℓ. Then the following diagram is commutative.*

$$\begin{array}{ccc} H^1_{\acute{e}t}(Y_1(Np^r))_0(1) & \xrightarrow{\varpi} & H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2)) \\ \downarrow & & \downarrow \\ \bigoplus_{(a,b)\in P_{Np^r}} \mathbb{Z}_p & \xrightarrow{\bar{\varpi}} & H^2(\mathbb{Q}(\zeta_{Np^r})_\lambda, \mathbb{Z}_p(2)) \end{array}$$

Here $H^1_{\acute{e}t}(Y_1(Np^r))_0$ is as in 3.3.7, and the (a,b)-component of the left vertical arrow is the boundary map at the cusp associated to $\infty_{Np^r}(a,b)$.

Proof. It is enough to check the commutativity for the generators $[u:v]$ of $H^1_{\text{ét}}(Y_1(Np^r))_0$ (here $u, v \in \mathbb{Z}/Np^r\mathbb{Z}, u \neq 0, v \neq 0$, $(u,v) = 1$).

We have a commutative diagram

$$\begin{array}{ccc} K_2(\mathbb{Q}(\zeta_{Np^r})_\lambda) & \xrightarrow{\partial} & \kappa(\lambda)^\times \\ \downarrow & & \downarrow \\ H^2(\mathbb{Q}(\zeta_{Np^r})_\lambda, \mathbb{Z}_p(2)) & \cong & H^1(\kappa(\lambda), \mathbb{Z}_p(1)) \end{array}$$

where $\kappa(\lambda)$ is the residue field of λ and ∂ is the tame symbol. We compute the tame symbol of $\{1-\zeta^u_{Np^r}, 1-\zeta^v_{Np^r}\}$ at λ. As is well known, $1-\zeta^u_{Np^r}$ is a λ-adic unit unless the order of $\zeta^u_{Np^r}$ is a power of ℓ. Hence the tame symbol is trivial unless one of $\zeta^u_{Np^r}$ and $\zeta^v_{Np^r}$ is of order a power of ℓ. If $u/Np^r \equiv u'/\ell^{e-s} \bmod \mathbb{Z}$ with $u' \in \mathbb{Z}$, $(u', \ell) = 1$ and $0 \leq s \leq e$, then v is prime to p and hence $\zeta^v_{Np^r}$ can not be of order a power of ℓ. Hence the tame symbol is the ℓ^{e-s}-th power of the residue class $1-\zeta^v_{Np^r}$ which is the residue class of $1-\zeta^{v/\ell^e}_{N'p^r}$. Here N' is the prime to ℓ-part of N and $\zeta^{1/\ell^e}_{N'p^r}$ is the unique ℓ^e-th root of $\zeta_{N'p^r}$ which is a $N'p^r$-th root of 1.

On the other hand, the boundary of $[u:v]_r$ is as in Lemma 3.3.11.

Theorem 5.3.3 follows from these. □

5.3.4. Note that the map

$$\varprojlim_r H^2(\mathbb{Z}[1/p, \zeta_{Np^r}], \mathbb{Z}_p(2)) \to \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))$$

is injective.

Theorem 5.3.5. *(1) Under the homomorphism ϖ in Theorem 5.2.3, the image of $H(1)$ in $\varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))^+$ is contained in the image of the injection in 5.3.4, and consequently, we have a Λ-homomorphism*

$$\varpi : H(1)/IH(1) \to \varprojlim_r H^2(\mathbb{Z}[1/p, \zeta_{Np^r}], \mathbb{Z}_p(2))^+.$$

Here I denotes the Eisenstein ideal of $\mathfrak{h}$.

(2) For each $r \geq 1$, there exists a unique homomorphism

$$\varpi_r : H^1_{\acute{e}t}(X_1(Np^r))(1)/IH^1_{\acute{e}t}(X_1(Np^r))(1) \to H^2(\mathbb{Z}[1/p, \zeta_{Np^r}], \mathbb{Z}_p(2))^+$$

which commutes with the homomorphism in (1). This map is compatible with the map ϖ_r in 5.2.1 of Sharifi.

Proof. The uniqueness in Theorem 5.3.5 follows from the surjectivity of the projection $H \to H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}}$ (the case $k = 2$ of Proposition 1.5.8). By Theorem 5.3.3 and by the exact sequence

$$H^2(\mathbb{Z}[1/p, \zeta_{Np^r}], \mathbb{Z}_p(2)) \to H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2)) \to \oplus_\lambda H^2(\mathbb{Q}(\zeta_{Np^r})_\lambda, \mathbb{Z}_p(2))$$

where λ ranges over all prime divisors of N in $\mathbb{Q}(\zeta_{Np^r})$, we have Theorem 5.3.5 (1). Theorem 5.3.5 (2) follows from (1) by [40] Proposition 2.3.5. □

6 The map Υ

In this section, we define the map Υ from the ideal class group side to the modular symbol side.

6.1 Formulation from here

In the rest (sections 6–11) of this paper, we use the following notation, and make the following Assumptions 1 – 4.

6.1.1. We fix an integer $N \geq 1$ which is prime to p.

Assumption 1. $\varphi(N)$ (i.e. the order of $(\mathbb{Z}/N\mathbb{Z})^\times$) is prime to p.

6.1.2. Let

$$\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times]] = \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times].$$

The group element of Λ corresponding to $c \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ is denoted by $[c]$.

Let

$$K = \mathbb{Q}(\zeta_{Np^\infty}) := \cup_r \mathbb{Q}(\zeta_{Np^r}).$$

The isomorphism

$$\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \cong \mathrm{Gal}(K/\mathbb{Q}) \; ; \; c \mapsto \sigma_c \;\; (2.4.3)$$

induces a canonical isomorphism

$$\Lambda \cong \mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]].$$

We have also a canonical homomorphism

$$\Lambda \to \mathfrak{h} \; ; \; [c] \mapsto \langle c \rangle.$$

6.1.3. For a character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times$, let O_ψ be the subring of $\bar{\mathbb{Q}}_p$ generated over $\mathbb{Z}_p$ by the image of ψ. Note that O_ψ is the valuation ring of some finite extension of $\mathbb{Q}_p$.

For a Λ-module M and for a character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times$, define the ψ-component M_ψ of M by

$$M_\psi = M \otimes_{\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]} O_\psi,$$

where $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \to O_\psi$ is the homomorphism induced by ψ, and $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]$ acts on M via the ring homomorphism $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \to \Lambda$ which is defined by identifying $(\mathbb{Z}/Np\mathbb{Z})^\times$ with the torsion part of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ in the canonical way.

Then as a topological ring over O_ψ, the ψ-component Λ_ψ of Λ is isomorphic to the ring $O_\psi[[T]]$ of formal power series in one variable over O_ψ.

6.1.4. From now on, in the rest of this paper, we fix an even character

$$\theta : (\mathbb{Z}/Np\mathbb{Z})^\times = (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times.$$

Let

$$\chi = \theta\omega^{-1} : (\mathbb{Z}/Np\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times.$$

Here

$$\omega : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{Z}_p^\times$$

is the Teichmüller character (that is, the composition $(\mathbb{Z}/Np\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$ where the first arrow is the projection and the second arrow is the canonical lifting). So, χ is an odd character.

We make the following Assumption 2 – Assumption 4.

Assumption 2. θ is primitive.

Assumption 3. If the restriction of χ to $(\mathbb{Z}/p\mathbb{Z})^\times \subset (\mathbb{Z}/Np\mathbb{Z})^\times$ is trivial, then the restriction $\chi|_{(\mathbb{Z}/N\mathbb{Z})^\times}$ of χ to $(\mathbb{Z}/N\mathbb{Z})^\times \subset (\mathbb{Z}/Np\mathbb{Z})^\times$ satisfies $\chi|_{(\mathbb{Z}/N\mathbb{Z})^\times}(p) \neq 1$.

Assumption 4. In the case $N = 1$, $(\chi, \theta) \neq (\omega, \omega^2)$.

6.1.5. Thus in the case $N = 1$, Assumptions 1–4 are just that $\theta \neq 1, \omega^2$. These excluded cases are in fact "trivial" cases, for in the case $N = 1$, $\mathfrak{h}_\psi/I_\psi = 0$ for $\psi = 1, \omega^2$.

6.1.6. From now, we denote simply by

$$\xi \in \Lambda_\theta$$

the image of ${}^{-1}\xi \in Q(\Lambda)$ (4.1.1) in $Q(\Lambda)_\theta$ (which belongs to Λ_θ).

6.1.7. As in Ohta [41] Corollary A.2.4 (see also Mazur-Wiles [32]), the inclusion map $\Lambda_\theta \to \mathfrak{h}_\theta$ (6.1.2) induces an isomorphism

$$\Lambda_\theta\,/(\xi) \xrightarrow{\cong} \mathfrak{h}_\theta/I_\theta.$$

6.1.8. Some authors adopt different formulations of the θ-component of the p-adic Hecke algebra. We hope the following comment helps the reader to see our formulation. Assume $N = 1$. If $k \geq 2$ is an even integer, the condition $p|\zeta(1-k)$ (which is equivalent to $X_{p^\infty,\chi} \neq 0$ for $\chi = \omega^{1-k}$) is equivalent in our formulation to the condition $\mathfrak{h}_\theta/I_\theta \neq 0$ for $\theta = \omega^{2-k}$. Thus the fact $691|\zeta(-11)$ tells that in the case $p = 691$, $\mathfrak{h}_\theta/I_\theta \neq 0$ for $\theta = \omega^{-10}$.

6.2 Drinfeld-Manin modification

6.2.1. For an $\mathfrak{H}$-module M, let

$$M_{DM} := M \otimes_{\mathfrak{H}} \mathfrak{h}.$$

We consder the cases

$$M \;=\; \tilde{H},\; M_\Lambda,\; H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}},\; M_2(Np^r)^{\text{ord}}_{\mathbb{Z}_p} \quad (r \geq 1).$$

(In the latter two cases, $(-)_{DM}$ coincides with $\otimes_{\mathfrak{H}(Np^r))^{\text{ord}}}\mathfrak{h}(Np^r)^{\text{ord}}$ by 1.5.5.) We use the notation DM for this operation $(-)_{DM}$ and call it the Drinfeld-Manin modification because it is closely related to Drinfeld-Manin splitting 1.9.3 as in 6.2.3 below.

6.2.2. Let $\tilde{H}_{DM'}$ (resp. $M_{\Lambda,DM'}$, resp. $H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{DM'}$, $M_2(Np^r)^{\text{ord}}_{\mathbb{Z}_p,DM'}$) be the image of $\tilde{H}$ (resp. M_Λ, resp. $H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}$, resp. $M_2(Np^r)_{\mathbb{Z}_p}$) in $H \otimes_\Lambda Q(\Lambda)$ (resp. $S_\Lambda \otimes_\Lambda Q(\Lambda)$, resp. $H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, resp. $S_2(Np^r)_{\mathbb{Q}_p}$) under the Drinfeld-Manin splitting.

Lemma 6.2.3. *The canonical maps $M_{DM,\theta,E} \to M_{DM',\theta,E}$ for $M = \tilde{H}$, M_Λ, $H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}$, $M_2(Np^r)^{\text{ord}}_{\mathbb{Z}_p,DM}$ are isomorphisms.*

Here $(-)_{\theta,E}$ denotes the Eisenstein component (1.9.2) of the θ-component. The part of this Lemma for $\tilde{H}$ is given in Lemma 4.1 in [51]. The proof of 6.2.3 is given in 6.2.8 below.

In this subsection 6.2, we present basic facts about Drinfeld-Manin modification and prove the following Proposition 6.2.4, whose Corollary 6.2.10 will play in section 6.3 an important role in the proof of Proposition 6.3.9..

Proposition 6.2.4. *$\tilde{H}_{DM,\theta,E}$ is a free Λ_θ-module.*

The proof of this 6.2.4 is given in 6.2.9 below.

6.2.5. As in Ohta [41] Theorem 1.5.5, we have exact sequences

$$0 \to H_{\theta,E} \to \tilde{H}_{\theta,E} \to \Lambda_\theta \to 0, \quad 0 \to S_{\Lambda,\theta,E} \to M_{\Lambda,\theta,E} \to \Lambda_\theta \to 0.$$

Here $\tilde{H}_{\theta,E} \to \Lambda_\theta$ is the boundary map at 0-cusps (3.3.7), and $M_{\Lambda,\theta,E} \to \Lambda_\theta$ is the constant term $\sum_{n=0}^{\infty} a_n q^n \mapsto a_0$.

6.2.6. Let $H^1_{\text{ét}}(Y_1(Np^r))_0 \subset H^1_{\text{ét}}(Y_1(Np^r))$ $(r \geq 1)$ and $\tilde{H}_0 = \varprojlim_r H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_0 \subset \tilde{H}$ be as in 3.3.7. Then by 6.2.5, the canonical injections $H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}}_{\theta,E} \to H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{0,\theta,E}$ and $H_{\theta,E} \to \tilde{H}_{0,\theta,E}$ are isomorphisms (Ohta [41] Proposition 3.1.2).

Since $[u:v]_r$ with $u, v \in \mathbb{Z}/Np^r\mathbb{Z} - \{0\}$ such that $(u,v) = (1)$ belongs to $H^1_{\text{ét}}(Y_1(Np^r))_0$, $[u:v]_{r,\theta,E}$ is regarded as an element of $H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}}_{\theta,E}$.

6.2.7. As $\mathfrak{H}$-modules, $\tilde{H}_{\theta,E}/H_{\theta,E}$ and $M_{\Lambda,\theta,E}/S_{\Lambda,\theta,E}$ are isomorphic to $\mathfrak{H}_\theta/I_{\mathfrak{H},\theta}$, where $I_{\mathfrak{H}}$ is the Eisenstein ideal of $\mathfrak{H}$ (1.9.1). From this, we obtain

$$\tilde{H}_{DM,\theta,E}/H_{\theta,E} \cong \mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta\,/(\xi), \quad M_{\Lambda,DM,\theta,E}/S_{\Lambda,\theta,E} \cong \mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta\,/(\xi).$$

Similarly we have

$$H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{DM,\theta,E}/H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}}_{\theta,E} \cong \mathfrak{h}(Np^r)_{\mathbb{Z}_p,\theta,E}/I \cong \Lambda_{r,\theta}/(\xi),$$

$$(M_2(Np^r)^{\text{ord}}_{\mathbb{Z}_p})_{DM,\theta,E}/S_2(Np^r)_{\mathbb{Z}_p,\theta,E} \cong \mathfrak{h}(Np^r)_{\mathbb{Z}_p,\theta,E}/I \cong \Lambda_{r,\theta}/(\xi).$$

Here I denotes the Eisenstein ideal, $\Lambda_r = \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$, and we denote the image of ξ in $\Lambda_{r,\theta}$ by the same letter ξ.

6.2.8. We prove Lemma 6.2.3.

Consider the Λ-adic Eisenstein series $E^{(2)}_\Lambda$ of weight 2 defined as

$$E^{(2)}_\Lambda = {}^{-1}\xi^* + \sum_{n=1}^{\infty} \sum_{d|n,(d,Np)=1} [d]q^n.$$

We have $\mathfrak{H}E^{(2)}_\Lambda = \Lambda\, E^{(2)}_\Lambda$ since $I_\mathfrak{H}$ kills $E^{(2)}_\Lambda$, and we have $M_{\Lambda,DM',\theta,E} = M_{\Lambda,\theta,E}/\,\Lambda_\theta\, E^{(2)}_{\Lambda,\theta}$. From this, we obtain $M_{\Lambda,DM',\theta,E}/S_{\Lambda,\theta,E} \xrightarrow{\simeq} \Lambda_\theta/(\xi)$. Comparing this with $M_{\Lambda,DM,\theta,E}/S_{\Lambda,\theta,E} \xrightarrow{\simeq} \Lambda_\theta/(\xi)$ (6.2.7), we obtain $M_{\Lambda,DM,\theta,E} \xrightarrow{\simeq} M_{\Lambda,DM',\theta,E}$. We similarly obtain $(M_2(Np^r)^{\text{ord}}_{\mathbb{Z}_p})_{DM,\theta,E} \xrightarrow{\simeq} (M_2(Np^r)^{\text{ord}}_{\mathbb{Z}_p})_{DM',\theta,E}$.

By Ohta [41] section 3.4, the exact sequences of $\mathfrak{H}$-modules $0 \to H_{\text{sub},\theta,E} \to \tilde{H}_{\theta,E} \to \tilde{H}_{\text{quo},\theta,E} \to 0$ and $0 \to H^1_{\text{ét}}(X_1(Np^r))^{\text{ord}}_{\text{sub},\theta,E} \to H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{\theta,E} \to H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{\text{quo},\theta,E} \to 0$ split. Hence $DM = DM'$ for $\tilde{H}_{\theta,E}$ and $H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{\theta,E}$ are reduced to those for $M_{\Lambda,\theta,E}$ and $M_2(Np^r)^{\text{ord}}_{\theta,E}$ by the relations $M_\Lambda = D(\tilde{H}_{\text{quo}})$ and $M_2(Np^r)^{\text{ord}} = D(H^1_{\text{ét}}(Y_1(Np^r))^{\text{ord}}_{\text{quo}})$.

6.2.9. We prove Proposition 6.2.4. Since the maximal ideal of the two dimensional regular local ring Λ_θ is generated by $f := 1 - \langle 1+Np\rangle \in \Lambda_\theta$ and p, it is sufficient to prove that $f : \tilde{H}_{DM,\theta,E} \to \tilde{H}_{DM,\theta,E}$ is injective and $\tilde{H}_{DM,\theta,E}/f\tilde{H}_{DM,\theta,E}$ is a free module over $\Lambda_\theta/(f) \cong O_\theta$. Since f is a non-zero-divisor of Λ_θ and $\tilde{H}_{DM,\theta,E} = \tilde{H}_{DM',\theta,E}$ is a Λ_θ-submodule of $H_{\theta,E} \otimes_{\Lambda_\theta} Q(\Lambda_\theta)$, we have the injectivity of f. Next, by the case $k=2$ of 1.5.8 and by 6.2.3, $\tilde{H}_{DM,\theta,E}/f\tilde{H}_{DM,\theta,E} \cong H^1_{\text{ét}}(Y_1(Np))^{\text{ord}}_{DM,\theta,E} = H^1_{\text{ét}}(Y_1(Np))^{\text{ord}}_{DM',\theta,E}$. Hence it is a finitely generated O_θ-submodule of $H^1_{\text{ét}}(Y_1(Np))^{\text{ord}}_{\theta,E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and hence is a free O_θ-module.

Note that by 6.2.7, in $\tilde{H}_{DM,\theta,E}$, $\xi \cdot \{0,\infty\}$ belongs to $H_{\theta,E}$.

Corollary 6.2.10. *In $\tilde{H}_{DM,\theta,E}$, the element $\xi \cdot \{0,\infty\}$ of $H_{\theta,E}$ is a part of a Λ_θ-basis of $H_{\theta,E}$.*

This follows from Proposition 6.2.4 and $\tilde{H}_{DM,\theta,E}/H_{\theta,E} \cong \Lambda_\theta/(\xi)$.

We describe the kernel of the canonical surjection $\tilde{H}_{\theta,E} \to \tilde{H}_{DM,\theta,E}$ by using special Siegel units.

Lemma 6.2.11. *Let c be an integer such that $(c, 6Np) = 1$. Then, $({}_cg_{0,1/Np^r})_{r\geq 1} \in \varprojlim_r \mathcal{O}(Y_1(Np^r)_{\mathbb{Z}[1/Np]})^\times$, where the transition maps of the inverse system are norm maps.*

Proof. This can be proved by using the distribution property of Siegel units (2.1.2). □

6.2.12. By the exact sequences $0 \to \mathbb{Z}/p^n\mathbb{Z}(1) \to G_m \xrightarrow{p^n} G_m \to 0$ (Kummer theory) on the étale site of $Y_1(Np^r)_{\mathbb{Z}[1/Np]}$, we have

$$(\{{}_cg_{0,1/Np^r}\})_{r\geq 1} \in \tilde{H}(1).$$

We can define $(\{g_{0,1/Np^r}\})_{r\geq 1} \in \tilde{H}_\theta(1)$ without c. In fact ${}_cg_{0,1/Np^r} = (c^2 - \langle c\rangle)g_{0,1/Np^r}$, and on the θ-component $\tilde{H}_\theta(1)$, we have $c^2 - \langle c\rangle = c^2 - \theta(c)$. The last operator is invertible for some c.

Lemma 6.2.13. *The image of* $(g_{0,1/Np^r})_r$ *under* $\tilde{H}_\theta(1) \to \tilde{H}_{\theta,E}(1)/H_{\theta,E}(1) \cong \Lambda_\theta$ *is* ξ.

This is well known and is checked easily by using the q-expansions of Siegel units.

Lemma 6.2.14. *The kernel of* $\tilde{H}_{\theta,E}(1) \to \tilde{H}_{DM,\theta,E}(1)$ *coincides with* $\Lambda_\theta \cdot \{g_{0,1/Np^\infty}\}$.

Proof. Let $U \subset \tilde{H}_{\theta,E}(1)$ be the image of $\varprojlim_n \mathcal{O}(Y_1(Np^r)_\mathbb{Q})^\times \otimes_\mathbb{Z} \mathbb{Z}/p^n\mathbb{Z}$ under Kummer theory. Then U is an $\mathfrak{H}$-submodule of $\tilde{H}_{\theta,E}(1)$. Furthermore U injects into $\tilde{H}_{\theta,E}(1)/H_{\theta,E}(1)$. Hence U is contained in the kernel of $\tilde{H}_{\theta,E}(1) \to \tilde{H}_{DM,\theta,E}(1)$. Hence $\Lambda_\theta \cdot \{g_{0,1/Np^\infty}\}$ is contained in this kernel. By 6.2.7, the canonical map from this kernel to $\tilde{H}_{\theta,E}(1)/H_{\theta,E}(1) \cong \Lambda_\theta$ is injective and the image coincides with $(\xi) \subset \Lambda_\theta$. Hence by 6.2.13, this kernel coincides with $\Lambda_\theta \cdot \{g_{0,1/Np^\infty}\}$. □

6.3 Structure of $H_\theta/I_\theta H_\theta$

The purpose of this section 6.3 is to introduce the results 6.3.2 and 6.3.4 on $H_\theta/I_\theta H_\theta$, related results 6.3.5 and 6.3.6, and their proofs. These are known results (Mazur-Wiles [32], Ohta [39], [40], Sharifi [51]). The reasons why we explain the proofs of the known results are: (1) The Galois action on $H_\theta/I_\theta H_\theta$ is described in [39], [40] for the model $X_1'(M)$, not for the model $X_1(M)$ which we use, so to avoid confusion, it seems good to explain well what happens for our model. (2) We carefully define a canonical basis of $H_\theta^+/I_\theta H_\theta^+$ as an $\mathfrak{h}_\theta/I_\theta$-module (6.3.18) and how to choose the base is related to the proofs of these results.

6.3.1. Define the subquotients $\mathcal{P}$ and $\mathcal{Q}$ of H_θ and a subquotient $\mathcal{R}$ of $Q(\Lambda_\theta) \otimes_{\Lambda_\theta} H_\theta$ as follows.

Let

$$\mathcal{P} = H_\theta^-/I_\theta H_\theta^- \subset H_\theta/I_\theta H_\theta, \quad \mathcal{Q} = (H_\theta/I_\theta H_\theta)/\mathcal{P}.$$

So, $\mathcal{Q}$ is canonically isomorphic to $H_\theta^+/I_\theta H_\theta^+$.

Let

$$\mathcal{R} = \tilde{H}_{DM,\theta,E}/H_{\theta,E}.$$

Here $(-)_{DM}$ denotes the Drinfeld-Manin modification as in section 6.2. Recall that $\mathcal{R} \cong \mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/(\xi)$ (6.2.7).

Proposition 6.3.2. *In* $H_\theta/I_\theta H_\theta$, $\mathcal{P}$ *is stable under the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. *The action of* $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *on* $\mathcal{P}$ *and that on* $\mathcal{R}$ *are given by* $\kappa(\sigma)^{-1}$ *where* κ *is the cyclotomic character, and the action of* σ *on* $\mathcal{Q}$ *is given by* $\langle\sigma\rangle^{-1}$. *Here* $\langle\sigma\rangle$ *is as in 1.7.14.*

Remark 6.3.3. (1) If we use the other model $X_1'(Np^r)$ $(r \geq 1)$ and define $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ in the same way as above, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{Q}$ trivially and acts on $\mathcal{P}$ and $\mathcal{R}$ by $\kappa(\sigma)^{-1}\langle\sigma\rangle$. This is seen from 1.4.5 (1).

(2) The stability of $\mathcal{Q}$ under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ follows from [51], Theorem 4.3. The proof of this stability given below is simpler.

Recall that we have an exact sequence $0 \to H_{\mathrm{sub}} \to H \to H_{\mathrm{quo}} \to 0$ and $0 \to H_{\mathrm{sub}} \to \tilde{H} \to \tilde{H}_{\mathrm{quo}} \to 0$ of representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ (1.7.10).

Proposition 6.3.4. *For any prime divisor ℓ of Np, the sequence $0 \to \mathcal{P} \to H_\theta/I_\theta H_\theta \to \mathcal{Q} \to 0$ splits uniquely as an exact sequence of representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

Proposition 6.3.5. *Let $(-)_E$ be the Eisenstein components. The canonical maps*

$$H^-_{\theta,E} \to H_{\mathrm{quo},\theta,\mathrm{E}}, \quad \tilde{H}^-_{\theta,E} \to \tilde{H}_{\mathrm{quo},\theta,\mathrm{E}}, \quad H_{\mathrm{sub},\theta,\mathrm{E}} \to H_{\theta,E}/H^-_{\theta,E}$$

are bijective.

Corollary 6.3.6. *Concerning the Eisenstein component $H_{\theta,E}$ of H_θ, we have*:

(1) As a module over $\mathfrak{h}_{\theta,E}$, $H^+_{\theta,E}$ is free of rank 1.

(2) As a module over $\mathfrak{h}_{\theta,E}$, $H^-_{\theta,E}$ is a dualizing module.

This follows from 6.3.5 and 1.7.13.

6.3.7. On $\mathcal{R}(1)$, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts trivially.

Proof. This follows from the fact $\mathcal{R}(1)$ is generated by the classes of 0-cusps (Ohta [41]) on which the Galois action is trivial. □

6.3.8. We construct a canonical Λ_θ-homomorphism $H_\theta/I_\theta H_\theta \to \Lambda_\theta\,/(\xi)$ by a method in [51] section 4 to use the Λ-adic pairing of Ohta

$$(-,-)_\Lambda : H_\theta \times H_\theta \to \Lambda_\theta$$

in 1.6.3. By 6.2.7, we have $\xi\{0,\infty\}_{DM,\theta,E} \in H_{\theta,E}$. Hence we have a homomorphism

$$H_\theta \to \Lambda_\theta\ ;\ x \mapsto (x, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda.$$

For $a \in I_\theta$,

$$(ax, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda = (x, a\xi\{0,\infty\}_{DM,\theta,E})_\Lambda = \xi(x, a\{0,\infty\}_{DM,\theta,E})_\Lambda$$

(note $I_\theta\{0,\infty\}_{\theta,DM} \subset H_{\theta,E}$ by 6.2.7). Hence we have a homomorphism

$$(-, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda : H_\theta/I_\theta H_\theta \to \Lambda_\theta\,/(\xi).$$

Proposition 6.3.9. *The map $(-, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda : H_\theta/I_\theta H_\theta \to \Lambda_\theta\,/(\xi)$ is surjective.*

Proof. By Corollary 6.2.10, there is a basis $(e_i)_{1\leq i\leq r}$ of $H_{\theta,E}$ as a free Λ_θ-module such that $e_1 = \xi\{0,\infty\}_{DM,\theta,E}$ in $\tilde{H}_{DM,\theta,E}$. Since the pairing $(-,-)_\Lambda : H_{\theta,E} \times H_{\theta,E} \to \Lambda_\theta$ is perfect (1.6.4,1.6.5), there is $x \in H_\theta$ such that $(x, e_1) = 1$. □

6.3.10. Let $\mathcal{P}' = \operatorname{Ker}(H_\theta/I_\theta H_\theta \to \Lambda_\theta/(\xi))$, and let $\mathcal{Q}' = (H_\theta/I_\theta H_\theta)/\mathcal{P}'$. (We will prove soon that $\mathcal{P}' = \mathcal{P}$.)

6.3.11. In $H_\theta/I_\theta H_\theta$, $\mathcal{P}'$ is stable under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. On $\mathcal{Q}'$, $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts as $\langle\sigma\rangle^{-1}$. (Here, $\langle\sigma\rangle$ is as in 1.7.14.)

Proof. This follows from 6.3.7 and 1.6.4 (3). □

6.3.12. Following Ohta [41] section 3.4, we give a splitting of the exact sequence $0 \to H_{\mathrm{sub},\theta,\mathrm{E}} \to H_{\theta,E} \to H_{\mathrm{quo},\theta,\mathrm{E}} \to 0$ of $\mathfrak{h}_\theta$-modules, where $(-)_E$ denotes the Eisenstein component (1.9.2).

We use the following (1)–(3) (1.7.14, 1.8.1).

(1) The action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $H_{\mathrm{quo}}(1)$ is unramified.

(2) The arithmetic Frobenius Fr_p acts on $H_{\mathrm{quo},\theta}(1)$ as $T^*(p)$.

(3) $H_{\mathrm{quo}} \otimes_{\mathfrak{h}} Q(\mathfrak{h})$ is a free $Q(\mathfrak{h})$-module of rank 1. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the determinant of the action of σ on the free $Q(\mathfrak{h})$-module $Q(\mathfrak{h}) \otimes_{\mathfrak{h}} H$ of rank 2 is $\kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}$, where $\kappa : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the cyclotomic character and $\langle\sigma\rangle$ is as in 1.7.14.

We define an element $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, elements $f, g \in \mathfrak{h}$, and an $\mathfrak{h}$-submodule S of H as follows.

Case 1. First, assume that the restriction of χ to $(\mathbb{Z}/p\mathbb{Z})^\times$ is not trivial. Let τ be an element of the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ whose image in $\mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ is a generator of the last group. Let

$$f = \kappa(\tau)^{-1} \in \mathbb{Z}_p^\times \subset \mathfrak{h}, \quad g = \langle\tau\rangle^{-1} \in \mathfrak{h}.$$

Case 2. Next, assume that the restriction of χ to $(\mathbb{Z}/p\mathbb{Z})^\times$ is trivial. Let τ be an element of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ whose restriction to unramified extensions of $\mathbb{Q}_p$ are the arithmetic Frobenius Fr_p and whose restrictions to $\mathbb{Q}_p(\zeta_{p^n})$ $(n \geq 0)$ are trivial. Let

$$f = T^*(p) \in \mathfrak{h}, \quad g = T^*(p)^{-1}\langle\tau\rangle^{-1} \in \mathfrak{h}.$$

In the case 1 (resp. 2), by the above (1) and (3) (resp. (2) and (3)), we see that the action of τ on H_{quo} (resp. H_{sub}) coincides with the action of f (resp. g).

In both cases 1, 2, define

$$S = \{x \in H \mid \tau x = fx\}.$$

We prove that

$$H_{\theta,E} = H_{\mathrm{sub},\theta,\mathrm{E}} \oplus S_{\theta,E}.$$

We may and do assume that the Eisenstein component $\mathfrak{h}_{\theta,E}$ of $\mathfrak{h}_\theta$ is not the zero ring. Then $\mathfrak{h}_{\theta,E}$ is a local ring (this follows from $\mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/(\xi)$ (6.1.7)). It is sufficient to prove that the classes of f and g in the residue field of $\mathfrak{h}_{\theta,E}$ are different.

Let a be the element of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\tau(\zeta_{Np^r}) = \zeta_{Np^r}^a$ for all $r \geq 0$. First consider the case 1. In this case, a has the shape $(b, 1) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ where $b \bmod p$ is a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$, and the class of f (resp. g) in the residue field of $\mathfrak{h}_{\theta,E}$ is equal to the class of b^{-1} (resp. $\theta|_{(\mathbb{Z}/p\mathbb{Z})^\times}(b)^{-1}$). Since the restriction of χ to $(\mathbb{Z}/p\mathbb{Z})^\times$ is non-trivial, the restriction of θ to $(\mathbb{Z}/p\mathbb{Z})^\times$ is not ω, and hence the classes of $\theta|_{(\mathbb{Z}/p\mathbb{Z})^\times}(b)^{-1}$ and b^{-1} in the residue field of $\mathfrak{h}_{\theta,E}$ are different. Hence the residue classes of f and g are different. Next consider the case 2. In this case, a has the shape $(1,p)$. The class of g in the residue field of $\mathfrak{h}_{\theta,E}$ is equal to $\theta_{(\mathbb{Z}/N\mathbb{Z})^\times}(p)$, a root of 1 which is not 1 and whose order is prime to p. On the other hand, $f = T^*(p)$ is congruent to 1 modulo the maximal ideal of $h_{\theta,E}$. Hence the residue classes of f and g are different.

6.3.13. We prove that the composition $H_{\mathrm{sub},\theta}/I_\theta H_{\mathrm{sub},\theta} \to H_\theta/I_\theta H_\theta \to \mathcal{Q}'$ is an isomorphism. Let $\tau \in \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $f, g \in \mathfrak{h}$ be as in 6.3.12.

By 6.3.11, τ acts on $\mathcal{Q}'$ by g. Since we have a natural surjection from $H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta}$ to the cokernel of this composition which is compatible with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and since τ acts on $H_{\mathrm{quo},\theta}$ by f, the action of τ on the cokernel is given by f. Hence the cokernel is zero. Thus $H_{\mathrm{sub},\theta}/I_\theta H_{\mathrm{sub},\theta} \to \mathcal{Q}' \cong \mathfrak{h}_\theta/I_\theta$ is a surjective $\mathfrak{h}_\theta$-homomorphism between free $\mathfrak{h}_\theta/I_\theta$-modules of rank 1, and hence it is an isomorphism.

6.3.14. It follows that the canonical map $\mathcal{P}' \to H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta}$ is an isomorphism. Hence on $\mathcal{P}'$, τ acts by f. Hence we have

$$\mathcal{P}' = S_\theta/I_\theta S_\theta \quad \text{in} \;\; H_\theta/I_\theta H_\theta.$$

6.3.15. We prove that the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{P}'$ is given by κ^{-1} where κ is the cyclotomic character.

For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, let

$$a(\sigma) \in \mathrm{Hom}_{\,\mathfrak{h}_{\theta,E}}(H_{\mathrm{sub},\theta,\mathrm{E}}, H_{\mathrm{sub},\theta,\mathrm{E}}), \quad b(\sigma) \in \mathrm{Hom}_{\,\mathfrak{h}_{\theta,E}}(H_{\mathrm{sub},\theta,\mathrm{E}}, S_{\theta,E}),$$

$$c(\sigma) \in \mathrm{Hom}_{\,\mathfrak{h}_{\theta,E}}(S_{\theta,E}, H_{\mathrm{sub},\theta,\mathrm{E}}), \quad d(\sigma) \in \mathrm{Hom}_{\,\mathfrak{h}_{\theta,E}}(S_{\theta,E}, S_{\theta,E})$$

be the components of $\sigma : H_{\theta,E} \to H_{\theta,E}$. Since $S_{\theta,E}/I_\theta S_{\theta,E} = S_\theta/I_\theta S_\theta$ is stable in $H_\theta/I_\theta H_\theta$ under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we have $c(\sigma) \equiv 0 \bmod I_{\theta,E}$. Since $H_{\mathrm{sub},\theta,\mathrm{E}}$ is free of rank 1 over $\mathfrak{h}_{\theta,E}$, we have $a(\sigma) \in \mathfrak{h}_{\theta,E}$. Since $S_{\theta,E}$ is a dualizing module of $\mathfrak{h}_{\theta,E}$ (this is because $S_{\theta,E} \cong H_{\mathrm{quo},\theta,\mathrm{E}}$), we have $d(\sigma) \in \mathfrak{h}_{\theta,E}$. We have that $Q(\mathfrak{h}_{\theta,E}) \otimes_{\mathfrak{h}_{\theta,E}} S_{\theta,E}$ is a free $Q(\mathfrak{h}_{\theta,E})$-module of rank 1, and $\det(\sigma)$ of $\sigma : Q(\mathfrak{h}_{\theta,E}) \otimes_{\mathfrak{h}_{\theta,E}} H_{\theta,E} \to Q(\mathfrak{h}_{\theta,E}) \otimes_{\mathfrak{h}_{\theta,E}} H_{\theta,E}$ is $\kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}$. That is, $a(\sigma)d(\sigma) - c(\sigma)b(\sigma) = \kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}$. Here note that $c(\sigma)b(\sigma) \in \mathfrak{h}_{\theta,E}$. Since $c(\sigma)b(\sigma) \equiv 0 \bmod I_{\theta,E}$, we have $a(\sigma)d(\sigma) \equiv \kappa(\sigma)^{-1}\langle\sigma\rangle^{-1} \bmod I_{\theta,E}$. Since $a(\sigma) \equiv \langle\sigma\rangle^{-1} \bmod I_{\theta,E}$, we have $d(\sigma) \equiv \kappa(\sigma)^{-1} \bmod I_{\theta,E}$.

6.3.16. By 6.3.15, the action of the complex conjugation on $\mathcal{P}'$ is -1. Since the complex conjugation acts on $\mathcal{Q}'$ trivially by 6.3.11, this proves $\mathcal{P}' = (H_\theta/I_\theta H_\theta)^- = \mathcal{P}$.

This completes the proofs of Propositions 6.3.2 and 6.3.5.

6.3.17. We prove Proposition 6.3.4. The case $\ell = p$ follows from 6.3.5. Assume $\ell | N$. For an element σ of the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, σ acts on $\mathcal{Q}$ by $\theta(\langle\sigma\rangle)^{-1}$ and acts on $\mathcal{P}$ trivially. Since the restriction of θ to $(\mathbb{Z}/N\mathbb{Z})^\times$ is primitive, we have 6.3.4.

6.3.18. As a consequence of this section 6.3, we have isomorphisms

$$H_\theta^+/I_\theta H_\theta^+ \xrightarrow{\cong} \mathcal{Q} \xrightarrow{\cong} \Lambda_\theta\,/(\xi) \cong \mathfrak{h}_\theta/I_\theta$$

where the second isomorphism is given by $(-, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda$. We call the $\mathfrak{h}_\theta/I_\theta$-bases of $H_\theta^+/I_\theta H_\theta^+$ and $\mathcal{Q}$ corresponding to $1 \in \mathfrak{h}_\theta/I_\theta$, the canonical bases.

6.4 From cyclotomic symbols to modular symbols

6.4.1. Let

$$X_{Np^\infty} := \varprojlim_r \mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r}))\{p\}.$$

It is regarded as a Λ-module via the canonical isomorphism $\Lambda \cong \mathbb{Z}_p[[\mathrm{Gal}(K/\mathbb{Q})]]$ where $K = \mathbb{Q}(\zeta_{Np^\infty})$.

We regard, via class field theory,

$$(1) \qquad X_{Np^\infty} = \mathrm{Gal}(L/K)$$

where L is the largest unramified pro-p abelian extension of K.

6.4.2. As in [51] Lemma 4.11, we have canonical Λ_χ-isomorphisms

$$X_{Np^\infty,\chi} \xrightarrow{\cong} \varprojlim_r H^2(\mathbb{Z}[1/p, \zeta_{Np^r}], \mathbb{Z}_p(1))_\chi \xrightarrow{\cong} \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(1))_\chi$$

and a Λ_θ-isomorphism

$$X_{Np^\infty,\chi}(1) \cong \mathcal{S}_\theta$$

where $\mathcal{S}$ is as in 5.2.6.

6.4.3. We define the reciprocity map

$$\Upsilon : X_{Np^\infty,\chi} \to \mathcal{P} = H_\theta^-/I_\theta H_\theta^-.$$

Consider the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ on $H_\theta/I_\theta H_\theta$. It acts trivially on $\mathcal{P}$ and on $\mathcal{Q}$ (6.3.2). Hence we have a homomorphism

$$\mathrm{Gal}(\bar{\mathbb{Q}}/K) \to \mathrm{Hom}_{\mathfrak{h}_\theta}(\mathcal{Q}, \mathcal{P})\ ;\ \sigma \mapsto (x \bmod \mathcal{P} \mapsto (\sigma - 1)x) \quad (x \in H_\theta/I_\theta H_\theta).$$

This action is unramified at prime numbers which do not divide Np, for the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on H is unramified at all primes which do not divide Np. By Propositions 6.3.2 and 6.3.4, the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ on $H_\theta/I_\theta H_\theta$ is unramified also at all prime divisors of Np.

By 6.4.1 (1), this defines a homomorphism $\mathrm{Gal}(L/K) \to \mathrm{Hom}_{\mathfrak{h}_\theta}(\mathcal{Q}, \mathcal{P}) = \mathcal{P}$, where the last $=$ uses the canonical basis of $\mathcal{Q}$ as an $\mathfrak{h}_\theta/I_\theta$-module (6.3.18). That is, we obtain a homomorphism

$$\Upsilon : X_{Np^\infty} \to \mathcal{P} = H_\theta^-/I_\theta H_\theta^-, \quad \Upsilon(\sigma) = \sigma e_B - e_B$$

where e_B is a lifting of the canonical $\mathfrak{h}_\theta/I_\theta$-basis of $\mathcal{Q}$ to $H_\theta/I_\theta H_\theta$.

By considering the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{P}$ and $\mathcal{Q}$ given in Proposition 6.3.2, we see that the above homomorphism Υ induces $X_{Np^\infty,\chi} \to H_\theta^-/I_\theta H_\theta^-$.

6.4.4. Note that in section 5, we obtained a homomophism $\varpi : (H_\theta^-/I_\theta H_\theta^-)(1) \to \mathcal{S}_\theta$ (cf. 5.2.3). We denote the induced homomorphism $H_\theta^-/I_\theta H_\theta^- \to X_{Np^\infty,\chi}$ also by ϖ. The two homomorphisms

$$\varpi : H_\theta^-/I_\theta H_\theta^- \to X_{Np^\infty,\chi}, \quad \Upsilon : X_{Np^\infty,\chi} \to H_\theta^-/I_\theta H_\theta^-$$

will be our main subjects in the next section 7.

7 Conjectures of Sharifi

7.1 The conjectures

In this section 7.1, we introduce several conjectures of Sharifi and explain the relations between these conjectures. In fact, we make some of these conjectures slightly stronger (see 7.1.14).

Let the situation be as in section 6.1.

Conjecture 7.1.1. *(1) The composition*

$$H_\theta^-/I_\theta H_\theta^- \xrightarrow{\varpi} X_{Np^\infty,\chi} \xrightarrow{\Upsilon} H_\theta^-/I_\theta H_\theta^-$$

is the identity map.

(2) The composition

$$X_{Np^\infty,\chi} \xrightarrow{\Upsilon} H_\theta^-/I_\theta H_\theta^- \xrightarrow{\varpi} X_{Np^\infty,\chi}$$

is the identity map.

Since X_{Np^∞} is a torsion free $\mathbb{Z}_p$-module by Ferrero-Washington [7], the map ϖ induces a homomorphism $\varpi : (H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor}) \to X_{Np^\infty,\chi}$ where (tor) is the p-primary torsion. Consider also the homomorphism $\Upsilon : X_{Np^\infty,\chi} \to (H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor})$ induced by Υ.

Conjecture 7.1.1 has a slightly weaker version.

Conjecture 7.1.2. *(1) The composition* $\Upsilon\circ\varpi : (H_\theta^-/I_\theta H_\theta^-)/(tor) \to X_{Np^\infty,\chi} \to (H_\theta^-/I_\theta H_\theta^-)/(tor)$ *is the identity map.*

(2) The maps $\varpi : (H_\theta^-/I_\theta H_\theta^-)/(tor) \to X_{Np^\infty,\chi}$ *and* $\Upsilon : X_{Np^\infty,\chi} \to (H_\theta^-/I_\theta H_\theta^-)/(tor)$ *are isomorphisms.*

7.1.3. Conjecture 7.1.2 (1) implies Conjecture 7.1.1 (2) and Conjecture 7.1.2 (2).

This is shown as follows. The following arguments by using Fitting ideals and characteristic ideals are standard in Iwasawa theory.

For a commutative ring R and for an R-module M of finite presentation, the (0-th) Fitting ideal $\mathrm{Fitt}_R(M)$ of M is defined as follows. Take a presentation $R^m \xrightarrow{f} R^n \to M \to 0$ (exact) of M. Then $\mathrm{Fitt}_R(M)$ is the ideal of R generated by all (n,n)-minors of f (i.e. the determinants of the (n,n)-matrices which appear as parts of the matrix f). Then $\mathrm{Fitt}_R(M)$ is independent of the choice of the presentation of M. (See for example, [36].) As is easily seen, we have

(1) If $R \to R'$ is a homomorphism of commutative rings and $M' = R' \otimes_R M$, $\mathrm{Fitt}_{R'}(M')$ coincides with the ideal of R' generated by the image of $\mathrm{Fitt}_R(M)$.

We apply (1) to the case $R = \mathfrak{h}_\theta$, $M = H_\theta^-$, and $R' = Q(R)$ (1.5.3). In this case, $M' \cong R'$ as an R'-module and hence $\mathrm{Fitt}_{R'}(M') = 0$. Hence by (1), we have

(2) $\mathrm{Fitt}_{\mathfrak{h}_\theta}(H_\theta^-) = 0$.

We next apply (1) to the case $R = \mathfrak{h}_\theta$, $M = H_\theta^-$, and $R' = \mathfrak{h}_\theta/I_\theta$. By (1) and (2), we have $\mathrm{Fitt}_{\mathfrak{h}_\theta/I_\theta}(H_\theta^-/I_\theta H_\theta^-) = 0$. Since $\Lambda_\theta/(\xi) \cong \mathfrak{h}_\theta/I_\theta$ (6.1.7), we have

(3) $\mathrm{Fitt}_{\Lambda_\theta/(\xi)}(H_\theta^-/I_\theta H_\theta^-) = 0$.

By applying (1) to the case $R = \Lambda_\theta$, $M = H_\theta^-/I_\theta H_\theta^-$ and $R' = \Lambda_\theta/(\xi)$, and by using (3), we have

(4) $\mathrm{Fitt}_{\Lambda_\theta}(H_\theta^-/I_\theta H_\theta^-) \subset (\xi)$.

For a finitely generated torsion Λ_θ-module M, the characteristic ideal $\mathrm{Char}_{\Lambda_\theta}(M)$ of M is defined to be $\prod_{\mathfrak{p}} \mathfrak{p}^{e(\mathfrak{p})}$ where $\mathfrak{p}$ ranges over all prime ideals of Λ_θ of height one and $e(\mathfrak{p})$ denotes the length of the localization $M_{\mathfrak{p}}$ as a $\Lambda_{\mathfrak{p}}$-module. (See for example, [59].) The relation with $\mathrm{Fitt}_{\Lambda_\theta}(M)$ is that $\mathrm{Char}_{\Lambda_\theta}(M)$ is the unique principal ideal of Λ_θ such that $\mathrm{Fitt}_{\Lambda_\theta}(M) \subset \mathrm{Char}_{\Lambda_\theta}(M)$ and such that the quotient $\mathrm{Char}_{\Lambda_\theta}(M)/\mathrm{Fitt}_{\Lambda_\theta}(M)$ is finite. By (4) and by $\mathrm{Char}_{\Lambda_\theta}((H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor})) = \mathrm{Char}_{\Lambda_\theta}(H_\theta^-/I_\theta H_\theta^-)$, we have

(5) $\mathrm{Char}_{\Lambda_\theta}((H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor})) \subset (\xi)$.

Assume that Conjecture 7.1.2 (1) is true. Let P be the kernel of $\Upsilon : X_{Np^\infty,\chi} \to (H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor})$. Then $X_{Np^\infty,\chi} \cong (H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor}) \oplus P$ as a Λ_θ-modules. (Here we regard $X_{Np^\infty,\chi}$ as a Λ_θ-module via the isomorphism $\Lambda_\theta \cong \Lambda_\chi$ given by the (-1) Tate twist.) Hence we have $\mathrm{Char}_{\Lambda_\theta}(X_{Np^\infty,\chi}) = \mathrm{Char}_{\Lambda_\theta}((H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor})) \cdot \mathrm{Char}_{\Lambda_\theta}(P)$. By Iwasawa Main conjecture proved by Mazur-Wiles, we have $\mathrm{Char}_{\Lambda_\theta}(X_{Np^\infty,\chi}) = (\xi)$. Hence by (5), we have $(\xi) \subset (\xi)\mathrm{Char}_{\Lambda_\theta}(P)$. This proves $\mathrm{Char}_{\Lambda_\theta}(P) = (1)$, that is, P is finite. Since $X_{Np^\infty,\chi}$ has no p-torsion (Ferrero-Washington [7]), we have $P = 0$. Hence Conjecture 7.1.1 (2) and Conjecture 7.1.2 (2) are true.

Remark 7.1.4. By Ohta [42] Theorem II, if the Eisenstein component (1.9.2) $\mathfrak{H}_{\theta,E}$ of the Hecke algebra $\mathfrak{H}_\theta$ is Gorenstein, then $\Upsilon : X_{Np^\infty,\chi} \to H_\theta^-/I_\theta H_\theta^-$ is an isomorphism.

Remark 7.1.5. $H_\theta^-/I_\theta H_\theta^-$ is p-torsion free if the Eisenstein component $\mathfrak{h}_{\theta,E}$ of $\mathfrak{h}_\theta$ or the Eisenstein component of $\mathfrak{H}_\theta$ is Gorenstein.

In fact, if the Eisenstein component of $\mathfrak{h}_\theta$ is Gorenstein, then since $H_{\mathrm{quo},\theta}$ is the dualizing module of $\mathfrak{h}_\theta$, the Eisenstein component $H_{\mathrm{quo},\theta,\mathrm{E}}$ is a free module of rank 1 over $\mathfrak{h}_{\theta,E}$, and hence $H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta}$ is a free $\mathfrak{h}_\theta/I_\theta$-module of rank 1. By this and by 6.3.5 and 6.1.7, we have

$$H_\theta^-/I_\theta H_\theta^- \cong H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta} \cong \mathfrak{h}_\theta/I_\theta \cong \Lambda_\theta/(\xi).$$

Since $\Lambda_\theta/(\xi)$ has no p-torsion (Ferrero-Washington [7]), $H_\theta^-/I_\theta H_\theta^-$ has no p-torsion.

Next, if $\mathfrak{H}_\theta$ is Gorenstein, then $H_\theta^-/I_\theta H_\theta^- \cong X_{Np^\infty,\chi}$ by 7.1.4, and by the fact $X_{Np^\infty,\chi}$ is p-torsion free (Ferrero-Washington [7]), $H_\theta^-/I_\theta H_\theta^-$ is p-torsion free.

Conjecture 7.1.6.

$$\Upsilon(\mathcal{C}_\chi) = \mathcal{L}_\theta^- \mod I_\theta$$

where $\mathcal{C}_\chi \in X_{Np^\infty,\chi}[[\mathbb{Z}_p^\times]]$ is the χ-component of the cyclotomic symbol element $\mathcal{C}$ in 5.2.12 and $\mathcal{L}_\theta^- \in H_\theta^-[[\mathbb{Z}_p^\times]]$ is the component of the p-adic L-function $\mathcal{L} \in H[[\mathbb{Z}_p^\times]]$ of Mazur-Kitagawa in two variables. Furthermore, $\Upsilon(\mathcal{C}^\star_{N,M,\chi}) = \mathcal{L}^{\star,-}_{N,M,\theta} \mod I_\theta$ for any divisor $M \geq 1$ of N. (See 5.2.12).

7.1.7. Conjecture 7.1.1 (1) implies Conjecture 7.1.6. Conjecture 7.1.2 (1) implies Conjecture 7.1.6 modulo p-primary torsion.

This is because ϖ sends $\mathcal{L}^{\star,-}_{N,M}$ to $\mathcal{C}^{\star}_{N,M}$ by Proposition 5.2.12.

Conjecture 7.1.8. *(Conjecture of McCallum-Sharifi; See [34], [51].) For $r \geq 1$, as an abelian group, $H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta$ is generated by elements of the form $\{1 - \zeta^u_{Np^r}, 1 - \zeta^v_{Np^r}\}$ ($u, v \in \mathbb{Z}/Np^r\mathbb{Z} - \{0\}$, $(u, v, Np) = 1$).*

7.1.9. Since we have isomorphisms

$$H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta / p^r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta$$

$$\cong H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], (\mathbb{Z}/p^r\mathbb{Z})(2))_\theta$$

$$\cong H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], (\mathbb{Z}/p^r\mathbb{Z})(1))_\chi \cong (\mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r}))/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r})))_\chi,$$

Conjecture 7.1.8 is rewritten as a conjecture on ideal class groups: For any $r \geq 1$, $(\mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r}))/p^r\mathrm{Cl}(\mathbb{Q}(\zeta_{Np^r})))_\chi$ is generated by the images of $\{1 - \zeta^u_{Np^r}, 1 - \zeta^v_{Np^r}\}$ ($u, v \in \mathbb{Z}/Np^r\mathbb{Z} - \{0\}$, $(u, v, Np) = 1$).

7.1.10. Conjecture 7.1.2 (2) implies Conjecture 7.1.8. In fact, since the canonical projection $\mathcal{S}_\theta \to H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta$ is surjective, the surjectivity of $\varpi : H^-_\theta / I_\theta H^-_\theta \to X_{Np^\infty,\chi} \cong \mathcal{S}_\theta$ in Conjecture 7.1.2 shows that $\varpi_r : H^1(X_1(Np^r))^{\mathrm{ord}}_\theta \to \mathcal{S}_\theta$; $[u : v]_r \mapsto \{1 - \zeta^u_{Np^r}, 1 - \zeta^v_{Np^r}\}$ ($u, v \in (\mathbb{Z}/Np^r\mathbb{Z}) - \{0\}$) is surjective. This shows that Conjecture 7.1.8 is true.

7.1.11. In [51] 1.2, Sharifi presents a conjecture which is a consequence of his conjecture 7.1.6.

Let $f = \sum_{n=1}^\infty a_n q^n$ be a normalized Hecke-eigen p-stabilized newform of weight $k \geq 2$ of level Np^r ($r \geq 1$) and of character ϵ. We assume that when regard $(\mathbb{Z}/Np\mathbb{Z})^\times$ as a torsion part of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, θ coincides with the character of $(\mathbb{Z}/Np\mathbb{Z})^\times$ induced by $\omega^{2-k}\epsilon^{-1}$ on this torsion part.

Let $F = \mathbb{Q}(a_n\,; n \geq 1)$, $L = \mathbb{Q}_p(a_n\,; n \geq 1)$, and $f^* = \sum_{n=1}^\infty \bar{a}_n q^n$ be as in section 4.5. We define the Eisenstein ideal $I(f)$ of O_L associated to f to be the ideal generated by $1 - a_\ell + l^{k-1}\epsilon(\ell)$ for all prime numbers $\ell \neq p$, and by $1 - a_p$.

Let $V(f^*)_{O_L}$ be the image of $V_k(X_1(Np^r))_{O_L} \to V(f^*)_L$. Then $V(f^*)_{O_L}$ is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-stable O_L-lattice in $V(f^*)_L$. Let

$$\bar{V}(f^*) = V(f^*)_{O_L} / I(f) V(f^*)_{O_L}.$$

This is a free $O_L/I(f)$-module of rank 2, and $\bar{V}(f^*)^+$ and $\bar{V}(f^*)^-$ are free $O_L/I(f)$-modules of rank 1. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{P} \otimes_{\mathbb{Z}_p} O_L & \to & H_\theta / I H_\theta \otimes_{\mathbb{Z}_p} O_L & \to & \mathcal{Q} \otimes_{\mathbb{Z}_p} O_L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bar{V}(f^*)^- & \to & \bar{V}(f^*) & \to & \bar{V}(f^*)^+ & \to & 0.
\end{array}$$

Here the middle vertical arrow is induced from the map $H \to V_k(X_1(Np^r))_{\mathbb{Z}_p}$ in 1.5.7. The vertical arrows are surjective. It follows that the lower horizontal sequence is stable under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The canonical basis of $\mathcal{Q}$ as an $\mathfrak{h}_\theta/I_\theta$-module (6.3.18) defines a basis of $\bar{V}(f^*)^+$ as an $O_L/I(f)$-module. Hence from the lower horizontal sequence, we obtain the f^*-version

$$X_{Np^\infty,\chi} \longrightarrow \bar{V}(f^*)^-$$

of Υ. As is checked easily this maps factors as

$$X_{Np^\infty,\chi} \longrightarrow H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta \longrightarrow \bar{V}(f^*)^-.$$

We denote this second arrow by Υ_f.

Conjecture 7.1.12. *Let $\psi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}^\times$ be a character with finite image. Let r be an integer such that $1 \le r \le k-1$ and such that $(-1)^r\psi(-1) = -1$. Let L_ψ be the subfield of $\bar{\mathbb{Q}}_p$ generated over L by the image of ψ. Then*

$$\Upsilon_f(\sum_{a \in (\mathbb{Z}/Np^r\mathbb{Z})^\times} \psi(a)a^{r-1}\{1-\zeta_{Np^r}^a, 1-\zeta_{Np^r}\}) = L_p(f,\psi,r) \mod I(f)$$

in $\bar{V}(f^*) \otimes_{O_L} O_{L_\psi}$. Here $L_p(f,\psi,r)$ is a value of $L_p(f) \in V(f^*)_{O_L}[[\mathbb{Z}_p^\times]]$ as in 4.5.1.

Conjecture 7.1.12 follows from Conjecture 7.1.6 by 4.5.5.

7.1.13. Here we explain that the conjectures in Introduction are contained in the conjectures here. In Introduction, we considered the case $N = 1$. In this case, the $(\theta = \omega^i)$-component of $\mathfrak{h}/I$ for $i \neq 0, 2$ is treated in the conjectures in this section 7.1. The ω^i-components of $\mathfrak{h}/I$ for $i = 0, 2$ are zero and yield no problem.

7.1.14. We made modifications of the formulations of the conjectures of Sharifi concerning the following points 1–3.

1. The map ϖ_r (5.2.1) is defined by Sharifi in [51] for each r, but the inverse limit ϖ of ϖ_r is not given in [51] and he does not present a conjecture in the style using ϖ. Our Conjecture 7.1.1 is a version by using ϖ of Remark at the end of [51] section 5 in which he uses ϖ_r.

2. As in section 6.4, we define the homomorphism Υ by taking a special basis of $H_\theta^+/I_\theta H_\theta^+$ over $\mathfrak{h}_\theta/I_\theta$. The original versions of Conjectures 7.1.1 and 7.1.6 in the paper [51] are that for some basis e of $H_\theta^+/I_\theta H_\theta^+$ over $\mathfrak{h}_\theta/I_\theta$, if we define the map $\Upsilon : X_{Np^\infty,\chi} \to H_\theta^-/I_\theta H_\theta^-$ as $\sigma \mapsto \sigma e - e$ by using e, then the statements of the conjectures hold. However, we learned from Sharifi that when he wrote [51], he was also planing to formulate the conjectures by using the same special basis e as in this paper.

3. In the original version of Conjecture 7.1.12 in [51], congruences are considered mod a maximal ideal containing $I(f)$.

7.2 Our results

Let the situation be as in section 6.1. We state our results 7.2.3, 7.2.6 and 7.2.8 on the conjectures of Sharifi.

7.2.1. Derivation in the Iwasawa algebra.

Let R be the valuation ring of a finite extension of $\mathbb{Q}_p$. Let Γ be a topological group which is isomorphic to the additive topological group $\mathbb{Z}_p$. For $f \in R[[\Gamma]]$ and for a generator t of Γ, we define $df/dt \in R[[\Gamma]]$ by identifying $R[[\Gamma]]$ with the ring of formal power series $R[[t-1]]$. If s is another generator of Γ, we have

$$tdf/dt = csdf/ds$$

where c is the element of $\mathbb{Z}_p$ such that $s = t^c$.

7.2.2. We define the derivative $\xi' \in \Lambda_\theta$ of $\xi \in \Lambda_\theta$ (6.1.6). We take $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_{Np}))$. We take the generator t of Γ such that $(1-p^{-1})\log(\kappa(t)) = 1$. Here κ is the cyclotomic character. By identifying Λ_θ with $O_\theta[[\Gamma]]$ via the canonical isomorphism $O_\theta[[\Gamma]] \xrightarrow{\cong} \Lambda_\theta$, we define

$$\xi' := td\xi/dt.$$

Theorem 7.2.3. *(1) The map $\Upsilon \circ \varpi : H_\theta^-/I_\theta H_\theta^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H_\theta^-/I_\theta H_\theta^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ satisfies $\xi'\Upsilon \circ \varpi = \xi'$.*

(2) Let $a \in H_\theta^-/I_\theta H_\theta^-$ be the class of $(1-T^*(p))\{0,\infty\}_{\theta,E}$ and let $b = (\{p, 1-\zeta_{Np^r}\})_r \in X_{Np^\infty,\chi}$. Then

$$\varpi(a) = b, \quad \Upsilon(b) = a.$$

7.2.4. The relations of the elements a and b in Theorem 7.2.3 (2) are studied by Sharifi in [49]. We will deduce Theorem 7.2.3 (2) from his theory in [49] and Theorem 9.6.3 in this paper.

7.2.5. We will consider the following conditions.

C(ξ): ξ has no multiple zero. That is, we have no multiple factor in the prime decomposition of ξ in Λ_θ.

C($\mathfrak{h}$): As a module over $\mathfrak{h}_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $H_\theta^-/I_\theta H_\theta^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by one element.

C($T^*(p)$): As a module over $\mathfrak{h}_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $H_\theta^-/I_\theta H_\theta^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by the class of $(1-T^*(p))\{0,\infty\}$.

Of course, $C(\mathfrak{h})$ is satisfied if $C(T^*(p))$ is satisfied.

From Theorem 7.2.3, we will deduce the following Theorems 7.2.6 and 7.2.8.

Theorem 7.2.6. *Assume that one of the conditions $C(\xi)$ and $C(T^*(p))$ is satisfied. Then:*

(1) Conjecture 7.1.1 (2) and Conjecture 7.1.2 are true. Thus the compositions $X_{Np^\infty,\chi} \xrightarrow{\Upsilon} H_\theta^-/I_\theta H_\theta^- \xrightarrow{\varpi} X_{Np^\infty,\chi}$ and $(H_\theta^-/I_\theta H_\theta^-)/(tor) \xrightarrow{\varpi} X_{Np^\infty,\chi} \xrightarrow{\Upsilon} (H_\theta^-/I_\theta H_\theta^-)/(tor)$ are the identity maps.

(2) The conjecture 7.1.6 for the L-function in two variables is true modulo the p-primary torsion of $H_\theta^-/I_\theta H_\theta^-$. The conjecture 7.1.12 on the ratio of L-values is true if $H_\theta^-/I_\theta H_\theta^-$ has no p-torsion.

By this theorem and by 7.1.5, we have

Corollary 7.2.7. *All conjectures in section 7.1 are true if the following conditions (i) and (ii) are satisfied.*

(i) Either $C(\xi)$ or $C(T^*(p))$ is satisfied.

(ii) Either $\mathfrak{h}_{\theta,E}$ or $\mathfrak{H}_{\theta,E}$ is Gorenstein.

Theorem 7.2.8. *Assume that one of the conditions $C(\xi)$ and $C(\mathfrak{h})$ is satisfied. Then:*

(1) Conjecture 7.1.2 (2) is true. That is, the maps $\varpi : (H_\theta^-/I_\theta H_\theta^-)/(tor) \to X_{Np^\infty,\chi}$ and $\Upsilon : X_{Np^\infty,\chi} \to (H_\theta^-/I_\theta H_\theta^-)/(tor)$ are isomorphisms.

(2) The conjecture 7.1.8 of McCallum-Sharifi is true.

Remark 7.2.9. (1) As far as the authors know, there is no known example of multiple zero of ξ.

(2) In the case $N = 1$, the λ-invariant $\mathrm{rank}_{O_\theta}(\Lambda_\theta/(\xi))$ is 0 or 1 in all known examples (see the sentences after 3.1 of Greenberg [15]).

7.2.10. The following conditions (i)–(iii) are equivalent.

(i) $C(\mathfrak{h})$

(ii) $H_\theta^-/I_\theta H_\theta^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a free $\mathfrak{h}_\theta/I_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module of rank 1.

(iii) For any prime ideal $\mathfrak{p}$ of $\mathfrak{h}_\theta$ of height one such that $I_\theta \subset \mathfrak{p}$, the local ring $\mathfrak{h}_{\theta,\mathfrak{p}}$ is Gorenstein.

The equivalence is seen as follows. (ii) implies (i) clearly. Note that for a prime ideal $\mathfrak{p}$ of $\mathfrak{h}_\theta$ of height one which contains I_θ, $H_{\theta,\mathfrak{p}}^-$ is a dualizing module over $\mathfrak{h}_{\theta,\mathfrak{p}}$ (6.3.6). Hence for such $\mathfrak{p}$, $\mathfrak{h}_{\theta,\mathfrak{p}}$ is Gorenstein if and only if $H_{\theta,\mathfrak{p}}^-$ is a free $\mathfrak{h}_{\theta,\mathfrak{p}}$-module of rank 1. If (i) is satisfied, then for such $\mathfrak{p}$ by Nakayama's lemma, the $\mathfrak{h}_{\theta,\mathfrak{p}}$-module $H_{\theta,\mathfrak{p}}^-$ is generated by one element. Since the last module is a faithful module, the last module is free of rank 1 and hence (iii) is satisfied. If (iii) is satisfied, then for any such $\mathfrak{p}$, $H_{\theta,\mathfrak{p}}^-$ is free of rank 1 as an $\mathfrak{h}_{\theta,\mathfrak{p}}$-module and hence $(H_\theta^-/I_\theta H_\theta^-)_\mathfrak{p}$ is free of rank 1 as an $(\mathfrak{h}_\theta/I_\theta\mathfrak{h}_\theta)_\mathfrak{p}$-module. This shows that the condition (ii) is satisfied.

7.2.11. If a conjecture of Greenberg [15] Conjecture 3.4 is true, $X_\chi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by one element as a $\Lambda_\chi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module. Hence it is natural to believe that the condition $C(\mathfrak{h})$ is always satisfied.

7.2.12. The following conditions (i)—(iii) are equivalent.

(i) $H_\theta^-/I_\theta H_\theta^-$ is generated by one element as a module over $\mathfrak{h}_\theta$.

(ii) $H_\theta^-/I_\theta H_\theta^-$ is a free $\mathfrak{h}_\theta/I_\theta$-module of rank 1.

(iii) The Eisenstein component of $\mathfrak{h}_\theta$ is Gorenstein.

This is proved by the arguments as in 7.2.10.

7.2.13. Our result Theorem 7.2.8 (1) assuming C($\mathfrak{h}$) shows that if $\mathfrak{h}_\theta$ is Gorenstein, then the Λ_χ-module $X_{Np^\infty,\chi}$ is generated by one element. But this is known (Harder-Pink [16], Kurihara [26]).

7.2.14. In the case $N = 1$, the theorems stated in Introduction are deduced from our results stated above. In fact, the $(\theta = \omega^i)$-components $(i \in \mathbb{Z}/(p-1)\mathbb{Z})$ of $\mathfrak{h}/I$ for $i \neq 0, 2$ are treated above, and the ω^i-components of $\mathfrak{h}/I$ for $i = 0, 2$ are zero in the case $N = 1$.

Concerning (ii) in Theorem 0.15, $(1-T^*(p))\{0, \infty\} \in H$ is defined since $(1-T^*(p))\tilde{H} \subset H$ in the case $N = 1$ by 3.3.7 and 4.4.1.

7.2.15. In the rest of this paper, we prove the above theorems. Here is an outline.

Identify $H_\theta^-/I_\theta H_\theta^-$ with $S_\Lambda/I_\theta S_\Lambda$ via the canonical isomorphism (cf. 8.1.1).

In section 8, we study the p-adic L-function $\mathcal{M}$ in two variables and prove

$\mathcal{M}_{s=0}(\gamma) = \xi'\gamma$ in $H_\theta/I_\theta H_\theta$ for $\gamma \in H_\theta^-$. (Theorem 8.1.2).

By studying Galois cohomology in section 9, we prove in section 10.1

$\mathcal{M}_{s=0}(\gamma) = \xi' \cdot \Upsilon(\varpi(\gamma))$ in $H_\theta/I_\theta H_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $\gamma \in H_\theta^-$. (Theorem 10.1.3)

These prove Theorem 7.2.3 (1).

Our proof of Theorem 7.2.3 (2) is by the method of Sharifi in his paper [49].

As is explained in section 10, we can deduce Theorems 7.2.6 and 7.2.8 from Theorem 7.2.3.

8 Some results on p-adic L-functions in two variables

8.1 $\mathcal{M}(\gamma) \mod I$ for $\gamma \in H^-$

8.1.1. By the coincidence of the Frobenius and $T^*(p)$ on $H_{\mathrm{quo},\theta}(1)$ (1.8.1) and by 4.2.3, we have $D(H_{\mathrm{quo},\theta}(1))/I_\theta D(H_{\mathrm{quo},\theta}(1)) = H_{\mathrm{quo},\theta}(1)/I_\theta H_{\mathrm{quo},\theta}(1)$.

We use the isomorphisms $H_\theta^-/I_\theta H_\theta^- \overset{\cong}{\to} H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta} \cong S_{\Lambda,\theta}/I_\theta S_{\Lambda,\theta}$ as identifications.

Let ${}^0\mathcal{M}$ be as in 4.3.3 and let $\mathcal{M}_{s=0}(\gamma)$ be as in 4.3.7.

Theorem 8.1.2. *For $\gamma \in H_\theta^-$ we have*

(1) ${}^0\mathcal{M}(\gamma) \equiv A_0\tilde{\xi} \cdot \gamma \mod I_\theta$ in $(H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta})[[\mathbb{Z}_p^\times]]$ with $A_0 \in Q(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$ and $\tilde{\xi} \in \Lambda_\theta[[\mathbb{Z}_p^\times]]$ defined as follows. First, A_0 is as in the case $s = 0$ of Proposition 4.3.6 (2). That is, A_0 is the image of the p-adic zeta function ${}^1\xi^*_{p^\infty}$ under $[a] \mapsto \sigma_a$. Next, $\tilde{\xi}$ is the image of ${}^{-1}\xi_{Np^\infty}$ under the composition $Q(\Lambda) \to Q(\Lambda[[\mathbb{Z}_p^\times]]) \to Q(\Lambda_\theta[[\mathbb{Z}_p^\times]])$ where the first arrow is induced by the ring homomorphism $\Lambda \to \Lambda[[\mathbb{Z}_p^\times]]$; $[a] \mapsto [a]\sigma_a$, and the second arrow is the projection.

(2) $\mathcal{M}_{s=0}(\gamma) \equiv \xi'\gamma \mod I_\theta$ in $H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta}$.

Proof. We prove (1). Let $(-)_E$ be the Eisenstein component.

Take an isomorphism $D(T) \cong T$ between two functors from the category of pro-p abelian groups with unramified actions of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ to the category of pro-p abelian groups (1.7.6). We have a commutative diagram

$$\begin{array}{ccccccc} H^-_{\theta,E} & \overset{\cong}{\to} & H_{\mathrm{quo}.\theta,\mathrm{E}} & \cong & D(H_{\mathrm{quo},\theta,\mathrm{E}}) & = & S_{\Lambda,\theta,E} \\ \cap & & \cap & & \cap & & \cap \\ \tilde{H}^-_{DM,\theta,E} & \overset{\cong}{\to} & \tilde{H}_{\mathrm{quo.DM},\theta,\mathrm{E}} & \cong & D(\tilde{H}_{\mathrm{quo,DM},\theta,\mathrm{E}}) & = & M_{\Lambda,DM,\theta,E}. \end{array}$$

Denote the isomorphism $H^-_{\theta,E} \xrightarrow{\cong} D(H_{\mathrm{quo},\theta,\mathrm{E}})$ in this diagram by h. Since $D(H_{\mathrm{quo},\theta,\mathrm{E}})$ is a dualizing $\mathfrak{h}_{\theta,E}$-module, the homomorphism ${}^0\mathcal{M} : H^-_{\theta,E} \to D(H_{\mathrm{quo},\theta,\mathrm{E}})[[\mathbb{Z}_p^\times]]$ has the form fh for some $f \in \mathfrak{h}_{\theta,E}[[\mathbb{Z}_p^\times]]$. We study $f \bmod I_{\theta,E}$.

If we denote by $\tilde{h}_{DM}$ the isomorphism $\tilde{H}^-_{DM,\theta,E} \xrightarrow{\cong} D(\tilde{H}_{\mathrm{quo,DM},\theta,\mathrm{E}})$ in the diagram, the homomorphism ${}^0\mathcal{M} : \tilde{H}^-_{DM,\theta,E} \to D(\tilde{H}_{\mathrm{quo,DM},\theta,\mathrm{E}})[[\mathbb{Z}_p^\times]]$ has also the form $f\tilde{h}_{DM}$ for the same f.

The isomorphisms h and $\tilde{h}_{DM}$ are compatible with the isomorphisms

$$\tilde{H}^-_{DM,\theta}/I_\theta H^-_\theta \cong \Lambda_\theta/(\xi)\ ;\ \{0,\infty\} \mapsto 1,$$

$$M_{\Lambda,DM,\theta,E}/S_{\Lambda,\theta,E} \cong \Lambda_\theta/(\xi)\ ;\ \sum_{n\geq 0} a_n q^n \mapsto a_0.$$

Hence to understand $f \bmod I_{\theta,E}$, it is sufficient to consider the constant term of ${}^0\mathcal{M}(\{0,\infty\})$. Recall that ${}^0\mathcal{M}(\{0,\infty\}) =$ the ordinary component of AB with A, B as in the case $s=0$ of Proposition 4.3.6. By $q \mapsto 0$, A is sent to A_0 and B is sent to B_0. Since $\langle a\rangle \in \mathfrak{H}_\theta$ ($a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$) acts on M_Λ as the multiplication by $[a^{-1}]$ on $\Lambda[[q]]$, this proves (1).

We prove (2). Similarly, it is sufficient to consider the constant term of $\mathcal{M}_{s=0}(\{0,\infty\})$ in the Drinfeld-Manin modification 6.2.1. In the Drinfeld-Manin modification, AB is equal to $A(B - B(0))$, where $B(0) \in M_\Lambda \otimes_\Lambda Q(\Lambda)$ is the image of B under $\sigma_a \mapsto 1$. The constant term of $A(B-B(0))$ is equal to $A_0(B_0 - B_0(0))$ where $B_0(0)$ is the image of B_0 under $\sigma_a \mapsto 1$. Let u be the element of $1+p\mathbb{Z}_p$ such that $(1-p^{-1})\log(u) = 1$, and let $\gamma = \sigma_u \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Then A_0 has the shape $U/(\gamma-1)$ with $U(0) = 1$. We have $A_0(B_0 - B_0(0)) = U \cdot (B_0 - B_0(0))/(\gamma-1)$. By the Λ_θ-homomorphism $\Lambda_\theta[[\mathbb{Z}_p^\times]] \to \Lambda_\theta\ ;\ \sigma_a \mapsto 1$ (for all $a \in \mathbb{Z}_p^\times$), U is sent to $U(0) = 1$ and $(B_0 -_0 (0))/(\gamma-1)$ is sent to the derivative $({}^{-1}\xi^*)'$ of ${}^{-1}\xi^*$. Since $\langle a\rangle$ acts on M_Λ as the multiplication by $[a^{-1}]$ on $\Lambda[[q]]$, this proves (2). □

8.2 Related results

8.2.1. Let ${}^0\mathcal{L} \in H[[\mathbb{Z}_p^\times]]$ be the image of $\mathcal{L} \in H[[\mathbb{Z}_p^\times]]$ under the isomorphism $H[[\mathbb{Z}_p^\times]] \to H[[\mathbb{Z}_p^\times]]\ ;\ x \otimes [a] \mapsto a^{-1}x \otimes [a]$. Let $\mathcal{L}_{s=0} \in H$ be the image of ${}^0\mathcal{L}$ under $H[[\mathbb{Z}_p^\times]] \to H\ ;\ x \otimes [a] \mapsto x$. Then $\mathcal{L}_{s=0} \in H^+$.

Theorem 8.2.2. *(1) Under the canonical isomorphism* $(H^+_\theta/I_\theta H^+_\theta)[[\mathbb{Z}_p^\times]] \xrightarrow{\cong} (\Lambda_\theta/(\xi))[[\mathbb{Z}_p^\times]]$ *which we take as an identification, we have* ${}^0\mathcal{L}^+_\theta \bmod I_\theta = 2\sigma_{-1}\cdot A_0\tilde{\xi}$ *where* A_0 *and* $\tilde{\xi}$ *are as in Theorem 8.1.2 (1).*

(2) $\mathcal{L}_{s=0,\theta} \bmod I_\theta = 2\xi'$ *in* $H^+_\theta/I_\theta H^+_\theta \cong \Lambda_\theta/(\xi)$.

Proof. We prove (1). We compute the image of ${}^0\mathcal{L}$ under the map $(-,\xi\{0,\infty\}_{DM,\theta,E})_\Lambda : H_\theta/I_\theta H_\theta \to \Lambda_\theta/(\xi)$. It is the image under $\Lambda \to \Lambda_\theta\ [a] \mapsto [a^{-1}]$ of the coefficient of q in $(({}^0\mathcal{L}, \xi\{0,\infty\}_{DM,\theta,E}))_\Lambda$. By Theorem 4.4.3, we have

$$\begin{aligned}({}^0\mathcal{L}, \xi\{0,\infty\}_{DM,\theta,E})) &= -\sigma_{-1}{}^0\mathcal{M}(\xi\{0,\infty\}_{DM,\theta,E}) = -\sigma_{-1}\xi\cdot{}^0\mathcal{M}(\{0,\infty\})_{DM,\theta,E}\\ &= (\sigma_{-1}\cdot\xi AB)_{DM,\theta,E}\end{aligned}$$

where A and B are as in the case $s = 0$ of 4.3.6. In the Drinfeld-Manin modification, $\sigma_{-1} \cdot \xi AB$ is equivalent to $\sigma_{-1} \cdot (\xi AB - ABE_\Lambda^{(2)}) \in S_\Lambda \otimes_\Lambda Q(\Lambda)$ where $E_\Lambda^{(2)}$ is the Λ-adic Eisenstein series of weight 2 in 6.2.8. We take the coefficient of q. Since we take mod ξ, ξAB is neglected. Since the coefficient of q in $E^{(2)}$ is 2, we obtain $2\sigma_{-1} \cdot A_0 B_0$.

(2) is obtained from (1) just as Theorem 8.1.2 (2) was obtained by using 8.1.2 (1). □

The conditions C(ξ) and C($T^*(p)$) have the following similar interpretations. Let $\mathcal{L}_{s=1}$ be the image of $\mathcal{L}$ under $H[[\mathbb{Z}_p^\times]] \to H\ ;\ x \otimes [a] \mapsto x$.

Proposition 8.2.3. *(1) $C(\xi) \Leftrightarrow \mathcal{L}_{s=0}$ generates the $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module $H_\theta^+/I_\theta H_\theta^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

(2) $C(T^*(p)) \Leftrightarrow \mathcal{L}_{s=1}$ generates the $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-module $H_\theta^-/I_\theta H_\theta^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. The equivalence (1) follows from Theorem 8.2.2 (2).
(2) is clear since $\mathcal{L}_{s=1} = (1 - T^*(p)^{-1})\{0, \infty\}$. □

Proposition 8.2.4. *Let f be as in 7.1.11. Let $\psi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}^\times$ and let $L_\psi \subset \bar{\mathbb{Q}}_p$ as before. Let r be an integer such that $1 \leq r \leq k-1$.*

(1) Assume $(-1)^r\psi(-1) = 1$. Then we have:

$$L_p(f, \psi, r) \equiv 2L(1-r, \psi)L(1-k+r, \psi^{-1}\epsilon) \mod I(f)O_{L_\psi}$$

in $O_{L_\psi}/I(f)O_{L_\psi}$.

(2) $L_p(f, 0) \equiv 2L'_p(1-k, \epsilon) \mod I(f)O_{L_\psi}$. Here L'_p is the derivative of the p-adic Dirichlet L-function.

Here $L_p(f, \psi, r)$, which is originally an element of $V(f^*)^+_{O_{L_\psi}}$, is regarded as an element of O_{L_ψ} by using an O_{L_ψ}-basis of $V(f^*)^+_{O_{L_\psi}}$ whose mod $I(f)$ comes from the canonical $\mathfrak{h}_\theta/I_\theta$-basis of $H_\theta^+/I_\theta H_\theta^+$.

Proof. This follows from Theorem 8.2.2. □

8.2.5. For example, let f be the p-stabilization of $\Delta = q\prod_{n=1}^\infty (1-q^n)^{24} \in S_{12}(1)_\mathbb{Z}$ where $p = 691$. In this case, $F = \mathbb{Q}$, $L = \mathbb{Q}_p$, $I(f) = p\mathbb{Z}_p$. Proposition 8.2.4 tells that for $0 < r < 12$, r even,

$$L_p(\Delta, r) \equiv 2\zeta(1-r)\zeta(1+r-k) \bmod p \quad \text{for} \ \ p = 691.$$

The ratio of $(r-1)!L(\Delta, r)/(2\pi i)^r$ for $r = 2, 4, 6$ is $1 : -\frac{5^2}{2^4 \cdot 3} : \frac{5}{2^2 \cdot 3}$ as in Introduction of Manin [28]. On the other hand, $\zeta(-1)\zeta(-9) : \zeta(-3)\zeta(-7) : \zeta(-5)^2 = \frac{1}{2^4 \cdot 3^2 \cdot 11} : \frac{1}{2^7 \cdot 3^2 \cdot 5^2} : \frac{1}{2^4 \cdot 3^5 \cdot 7^2}$. These ratios mod 691 coincide.

8.2.6. Note that the values $L_p(f, \psi, r)$ of p-adic L-functions with $(-1)^r\psi(-1) = 1$ appear in Proposition 8.2.4 whereas those with $(-1)^r\psi(-1) = -1$ appear in Conjecture 7.1.12.

9 Study of Galois cohomology

In this section 9, we study the maps

$$H^-_\theta/I_\theta H^-_\theta \xrightarrow{\varpi} X_{Np^\infty,\chi} \xrightarrow{\Upsilon} H^-_\theta/I_\theta H^-_\theta$$

by using Galois cohomology theory.

In section 9.1, we give some Galois cohomological understandings of $X_{Np^\infty,\chi}$. In section 9.2, we give some Galois cohomological understanding of "the evaluation at ∞" in section 5 which was important in our study of the map ϖ. In section 9.3, we show that the derivative of p-adic zeta function appears in Galois cohomology. In section 9.4, we give some Galois cohomological understandings of the map Υ. In section 9.5, we relate the derivatie of p-adic zeta function to this understanding of Υ. In section 9.6, we compute the Galois cohomology class of an extension of $\mathcal{R}$ by $\mathcal{Q}$ which appears in cohomology of modular curves.

See Sharifi [52] for results related to the results of this section.

9.1 Galois cohomology of $\mathcal{Q}(2)$

9.1.1. Recall that $X_{Np^\infty,\chi} \cong \mathcal{S}_\theta(-1)$ (6.4.2). In this section 9.1, we define the following two isomorphisms

$$(1)\ H^1(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \cong \mathcal{S}_\theta, \qquad (2)\ H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \cong \mathcal{S}_\theta.$$

9.1.2. Let $\Lambda^\sharp_\theta$ be Λ_θ on which $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts as follows. For $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, σ acts as the multiplication by $[a]^{-1} \in \Lambda$ where $a \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ is determined by $\sigma(\zeta_{Np^r}) = \zeta_{Np^r}^a$ $(r \geq 1)$. We have

$$H^2(\mathbb{Z}[1/Np], \Lambda^\sharp_\theta(2)) \cong \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta = \mathcal{S}_\theta.$$

Lemma 9.1.3. $H^1(\mathbb{Z}[1/Np], \Lambda^\sharp_\theta(2)) = 0$.

Proof. We have

$$H^1(\mathbb{Z}[1/Np], \Lambda^\sharp_\theta(2)) \cong \varprojlim_r H^1(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta \cong \varprojlim_r (\mathbb{Z}[1/Np, \zeta_{Np^r}]^\times \otimes \mathbb{Z}_p)_\chi(1)$$

$$\cong \varprojlim_r (\mathbb{Z}[\zeta_{Np^r}]^\times \otimes \mathbb{Z}_p)_\chi(1) \cong \mathbb{Z}_p(1)_\chi(1) = 0$$

where the third $\cong$ follows from Assumptions 2 and 3 in 6.1.4, the fourth $\cong$ follows from the fact χ is odd, and the fifth $\cong$ follows from the Assumptions 2 and 4 in 6.1.4. $\square$

9.1.4. Consider the exact sequence

$$0 \to \Lambda^\sharp_\theta(2) \xrightarrow{\xi} \Lambda^\sharp_\theta(2) \to (\Lambda^\sharp_\theta/(\xi))(2) \to 0.$$

By section 6.3, we have a canonical isomorphism $(-, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda : \mathcal{Q} \cong \Lambda^\sharp_\theta/(\xi)$ as a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over Λ_θ. Hence we have an exact sequence

$$0 \to \Lambda^\sharp_\theta(2) \xrightarrow{\xi} \Lambda^\sharp_\theta(2) \to \mathcal{Q}(2) \to 0.$$

By 9.1.2, this exact sequence induces an exact sequence

$$H^1(\mathbb{Z}[1/Np], \Lambda^\sharp_\theta(2)) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \to \mathcal{S}_\theta \xrightarrow{\xi} \mathcal{S}_\theta \to H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \to 0.$$

By classical Iwasawa theory, $\mathcal{S}_\theta$ is killed by ξ. Hence by Lemma 9.1.3, the last exact sequence gives the isomorphisms (1) and (2) in 9.1.1.

9.2 Evaluation at ∞ and Galois cohomology

The aim of this subsection is to prove the following proposition.

Proposition 9.2.1. *The evaluation at the ∞-cusp*

$$\varprojlim_r H^2(X_1(Np^r) \otimes \mathbb{Z}[1/Np], \mathbb{Z}_p(2)) \to \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta$$

coincides with the composition

$$\varprojlim_r H^2(X_1(Np^r) \otimes \mathbb{Z}[1/Np], \mathbb{Z}_p(2)) \to H^1(\mathbb{Z}[1/Np], H_\theta(2))$$

$$\to H^1(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \cong \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta$$

Here in this composition, the second arrow is given by the projection $H \to \mathcal{Q}$ and the last isomorphism is 9.1.1 (2).

9.2.2. We consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Lambda^\sharp_\theta(2) & \to & \tilde{H}_{c,\theta,E}(2) & \to & H_{\theta,E}(2) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Lambda^\sharp_\theta(2) & \xrightarrow{\xi} & \Lambda^\sharp_\theta(2) & \to & \Lambda^\sharp_\theta/(\xi)(2) & \to & 0. \end{array}$$

Here: The upper horizontal row is the Λ_θ-dual of the exact sequence $0 \to H_{\theta,E} \to \tilde{H}_{\theta,E} \to \Lambda_\theta \to 0$ (6.2.5) for the pairing $(-,-)_\Lambda$ (section 1.6). The middle vertical arrow is the pairing $(-,[g])_\Lambda$ with $[g] =$ the class of $(g_{0,1/Np^r})_r \in \tilde{H}_{\theta,E}(1)$ (6.2.12) for $(-,-)_\Lambda$ as in 1.6.7. The right vertical arrow is the map $(-,\xi\{0,\infty\}_{DM,\theta,E})_\Lambda$ in section 6.3, that is, the composition $H_{\theta,E}(2) \to \mathcal{Q}(2) \cong \Lambda^\sharp/(\xi)(2)$. The map ξ is the multiplication by ξ.

Lemma 9.2.3. *The above diagram in 9.2.2 is commutative and respects the action* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Proof. The compatibility with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is checked easily. The commutativity of the left square is by 6.2.13. We consider the right square.

Let $x \in H_{\theta,E}$ and let y be a lifting of x to $\tilde{H}_{c,\theta,E}$. Let λ be the image of x in $\Lambda_\theta/(\xi)$ under the right vertical arrow of the diagram. What we have to prove is $(y,[g])_\Lambda \bmod \xi = \lambda$.

By the boundary map $\tilde{H}_{\theta,E} \to \Lambda_\theta$, $\{0,\infty\}$ is sent to -1 and $[g]$ is sent to ξ. Hence $A =: \xi\{0,\infty\} + [g]$ belongs to $H_{\theta,E}$. Since $[g]$ dies in the Drinfeld-Manin modification (6.2.14), we have $A = \xi\{0,\infty\}_{DM,E}$. Hence $\lambda = (x,A)_\Lambda = (y,\xi\{0,\infty\})_\Lambda + (y,[g])_\Lambda$. But $(y,\xi\{0,\infty\})_\Lambda \equiv 0 \mod \xi$. □

Corollary 9.2.4. *The composition* $H^1(\mathbb{Z}[1/Np], H_{\theta,E}(2)) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \cong \mathcal{S}_\theta$ *coincides with the connecting map* $H^1(\mathbb{Z}[1/Np], H_{\theta,E}(2)) \to H^2(\mathbb{Z}[1/Np], \Lambda_\theta^\sharp(2))$ *of the upper exact sequence in the diagram in 9.2.2.*

Hence for the proof of Proposition 9.2.1, it is sufficient to prove the following Lemma 9.2.5.

Lemma 9.2.5. *The evaluation* $\varprojlim_r H^2(X_1(Np^r) \otimes \mathbb{Z}[1/Np], \mathbb{Z}_p(2)) \to \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta$ *at the* ∞*-cusp coincides with the composition* $\varprojlim_r H^2(X_1(Np^r) \otimes \mathbb{Z}[1/Np], \mathbb{Z}_p(2)) \to H^1(\mathbb{Z}[1/Np], H_{\theta,E}(2)) \to H^2(\mathbb{Z}[1/Np], \Lambda_\theta^\sharp(2))$*, where the last arrow is the connecting map of the upper exact sequence in the diagram in 9.2.2.*

9.2.6. We consider the upper exact sequence in the diagram in 9.2.2. Since the Atkin-Lehner involution, which appears in the definition of $(-,-)_\Lambda$, exchanges 0-cusps and ∞-cusps, and since only 0-cusps contribute to the exact sequence $0 \to H_{\theta,E} \to \tilde{H}_{\theta,E} \to \Lambda_\theta \to 0$, only ∞-cusps contribute to the dual exact sequence $0 \to \Lambda_\theta \to \tilde{H}_{c,\theta,E} \to H_{\theta,E} \to 0$. More precisely, the last exact sequence is obtained as follows. Let $F = \mathbb{Z}/p^n\mathbb{Z}(2)$ on the étale site of $X_1(Np^r)_{\bar{\mathbb{Q}}}$, $F'' = i_*(\mathbb{Z}/p^n\mathbb{Z})(2)$ where i is the inclusion morphism $\{\infty\text{-cusps}\} \to X_1(Np^r)_{\bar{\mathbb{Q}}}$, $F' = \mathrm{Ker}\,(F \to F'')$. Then from the exact sequence $0 \to F' \to F \to F'' \to 0$, we obtain an exact sequence $0 \to T \to H^1(X_1(Np^r)_{\bar{\mathbb{Q}}}, F') \to H^1(X_1(Np^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2) \to 0$, where T is the cokernel of $H^0(X_1(Np^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2) \to H^0(X_1(Np^r)_{\bar{\mathbb{Q}}}, F'')$. When we take the inverse limit for n and r, the inverse limit of the (θ, E)-component of T becomes $\Lambda_\theta^\sharp$, that of $H^1(X_1(Np^r)_{\bar{\mathbb{Q}}}, F')$ becomes $\tilde{H}_{c,\theta,E}(2)$, and that of $H^1(X_1(Np^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2)$ becomes $H_{\theta,E}(2)$.

Lemma 9.2.7. *Let* $\mathcal{C}_1$*,* $\mathcal{C}_2$*,* $\mathcal{C}_3$ *be abelian categories and let* $f : \mathcal{C}_1 \to \mathcal{C}_2$ *and* $g : \mathcal{C}_2 \to \mathcal{C}_3$ *be left exact functors. Assume that* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *have enough injective objects, and assume that* f *sends injective objects of* $\mathcal{C}_1$ *to injective objects of* $\mathcal{C}_2$*. Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence in* $\mathcal{C}_1$ *and assume that* $(R^i f)F'' = 0$ *for all* $i > 0$*. Let* S *be the kernel of* $(R^2(gf))F \to g(R^2 f)F$ *and let* T *be the cokernel of* $fF \to fF''$*. Then we have a commutative diagram*

$$\begin{array}{ccccc}
S & \subset & R^2(gf)F & \to & R^2(gf)F'' \\
 & & & & \| \\
\downarrow & & & & (R^2 g)fF'' \\
 & & & & \downarrow \\
(R^1 g)(R^1 f)F & & \to & & (R^2 g)T.
\end{array}$$

Here the left vertical arrow is by the spectral sequence $(R^i g)(R^j f) \Rightarrow R^{i+j}(gf)$ *and the lower horizontal arrow is the connecting map of the exact sequence* $0 \to T \to (R^1 f)F' \to (R^1 f)F \to 0$ *(which appears in the long exact sequence of* $R^i f$ *obtained from the exact sequence* $0 \to F' \to F \to F'' \to 0$*).*

Proof. Exercise. □

9.2.8. We prove Lemma 9.2.5.

We apply Lemma 9.2.7 by taking the following: $\mathcal{C}_1$ is the category of abelian sheaves on the étale site of $X_1(Np^r) \otimes \mathbb{Z}[1/Np]$, $\mathcal{C}_2$ is the category of abelian sheaves on the étale site of Spec($\mathbb{Z}[1/Np]$), $\mathcal{C}_3$ is the category of abelian groups, f is the direct image functor, g is the global section functor, $F = \mathbb{Z}/p^n\mathbb{Z}(2)$ on the étale site of $X_1(Np^r) \otimes \mathbb{Z}[1/Np]$ for some $n \geq 1$, $F'' = i_*(\mathbb{Z}/p^n\mathbb{Z})(2)$ where i is the inclusion morphism $\infty_{Np^r}(0,1)$: Spec($\mathbb{Z}[1/Np, \zeta_{Np^r}]^+$) $\to X_1(Np^r)$, and $F' = \mathrm{Ker}\,(F \to F'')$. The commutative diagram in Lemma 9.2.7 becomes

$$\begin{array}{ccc} H^2(X_1(Np^r) \otimes \mathbb{Z}[1/Np], (\mathbb{Z}/p^n\mathbb{Z})(2)) & \to & H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], (\mathbb{Z}/p^n\mathbb{Z})(2))^+ \\ \downarrow & & \downarrow \\ H^1(\mathbb{Z}[1/Np], H^1_{\text{ét}}(X_1(Np^r))(2)) & \to & H^2(\mathbb{Z}[1/Np], T) \end{array}$$

(S in the diagram in 9.2.7 becomes the same as $R^2(gf)F = H^2(X_1(Np'r) \otimes \mathbb{Z}[1/Np], (\mathbb{Z}/p^n\mathbb{Z})(2))$). In the present diagram, the upper horizontal arrow is the evaluation at the ∞-cusp, T is the cokernel of $H^0(X_1(Np^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2) \to H^0(\mathbb{Z}[1/Np, \zeta_{Np^r}] \otimes \bar{\mathbb{Q}}, \mathbb{Z}/p^n\mathbb{Z})(2)$ which is identified with the cokernel of $\mathbb{Z}/p^n\mathbb{Z}(2) \to \mathbb{Z}/p^n\mathbb{Z}[(\mathbb{Z}/Np^r\mathbb{Z})^\times/\{\pm 1\}](2)$, and the lower horizontal arrow is the connecting map of the exact sequence $0 \to T \to H^1(X_1(Np^r)_{\bar{\mathbb{Q}}}, F') \to H^1(X_1(Np^r)_{\bar{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z}) \to 0$. Taking the inverse limit for r and n, we obtain Lemma 9.2.5.

Now the proof of Proposition 9.2.1 is completed.

9.3 Derivative and Galois cohomology

In this 9.3, we prove the following proposition.

Proposition 9.3.1. *Consider the following two homomorphisms*

$$a, b : H^1(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \longrightarrow H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2)).$$

The homomorphism a is the composition

$$H^1(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \cong \mathcal{S}_\theta \cong H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2))$$

of the isomorphisms in (1), (2) in 9.1.1. The homomorphism b is the cup product $\cup(1 - p^{-1})\log(\kappa)$ with $(1 - p^{-1})\log(\kappa) \in H^1(\mathbb{Z}[1/Np], \mathbb{Z}_p)$, where κ is the cyclotomic character. Then we have

$$b = \xi' \cdot a.$$

9.3.2. Let G be a pro-finite group and assume the following (1) and (2).

(1) For any finite abelian group T of p-power order endowed with a continuous action of G, and for any $i \geq 0$, the cohomology group $H^i(G, T)$ is finite.

(2) We are given a closed normal subgroup H of G such that $\Gamma := G/H$ is isomorphic to $\mathbb{Z}_p$.

The assumption (1) tells the following. For a pro-p abelian group A endowed with a continuous action of G, and for any i, the continuous cohomology $H^i(G, A)$ coincides with $\varprojlim_{A'} H^i(G, A/A')$ where A' ranges over all open subgroups of A.

Let $\mathbb{Z}_p[[\Gamma]]^\sharp$ be $\mathbb{Z}_p[[\Gamma]]$ on which G acts as follows. For $\sigma \in G$, the action of σ on $\mathbb{Z}_p[[\Gamma]]^\sharp$ is the multiplication by $\bar{\sigma}^{-1}$, where $\bar{\sigma}$ denotes the image of σ in Γ.

Let f be a non-zero-divisor of $\mathbb{Z}_p[[\Gamma]]$ and let T be a pro-p abelian group endowed with a continuous action of G. Let M be the cokernel of the injective homomorphism $f = f \otimes 1 : \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T \to \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T$. Here $\hat{\otimes}$ is the topological tensor product as in 1.7.3.

Let t be a generator of Γ and let

$$l_t : \Gamma \xrightarrow{\cong} \mathbb{Z}_p$$

be the isomorphism which sends t to 1. We denote the element of $H^1(G, \mathbb{Z}_p)$ corresponding to l_t by the same letter l_t.

Proposition 9.3.3. *Let the notation be as above. Then for each $i \geq 0$, the cup product*

$$H^i(G, M) \to H^{i+1}(G, M)\ ; x \mapsto -l_t \cup x$$

with $-l_t \in H^1(G, \mathbb{Z}_p)$ coincides with the composition

$$H^i(G, M) \xrightarrow{\partial} H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T) \xrightarrow{tdf/dt} H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T) \to H^{i+1}(G, M)$$

where:

The first arrow ∂ is the connecting map of the exact sequence of G-modules $0 \to \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T \xrightarrow{f} \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T \to M \to 0$.

The second arrow tdf/dt is induced by the $\mathbb{Z}_p[[\Gamma]]$-linear G-homomorphism $tdf/dt \otimes 1 : \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T \to \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T$.

The third arrow is the canonical projection.

The proof of this proposition is given after we prove the following Lemmas 9.3.4 and 9.3.5.

Lemma 9.3.4. *For any pro-p abelian group A endowed with a continuous action of G, the map $x \mapsto -l_t \cup x : H^i(G, A) \to H^{i+1}(G, A)$ coincides with the composition*

$$H^i(G, A) \to H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} A) \to H^{i+1}(G, A)$$

where the first arrow is the connecting map of exact sequence $0 \to \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} A \xrightarrow{a} \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} A \to A \to 0$ with $a = t \otimes 1 - 1$ and the second map is the canonical projection.

Proof. This can be proved by explicit computations of the connecting map and the cup product of group cohomology. Here $-l_t$ appears because if $\sigma \in G$ and if we write $\bar{\sigma}^{-1} - 1 = (t-1)g$ where $\bar{\sigma}$ is the image of σ in Γ and $g \in \mathbb{Z}_p[[\Gamma]]^\sharp$, then the image of g in $\mathbb{Z}_p$ under the canonical projection coincides with $-l_t(\sigma)$. $\square$

Lemma 9.3.5. *Assume we have a commutative diagram of exact sequences of pro-p G-modules*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P' & \to & Q' & \to & R' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & Q & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P'' & \to & Q'' & \to & R'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Let $j \geq 1$. Assume $H^j(G,Q') \to H^j(G,R')$ is surjective and $H^j(G,P) \to H^j(G,Q)$ is injective. Then the connecting map $H^{j-1}(G,R'') \to H^j(G,P'')$ of $0 \to P'' \to Q'' \to R'' \to 0$ coincides with the minus of the connecting map of the snake lemma for

$$\begin{array}{ccccccc} & & H^j(G,P') & \to & H^j(G,Q') & \to & H^j(G,R') & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^j(G,P) & \to & H^j(G,Q) & \to & H^j(G,R) & & \end{array}$$

defined by using the map $H^{j-1}(G,R'') \to Ker(H^j(G,R') \to H^j(G,R))$ and the map $Coker(H^j(G,P') \to H^j(G,P)) \to H^j(G,P'')$.

Proof. For a pro-p G-module A, let $J(A)$ be the standard complex which computes the Galois cohomology of A. We have a commutative diagram of exact sequences of complexes

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J(P') & \to & J(Q') & \to & J(R') & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J(P) & \to & J(Q) & \to & J(R) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J(P'') & \to & J(Q'') & \to & J(R'') & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0. & & \end{array}$$

Let $\bar{x} \in H^{j-1}(G,R'')$. Let x be an element of $J(R'')^{j-1}$ which represents $\bar{x}$. Lift x to $y \in J(Q)^{j-1}$. Then the image of $\bar{x}$ under the connecting map $H^{j-1}(G,R'') \to H^j(G,P'')$ is the class of $dy_{J(Q'')} \in J(P'')^j$. Call this class (1). On the other hand, the image of $\bar{x}$ under the connecting map $H^{j-1}(G,R'') \to H^j(G,R')$ is the class of $dy_{J(R)} \in J(R')^j$. By the surjectivity of $H^j(G,Q') \to H^j(G,R')$, there exist $z \in J(Q')^j$ and $u \in J(R')^{j-1}$ such that $dz = 0$ and $z_{J(R')} = dy_{J(R)} + du$. Let $\tilde{u}$ be a lifting of u to $J(Q')^{j-1}$ and replace z by $z - d\tilde{u}$. Then we have $z_{J(R')} = dy_{J(R)}$. Hence the map $J(Q)^j \to J(R)^j$ sends $z_{J(Q)} - dy$ to 0. That is, $z_{J(Q)} - dy \in J(P)^j$. Hence the class of $z_{J(Q)}$ in $H^j(G,Q)$ is the image of the class of $z_{J(Q)} - dy \in J(P)^j$ in $H^j(G,P)$. The image in $H^j(G,P'')$ of this class is the class of $z_{J(Q'')} - dy_{J(Q'')} = -dy_{J(Q'')} \in J(Q'')^j$. Call this class (2). Then (1) $= -$ (2). □

9.3.6. We prove Proposition 9.3.3.

We apply Lemma 9.3.5 to the following situation: $P = P' = Q = Q' = \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T$, $P \to Q$ and $P' \to Q'$ are $t \otimes 1 \otimes 1 - 1$, and $P' \to P$ and $Q' \to Q$ are $1 \otimes f \otimes 1$, and $j = i + 1$. The sequences $R' \to R \to R''$ and $P'' \to Q'' \to R''$ are identified with $\mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T \xrightarrow{f} \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T \to M$ and $\mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} M \xrightarrow{t \otimes 1 - 1} \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} M \to M$, respectively. We have an isomorphism of G-modules

$$\mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp \cong \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} \mathbb{Z}_p[[\Gamma]] \ ; \ t_1 \otimes t_2 \mapsto t_1 \otimes t_1^{-1} t_2 \ \ (t_i \in \Gamma)$$

where G acts on the $\mathbb{Z}_p[[\Gamma]]$ without $\sharp$ trivially. By this isomorphism, the sequence $H^j(G,P) \to H^j(G,Q) \to H^j(G,R)$ is identified with

$$A \hat{\otimes} \mathbb{Z}_p[[\Gamma]] \xrightarrow{b} A \hat{\otimes} \mathbb{Z}_p[[\Gamma]] \xrightarrow{c} A$$

$$\text{where } A = H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T), \quad b = (t \otimes 1) \otimes t^{-1} - 1, \quad c(x \otimes y) = yx.$$

The map b is injective and the map c is surjective as is easily seen.

By Lemma 9.3.4 and Lemma 9.3.5, the map $H^i(G, M) \to H^{i+1}(G, M)$; $x \mapsto l_t \cup x$ coincides with the composition $H^i(G, M) \to H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} M) \to H^{i+1}(G, M)$ where the second arrow is the canonical projection and the first arrow is the connecting map of the snake lemma for

$$\begin{array}{cccccccccc} & & & & & & H^i(G,M) & & \\ & & & & & & \downarrow \partial & & \\ 0 & \to & A\hat{\otimes}\mathbb{Z}_p[[\Gamma]] & \xrightarrow{b} & A\hat{\otimes}\mathbb{Z}_p[[\Gamma]] & \xrightarrow{c} & A & \to & 0 \\ & & \downarrow 1 \otimes f & & \downarrow 1 \otimes f & & \downarrow f & & \\ 0 & \to & A\hat{\otimes}\mathbb{Z}_p[[\Gamma]] & \xrightarrow{b} & A\hat{\otimes}\mathbb{Z}_p[[\Gamma]] & \xrightarrow{c} & A & \to & 0 \\ & & \downarrow & & & & & & \\ & & H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} M). & & & & & & \end{array}$$

Here the map $H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T) \hat{\otimes} \mathbb{Z}_p[[\Gamma]] \to H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} M)$ is $u \otimes v \mapsto$ (the projection of vu).

Let $x \in H^i(G, M)$ and let $y = \partial(x) \in A$. Then y lifts to $y \otimes 1 \in A \hat{\otimes} \mathbb{Z}_p[[\Gamma]]$. Since $f(y) = 0$, $(1 \otimes f)(y \otimes 1) = (1 \otimes f - f \otimes 1)(y \otimes 1)$. Write

$$1 \otimes f - f \otimes 1 = (t \otimes t^{-1} - 1)g \quad \text{in } \mathbb{Z}_p[[\Gamma]] \hat{\otimes} \mathbb{Z}_p[[\Gamma]]$$

with $g \in \mathbb{Z}_p[[\Gamma]] \hat{\otimes} \mathbb{Z}_p[[\Gamma]]$. Then $(1 \otimes f - f \otimes 1)(y) = b(gy)$. Since the map

$$\mathbb{Z}_p[[\Gamma]] \hat{\otimes} \mathbb{Z}_p[[\Gamma]] \to \mathbb{Z}_p[[\Gamma]] \; ; \; u \otimes v \otimes h \mapsto uv \otimes h$$

sends g to $-tdf/dt$, the image of gy under

$$H^{i+1}(G, \mathbb{Z}_p[[\Gamma]]^\sharp \hat{\otimes} T) \hat{\otimes} \mathbb{Z}_p[[\Gamma]] \to H^i(G, M) \; ; \; u \otimes v \mapsto \text{projection of } vu$$

is the image of $(tdf/dt)y$. This proves the proposition.

9.3.7. Proposition 9.3.1 follows from Proposition 9.3.3.

9.4 The map Υ and Galois cohomology

Here, we give descriptions 9.4.3 and 9.4.4 of the map Υ by Galois cohomology theory.

9.4.1. We review the arithmetic duality in Galois cohomology theory (Poitou-Tate duality).

Let F be a finite extension of $\mathbb{Q}$ and let U be a dense open subscheme of $\mathrm{Spec}(O_F[1/p])$.

For a finite abelian group T of order a power of p endowed with a continuous action of $G := \pi_1(U)$, and for $i \in \mathbb{Z}$, let $H^i_{(c)}(U, T)$ be the i-th cohomology of the following complex $C_{(c)}(G, T)$: $C_{(c)}(G, T)$ is the simple complex associated to the double complex defined to be the "mapping fiber" of the canonical map $C(G, T) \to \oplus_{v \notin U} C(F_v, T)$. Here $C(G, T)$ is the standard complex of the topological G-module T which computes the continuous cohomology $H^i(G, T) = H^i(U, T)$, v ranges over all finite places of F which do not belong to U, and $C(F_v, T)$ denotes the standard complex of the topological $\mathrm{Gal}(\bar{F}_v/F_v)$-module T which computes the continuous cohomology $H^i(\mathrm{Gal}(\bar{F}_v/F_v), T) = H^i(F_v, T)$.

We have an evident long exact sequence

$$\cdots \to H^i_{(c)}(U, T) \to H^i(U, T) \to \oplus_{v \notin U} H^i(F_v, T) \to H^{i+1}_{(c)}(U, T) \to \ldots.$$

Via this exact sequence, $H^3_{(c)}(U, T)$ is isomorphic to the cokernel of $H^2(U, T) \to \oplus_{v \notin U} H^2(F_v, T)$. This isomorphism for $T = (\mathbb{Z}/p^n\mathbb{Z})(1)$ and the canonical isomorphisms $H^2(F_v, (\mathbb{Z}/p^n\mathbb{Z})(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$ for $v \notin U$ induce a canonical isomorphism $H^3_{(c)}(U, (\mathbb{Z}/p^n\mathbb{Z})(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$. The duality of Poitou-Tate formulated as in Mazur [29] says that for $n \geq 0$ such that p^n kills T, the cup product

$$H^i_{(c)}(U, T) \times H^{3-i}(U, T^\vee(1)) \to H^3_{(c)}(U, (\mathbb{Z}/p^n\mathbb{Z})(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$$

is a perfect duality of finite abelian groups. Here $(-)^\vee$ denotes the Pontryagin dual.

In particular, $H^3_{(c)}(U, T)$ is the dual of $H^0(U, T^\vee(1))$ and hence we have a canonical isomorphism

$$(1) \qquad H^3_{(c)}(U, T) \cong T(-1)_G \quad \text{(the co-invariant)}.$$

9.4.2. Let $\mathcal{P}$ and $\mathcal{Q}$ be as in 6.3.1.

Since $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts trivially on $\mathcal{P}(1)$, we have by (1) in 9.4.1

$$H^3_{(c)}(\mathbb{Z}[1/Np], \mathcal{P}(2)) \cong \mathcal{P}(1) \cong \mathcal{P}.$$

Proposition 9.4.3. *We have a commutative diagram*

$$\begin{array}{ccccc} H^2_{(c)}(\mathbb{Z}[1/Np], \mathcal{Q}(2)) & \to & H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2)) & \cong & X_{Np^\infty,\chi} \\ \downarrow & & & & \downarrow \Upsilon \\ \mathcal{P} & & = & & \mathcal{P}. \end{array}$$

Here the left vertical arrow is the minus of the connecting map

$$H^2_{(c)}(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \to H^3_{(c)}(\mathbb{Z}[1/Np], \mathcal{P}(2)) = \mathcal{P}$$

of the exact sequence $0 \to \mathcal{P}(2) \to (H_\theta/I_\theta H_\theta)(2) \to \mathcal{Q}(2) \to 0$.

(Since the upper horizontal arrow of the above diagram is surjective, this characterizes the map Υ.)

Proof. Let $\mathfrak{X} = \mathrm{Gal}(M/K)$ where $K = \mathbb{Q}(\zeta_{Np^\infty})$ and M is the largest pro-p abelian extension of $\mathbb{Q}(\zeta_{Np^\infty})$ which is unramified at any primes which do not divide Np. Since $\mathfrak{X}$ is the Pontryagin dual of $H^1(\mathbb{Z}[1/Np, \zeta_{Np^\infty}], \mathbb{Q}_p/\mathbb{Z}_p)$, Poitou-Tate duality (9.4.1) shows that $\mathfrak{X} \cong \varprojlim_r H^2_{(c)}(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(1))$. Hence

$$\mathfrak{X}_\chi \cong \mathfrak{X}(1)_\theta \cong \varprojlim_r H^2_{(c)}(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta \cong H^2_{(c)}(\mathbb{Z}[1/Np], \Lambda^\sharp_\theta(2)).$$

The composite map

$$\mathfrak{X}_\chi \cong H^2_{(c)}(\mathbb{Z}[1/Np], \Lambda^\sharp_\theta(2)) \to H^2(\mathbb{Z}[1/Np], (\Lambda^\sharp_\theta/(\xi))(2)) \cong X_{Np^\infty,\chi}$$

coincides with the canonical projection.

For the proof of 9.4.3, since the map $\mathfrak{X} \to H^2_{(c)}(\mathbb{Z}[1/Np], \mathcal{Q}(2))$ is surjective, it is sufficient to prove that the composition $\mathfrak{X} \to H^2_{(c)}(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \to H^3_{(c)}(\mathbb{Z}[1/Np], \mathcal{P}(2)) \cong \mathcal{P}$ induced by the left vertical arrow of 9.4.3 coincides with the composition $\mathfrak{X} \to X_{Np^\infty} \xrightarrow{\Upsilon} \mathcal{P}$. By Poitou-Tate duality, it is sufficient to prove that the composition

$$\mathcal{P}^\vee \cong H^0(\mathbb{Z}[1/Np], \mathcal{P}^\vee(-1)) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}^\vee(-1))$$

$$\to H^1(\mathbb{Z}[1/Np, \zeta_{Np^\infty}], \mathcal{Q}^\vee) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathfrak{X}, \mathcal{Q}^\vee) \to \mathfrak{X}^\vee$$

coincides with the minus of the Pontryagin dual of $\mathfrak{X} \to X_{Np^\infty} \xrightarrow{\Upsilon} \mathcal{P}$. Here $H^0(\mathbb{Z}[1/Np], \mathcal{P}^\vee(-1)) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}^\vee(-1))$ is the connecting map of the exact sequence

$$0 \to \mathcal{Q}^\vee(-1) \to (H_\theta/I_\theta H_\theta)^\vee(-1) \to \mathcal{P}^\vee(-1) \to 0$$

and the map $\mathrm{Hom}_{\mathrm{cont}}(\mathfrak{X}, \mathcal{Q}^\vee) \to \mathfrak{X}^\vee$ is defined by the evaluation at the canonical basis of $\mathcal{Q}$. The last coincidence is proved easily. □

Proposition 9.4.4. *Consider the diagram*

$$\begin{array}{ccccccccc}
 & & & & & & H^2_{(c)}(\mathbb{Z}[1/Np], \mathcal{Q}(2)) & & \\
 & & & & & & \downarrow & & \\
 & & H^2(\mathbb{Z}[1/Np], \mathcal{P}(2)) & \to & H^2(\mathbb{Z}[1/Np], (H_\theta/I_\theta H_\theta)(2)) & \to & H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \oplus_{\ell|Np} H^2(\mathbb{Q}_\ell, \mathcal{P}(2)) & \to & \oplus_{\ell|Np} H^2(\mathbb{Q}_\ell, (H_\theta/I_\theta H_\theta)(2)) & \to & \oplus_{\ell|Np} H^2(\mathbb{Q}_\ell, \mathcal{Q}(2)) & & \\
 & & \downarrow & & & & & & \\
 & & \mathcal{P} & & & & & &
\end{array}$$

in which rows are exact and compositions of vertical arrows in the column are zero. The map $\Upsilon : X_{Np^\infty,\chi} \to \mathcal{P}$ coincides with the map given by snake lemma.

Proof. We can show that the above map of the snake lemma coincidences with the minus of the connecting map $H^2_{(c)}(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \to H^3_{(c)}(\mathbb{Z}[1/Np], \mathcal{P}(2))$, by using the "complex version" of 9.3.5. By this, 9.4.4 is reduced to 9.4.3. Here the "complex version" of 9.3.5 is as follows: We replace pro-p G-modules in 9.3.5 by complexes of pro-p G-modules. This complex version is proved in the same way as 9.3.5. □

9.5 Derivative of p-adic Dirichlet L and Galois cohomology

Recall (7.2.2) that $\xi' = td\xi/dt$ where t is the generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}(\zeta_{Np}))$ such that $(1-p^{-1})\log(\kappa(t)) = 1$.

Proposition 9.5.1. *We have a commutative diagram*

$$\begin{array}{ccc}
H^1(\mathbb{Z}[1/Np], H_\theta(2)) & \to & \mathcal{S}_\theta = \varprojlim_r H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathbb{Z}_p(2))_\theta \\
\downarrow & & \downarrow \xi'\Upsilon \\
H^1(\mathbb{Q}_p, \mathcal{P}(2)) & \to & \mathcal{P}.
\end{array}$$

Here the upper horizontal arrow is the evaluation at ∞, the left vertical arrow is induced by $H_\theta \to H_{\mathrm{quo},\theta} \to H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta} \cong \mathcal{P}$, and the lower horizontal arrow is given by the cup product

$$\cup(1-p^{-1})\log(\kappa) : H^1(\mathbb{Q}_p, \mathcal{P}(2)) \to H^2(\mathbb{Q}_p, \mathcal{P}(2)) = H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathcal{P} = \mathcal{P}.$$

Before we give the proof of 9.5.1, we prove

Lemma 9.5.2. *Let ℓ be a prime divisor of N. Then for any $m \in \mathbb{Z}$, the canonical map $H^1(\mathbb{F}_\ell, H^0(\mathbb{Q}_{\ell,ur}, H_\theta(m))) \to H^1(\mathbb{Q}_\ell, H_\theta(m))$ is bijective.*

Proof. This map is injective, and the cokernel is isomorphic to $H^0(\mathbb{F}_\ell, H^1(\mathbb{Q}_{\ell,ur}, H_\theta(m)))$.

The action of the absolute Galois group of $\mathbb{Q}_{\ell,ur}(\zeta_N)$ on H_θ is trivial. Let G be the Galois group of $\mathbb{Q}_{\ell,ur}(\zeta_N)$ over $\mathbb{Q}_{\ell,ur}$. Since p does not divide $\varphi(N)$, the map $H^1(\mathbb{Q}_{\ell,ur}, H_\theta(m)) \to H^1(\mathbb{Q}_{\ell,ur}(\zeta_N), H_\theta(m))^G$ is bijective. Hence it is sufficient to prove that $H^0(\mathbb{F}_\ell, H^1(\mathbb{Q}_{\ell,ur}(\zeta_N), H_\theta(m))^G) = 0$. We have $H^1(\mathbb{Q}_{\ell,ur}(\zeta_N), H_\theta(m))^G = H_\theta(m-1)^G$. In the finite level, the action of the geometric Frobenius Fr_ℓ^{-1} at ℓ on H_θ is of weight 1 (since the restriction of θ to $(\mathbb{Z}/N\mathbb{Z})^\times$ is primitive, H_θ is potentially good at ℓ). Hence $H^0(\mathbb{F}_\ell, H_\theta(m-1)^G) = 0$. $\square$

9.5.3. We prove Proposition 9.5.1.

Consider the commutative diagram

$$\begin{array}{ccc}
H^1(\mathbb{Z}[1/Np], H_\theta(2)) & \to & H^2(\mathbb{Z}[1/Np], H_\theta(2)) \\
\downarrow & & \downarrow \\
\oplus_{\ell|Np} H^1(\mathbb{Q}_\ell, H_\theta(2)) & \to & \oplus_{\ell|Np} H^2(\mathbb{Q}_\ell, H_\theta(2)) \\
\downarrow & & \| \\
\oplus_{\ell|Np} H^1(\mathbb{Q}_\ell, \mathcal{P}(2)) & \to & \oplus_{\ell|Np} H^2(\mathbb{Q}_\ell, \mathcal{P}(2)) \\
 & & \downarrow \\
 & & \mathcal{P}
\end{array}$$

in which all the horizontal arrows are the cup product $\cup(1-p^{-1})\log(\kappa)$, and the vertical arrows $H^i(\mathbb{Q}_\ell, H_\theta(2)) \to H^i(\mathbb{Q}_\ell, \mathcal{P}(2))$ for $i = 1, 2$ are induced by $H_\theta(2) \to (H_\theta/I_\theta H_\theta)(2) \to \mathcal{P}(2)$ where the last arrow is obtained from the unique splitting of the exact sequence $0 \to \mathcal{P} \to H_\theta/I_\theta H_\theta \to \mathcal{Q} \to 0$ over $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

Let $x \in H^1(\mathbb{Z}[1/Np], H_\theta(2))$. Let $y \in H^2(\mathbb{Z}[1/Np], H_\theta(2))$ and $z \in \mathcal{P}$ be the images of x in this diagram.

For any prime number $\ell \neq p$, $(1-p^{-1})\log(\kappa) \in H^1(\mathbb{Q}_\ell, \mathbb{Z}_p)$ belongs to the unramified part $H^1(\mathbb{F}_\ell, \mathbb{Z}_p)$. Hence for $\ell|N$, by the case $m = 2$ of Lemma 9.5.2, the cup product $\cup(1-p^{-1})\log(\kappa) : H^1(\mathbb{Q}_\ell, H(2)) \to H^2(\mathbb{Q}_\ell, H(2))$ is zero (use the fact $H^2(\mathbb{F}_\ell, -) = 0$). Hence z coincides with the image of x under the composition $H^1(\mathbb{Z}[1/Np], H_\theta(2)) \to H^1(\mathbb{Q}_p, \mathcal{P}) \to \mathcal{P}$ of Proposition 9.5.1. It remains to prove that $z = \xi' \cdot \Upsilon(u)$ where $u \in \mathcal{S}_\theta$ is the image of x under the evaluation at ∞. Let v be the image of y under $H^2(\mathbb{Z}[1/Np], H_\theta(2)) \to H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2)) \cong \mathcal{S}_\theta$. By Proposition 9.3.3, we have $v = \xi' u$. On the other hand, Proposition 9.4.4 shows that $\Upsilon(v)$ is the image z of y under the composition of vertical arrows $H^2(\mathbb{Z}[1/Np], H_\theta(2)) \to \mathcal{P}$. Hence $z = \Upsilon(v) = \xi'\Upsilon(u)$.

9.6 A cuspidal extension class in Galois cohomology

9.6.1. Let $\mathcal{E} = \tilde{H}_{DM,\theta,E}/\mathrm{Ker}\,(H_{\theta,E} \to \mathcal{Q})$. We try to understand the extension $0 \to \mathcal{Q} \to \mathcal{E} \to \mathcal{R} \to 0$.

9.6.2. Because we have canonical bases of $\mathcal{Q}$ and $\mathcal{R}$ as $\Lambda_\theta/(\xi)$-modules, the class of this extension is understood as an element of $H^1(\mathbb{Z}[1/Np], (\Lambda_\theta^\sharp/\xi)(1))$. From the exact sequence $0 \to \Lambda_\theta^\sharp(1) \xrightarrow{\xi} \Lambda_\theta^\sharp(1) \to (\Lambda_\theta^\sharp/(\xi))(1) \to 0$, we have an exact sequence

$$\varprojlim_r (\mathbb{Z}[1/Np, \zeta_{Np^r}]^\times \otimes \mathbb{Z}_p)_\theta \xrightarrow{\xi} \varprojlim_r (\mathbb{Z}[1/Np, \zeta_{Np^r}]^\times \otimes \mathbb{Z}_p)_\theta \to H^1(\mathbb{Z}[1/Np], (\Lambda_\theta^\sharp/\xi)(1)).$$

The goal of this section 9.6 is to prove

Theorem 9.6.3. *The class of the extension $0 \to \mathcal{Q} \to \mathcal{E} \to \mathcal{R} \to 0$ coincides with the image of class of the family $(1-\zeta_{Np^r})_r$ under the canonical homomorphism $\varprojlim_r (\mathbb{Z}[1/Np, \zeta_{Np^r}]^\times \otimes \mathbb{Z}_p)_\theta \to H^1(\mathbb{Z}[1/Np], (\Lambda_\theta^\sharp/\xi)(1))$.*

Lemma 9.6.4. *Let X be a proper smooth curve over an algebraically closed field k, and let Y be a dense open set of X. Let $n \in \mathbb{Z}$ and assume n is invertible in k. Let D be a divisor on X with support in $X - Y$, and assume nD is a principal divisor (g). Then the map $H^1(X, \mathbb{Z}/n\mathbb{Z}(1)) \to H^1(Y, \mathbb{Z}/n\mathbb{Z}(1))$ sends the class of D in $H^1(X, \mathbb{Z}/n\mathbb{Z}(1)) \cong J_X[n]$ to the Kummer class of g.*

Proof. Let $L = \mathcal{O}(D)$ and let T be the $\mathbb{Z}/n\mathbb{Z}(1)$-torsor $\{u : L \xrightarrow{\cong} \mathcal{O}_X\ ;\ u^{\otimes n} = g\}$ on the étale site of X. Then the class of T in $H^1(X, \mathbb{Z}/n\mathbb{Z}(1))$ goes to the class of L in $H^1(X, G_m)[n]$. On Y, T is identified with the $\mathbb{Z}/n\mathbb{Z}(1)$-torsor $\{n\text{-th root of } g\}$ whose class is the Kummer class of g. This proves the lemma. $\square$

Lemma 9.6.5. *Let $X = X_1(Np^r)$, $Y = Y_1(Np^r)$ $(r \geq 1)$. By the map $H^1_{\acute{e}t}(Y)_{DM}/H^1_{\acute{e}t}(X) \to H^1_{\acute{e}t}(X) \otimes \mathbb{Q}/\mathbb{Z}$ induced by $H^1_{\acute{e}t}(Y)_{DM} \xrightarrow{\subseteq} H^1_{\acute{e}t}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, the class of a cuspidal divisor D of degree 0 in $H^1_{\acute{e}t}(Y)_{DM}/H^1_{\acute{e}t}(X)$ goes to the minus of the class of D in $J_{X,tor} = H^1_{\acute{e}t}(X) \otimes \mathbb{Q}/\mathbb{Z}$.*

Here $H^1_{\acute{e}t}(Y)_{DM}$ be the Drinfeld -Manin modification (6.2.1) of $H^1_{\acute{e}t}(Y)$.

Proof. Take $E \in H^1_{\text{ét}}(Y)$ whose class in $H^1_{\text{ét}}(Y)/H^1_{\text{ét}}(X)$ coincides with that of D. Since the class of D is torsion, we have $nD = (g)$ for some $n \neq 0$ and g. Hence the class of D in $H^1_{\text{ét}}(Y)/H^1_{\text{ét}}(X)$ coincides with that of the Kummer class $[g]$ of g. Hence $nD = [g] + x$ in $H^1_{\text{ét}}(Y)$ where $x \in H^1_{\text{ét}}(X)$. In $H^1_{\text{ét}}(Y)/nH^1_{\text{ét}}(Y)$, we have $x = -[g]$. Since $H^1_{\text{ét}}(X)/nH^1_{\text{ét}}(X) \to H^1_{\text{ét}}(Y)/nH^1_{\text{ét}}(Y)$ is injective and the class of D goes to $[g]$ by Lemma 9.6.4, we see that $x = -D$ in $H^1_{\text{ét}}(X)/nH^1_{\text{ét}}(X)$. On the other hand, in $H^1_{\text{ét}}(Y)_{DM}$, since $[g]$ vanishes there, $nE = [g] + x$ implies that $nE = x$. Thus in $H^1_{\text{ét}}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $nE = x$. This proves the lemma. $\square$

Note that

Lemma 9.6.6. *Let J be the Jacobian variety of $X_1(Np^r)$ and let GJ be the generalized Jacobian variety of $X_1(Np^r)$ with respect to the ∞-cusp of $X_1(Np^r)$. Consider the exact sequence* *(a)* $0 \to G_m \to T \to GJ \to J \to 0$

where T is the Weil restriction of G_m from the residue field of the ∞-cusp of $X_1(Np^r)$ to $\mathbb{Q}$, and let

(b) $0 \to T[\xi]_\theta \to GJ[\xi]_\theta \to J[\xi]_\theta \to 0$ *be the exact sequence obtained by taking the θ-component of the ξ-torsion part (the kernel of the action of ξ) of the exact sequence (a). Then via the perfect duality*

$$H^1_{\acute{e}t}(Y_1(Np^r))_\theta/\xi H^1_{\acute{e}t}(Y_1(Np^r))_\theta \times GJ[\xi]_\theta \to \mathbb{Q}_p/\mathbb{Z}_p \ ; \ (x,y) \mapsto (w_{Np^r}(x), y)$$

where $(-,-)$ is the usual pairing, the Eisenstein component of the exact sequence (b) is dual to the exact sequence

$$0 \to H^1_{\acute{e}t}(X_1(Np^r))_{\theta,E}/\xi H^1_{\acute{e}t}(X_1(Np^r))_{\theta,E} \to$$

$$H^1_{\acute{e}t}(Y_1(Np^r))_{\theta,E}/\xi H^1_{\acute{e}t}(Y_1(Np^r))_{\theta,E} \to (\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]_\theta/(\xi))(-1) \to 0.$$

(The last exact sequence is a quotient of the exact sequence $0 \to H_{\theta,E} \to \tilde{H}_{\theta,E} \to \Lambda_\theta \to 0$ in 6.2.5.)

9.6.7. Let $A = \mathbb{Z}_p[G]$ for a finite abelian group G. Let $a \in A$, $R = A/(a)$. Assume R is finite (that is, a is a non-zero-divisor of A).

Then we have a functorial isomorphism $\mathrm{Hom}_R(M,R) \cong Hom(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for an R-module M.

It is defined as follows. Note that a is invertible in $A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p[G]$. We define a homomorphism $f : R \to \mathbb{Q}/\mathbb{Z}$ as follows. Let $g : \mathbb{Q}_p[G] \to \mathbb{Q}_p$ be the map $\sum_{\sigma \in G} c_\sigma \sigma \mapsto c_1$ ($c_\sigma \in \mathbb{Q}_p$). Let $\tilde{f} : A \to \mathbb{Q}_p$ be the homomorphism defined by $\tilde{f}(b) = g(ba^{-1})$. Then for $b \in A$, we have $\tilde{f}(ab) = g(b) \in \mathbb{Z}_p$. Hence $\tilde{f}$ induces a homomorphism $f : R = A/(a) \to \mathbb{Q}_p/\mathbb{Z}_p$.

This homomorphism $f : R \to \mathbb{Q}_p/\mathbb{Z}_p$ induces $\mathrm{Hom}_R(M,R) \to \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ for any R-module M.

We prove that this homomorphism $\mathrm{Hom}_R(M,R) \to \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is an isomorphism. To prove it, we are reduced to the case $M = R$. That is, it is sufficient to prove that $R \to \mathrm{Hom}(R, \mathbb{Q}_p/\mathbb{Z}_p)$; $x \mapsto (y \mapsto f(xy))$ is bijective. This map is $A/(a) \to \mathrm{Hom}(A/(a), \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)[a] = (A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)[a]$ ($[a]$ denotes the kernel of a). The inverse of this composition is given by the connecting map of the snake lemma for the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \to & A & \to & A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \to & A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \to & 0 \\
 & & \downarrow a & & \cong\downarrow a & & \downarrow a & & \\
0 & \to & A & \to & A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \to & A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \to & 0.
\end{array}$$

By Lemma 9.6.5, we have

Lemma 9.6.8. *The canonical map $(-, \xi\{0,\infty\}_{DM,\theta,E})_\Lambda : H_\theta \to \Lambda_\theta/(\xi)$ (6.3.8) is described as the inverse limit of the following maps $H^1_{\acute{e}t}(X_1(Np^r)) \to \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]_\theta/(\xi)$ for $r \geq 1$:*

Consider the divisor $D = (0) - (\infty)$ on $X_1(Np^r)$. Let $A = \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]_\theta$. We denote the image of ξ in A by the same letter ξ. Let $R = A/(\xi)$. Let J be the Jacobian variety of $X_1(Np^r)$. In $J_{tor,\theta,E} = H^1_{ét}(X_1(Np^r))_{\theta,E} \otimes \mathbb{Q}/\mathbb{Z}$, the class class$(D)$ of D is killed by ξ. The θ-component of the above 9.6.7 *shows that $J[\xi]_{\theta,E}$ (which is the Pontryagin dual of $H^1_{ét}(X_1(Np^r))_{\theta,E}/\xi H^1_{ét}(X_1(Np^r))_{\theta,E}$ via the duality in Lemma* 9.6.6*) is the R-dual of $H^1_{ét}(X_1(Np^r))_{\theta,E}/\xi H^1_{ét}(X_1(Np^r)_{\theta,E}$. Hence, class$(D) \in J[\xi]_{\theta,E}$ defines $H^1_{ét}(X_1(Np^r))_{\theta,E} \to R$.*

Lemma 9.6.9. *Let C be the group of divisors on $X_1(Np^r) \otimes \bar{\mathbb{Q}}$ supported on cusps. Let $(C \otimes \mathbb{Z}_p)_{\theta,E}$ be the Eisenstein component of the θ-component of $C \otimes \mathbb{Z}_p$ as an $h(Np^r)_{\mathbb{Z}_p}$-module. Then we have*

(1) *As a module over the ring $\mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]_\theta$ of diamond operators, $(C \otimes \mathbb{Z}_p)_{\theta,E}$ is a free module of rank* 1*, and any* 0*-cusp is a generator of this module.*

(2) *For any cusp x of $X_1(Np^r) \otimes \bar{\mathbb{Q}}$ which is not a* 0*-cusp, the image of x in $(C \otimes \mathbb{Z}_p)_{\theta,E}$ is zero.*

Proof. This follows from Ohta [41]. (It can be deduced from 1.3.5). □

9.6.10. We prove Theorem 9.6.3.

By Lemmas 9.6.8 and 9.6.6, it is sufficient to prove the following statement (S).

(S) The image of the class of $(0) - (\infty)$ in $H^0(\mathbb{Q}(\zeta_{Np^r}), J[\xi]_{\theta,E})$ under the connecting map $H^0(\mathbb{Q}(\zeta_{Np^r}), J[\xi]_{\theta,E}) \to H^1(\mathbb{Q}(\zeta_{Np^r}), T[\xi]_{\theta,E})$ of the exact sequence (b) of Lemma 9.6.6 coincides with the Kummer class of of $1 - \zeta_{Np^r}$.

Take an integer c such that $(c, 6Np) = 1$ and such that the image of $c^2 - [c]$ in Λ_θ is invertible. By the q-expansion of ${}_cg_{0,1/Np^r}$ (2.1.1) and by Lemmas 6.2.13 and 9.6.9, there is $h \in h(Np^r)_{\mathbb{Z}}$ whose image in $h(Np^r)_{\mathbb{Z}_p,\theta,E}$ is invertible and which satisfies the following conditions (i)–(iii).

(i) h kills ∞-cusps.

(ii) $\mathrm{div}({}_cg^h_{0,1/Np^r}) = h \cdot (c^2 - \langle c \rangle) \cdot \xi \cdot (0)$.

(iii) ${}_cg^h_{0,1/Np^r}(\infty) = (1 - \zeta_{Np^r})^{h(c^2 - \sigma_c)}$.

We compute the image of the class of $h \cdot (c^2 - \langle c \rangle) \cdot ((0) - (\infty))$ in $H^0(\mathbb{Q}(\zeta_{Np^r}), J[\xi]_\theta)$ under the connecting map $H^0(\mathbb{Q}(\zeta_{Np^r}), J[\xi]_\theta) \to H^1(\mathbb{Q}(\zeta_{Np^r}), T[\xi]_\theta)$. Let s be an idele on $X_1(Np^r) \otimes \bar{\mathbb{Q}}$ satisfying the following conditions (iv) and (v).

(iv) Outside ∞-cusps, the divisor of s coincides with the divisor $h \cdot (c^2 - \langle c \rangle) \cdot ((0) - (\infty))$.

(v) The components of s^ξ (= the result of the action of ξ on s) at ∞-cusps are the images of ${}_cg^h_{0,1/Np^r}$ in the local fields of $X_1(Np^r) \otimes \bar{\mathbb{Q}}$ at ∞-cusps.

Then by (i) and (ii), the class of s^ξ in GJ coincides with the class of the principal idele ${}_cg^h_{0,1/Np^r}$ and hence vanishes. Thus $s \in GJ[\xi]$. Furthermore, the image of s in $J[\xi]$ coincides with $h \cdot (c^2 - \langle c \rangle) \cdot ((0) - (\infty))$. Hence the connecting map sends the class of $h \cdot (c^2 - \langle c \rangle) \cdot ((0) - (\infty))$ to the class of the Galois 1-cocyle $\sigma \mapsto \sigma(s)/s$ ($\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{Np^r}))$, which is the Kummer class of ${}_cg^h_{0,1/Np^r}(\infty) = (1 - \zeta_{Np^r})^{h(c^2 - \sigma_c)}$. This proves the statement (S).

10 Proofs of our results on Sharifi conjectures

In section, we complete the proofs of our results stated in section 7.2.

10.1 Study of $\xi' \cdot \Upsilon\varpi$

The goal of this subsection is to prove Theorem 7.2.3 (1).

10.1.1. We regard $H_\theta^-/I_\theta H_\theta^- = H_{\mathrm{quo},\theta}/I_\theta H_{\mathrm{quo},\theta} = D(H_{\mathrm{quo},\theta}(1))/I_\theta D(H_{\mathrm{quo},\theta}(1))$ via the canonical isomorphisms (6.3.5, 8.1.1).

Let $\gamma \in H_\theta^-$ and consider $\mathcal{M}_{s=0}(\gamma) \in D(H_{\mathrm{quo},\theta}(1))$. By the above fact, we can regard $\mathcal{M}_{s=0}(\gamma) \bmod I_\theta \in H_{\mathrm{quo},\theta}/IH_{\mathrm{quo},\theta} = H_\theta^-/I_\theta H_\theta^-$.

Proposition 10.1.2. *The composition*

$$H^1(\mathbb{Z}[1/Np], H_\theta(2)) \overset{\mathrm{Col}^\flat}{\to} D(H_{\mathrm{quo},\theta}(1)) \to H_\theta^-/I_\theta H_\theta^-$$

(the first arrow is $\mathrm{Col}^\flat$ *and the second arrow is the canonical projection) coincides with the composition*

$$H^1(\mathbb{Z}[1/Np], H_\theta(2)) \overset{\infty}{\to} \mathcal{S}_\theta \overset{\xi'\Upsilon}{\to} H_\theta^-/I_\theta H_\theta^-$$

(the first arrow is the evaluation at ∞ *and the second arrow is* $\xi' \cdot \Upsilon$*).*

Proof. This follows from Proposition 9.5.1 and Proposition 4.2.4. □

Theorem 10.1.3. *Let* $\gamma \in H_\theta^-$*. Then*

$$\mathcal{M}_{s=0}(\gamma) = \xi' \cdot \Upsilon\varpi(\gamma) \bmod I_\theta \quad \textit{in} \;\; (H_\theta^-/I_\theta H_\theta^-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Proof. The first composite map of 10.1.2 sends $z^\sharp_{1,Np^\infty}(\gamma) \in H^1(\mathbb{Z}[1/p], H_\theta(2)) \otimes_\Lambda \Lambda\,\mu^{-1}$ ($\gamma \in H_\theta^-$, $\mu = p((1+Np)^2 - \langle 1+Np\rangle) \in \Lambda$ as in Theorem 3.3.9) to $\mathcal{M}_{s=0}(\gamma) \bmod I_\theta$ by 4.3.8. On the other hand, the second composite map of 10.1.2 sends $\mu \cdot z^\sharp_{Np^\infty}(\gamma)$ to $\mu\xi' \cdot \Upsilon\varpi(\gamma)$. Note that μ is invertible in $\mathfrak{h}_\theta/I_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by 5.2.8. Hence by Theorem 10.1.2, we obtain Theorem 10.1.3. □

10.1.4. Recall that for $\gamma \in H_\theta^-$, we have

$$\mathcal{M}_{s=0}(\gamma) = \xi' \cdot \gamma \bmod I_\theta \quad \text{in} \;\; H_\theta^-/I_\theta H_\theta^-$$

by Theorem 8.1.2 (2). By comparing this with Theorem 10.1.3, we obtain Theorem 7.2.3 (1).

10.2 Proofs related to the conditions $\mathrm{C}(\xi)$ and $\mathrm{C}(\mathfrak{h})$

10.2.1. We prove Theorems 7.2.6 and 7.2.8 assuming $\mathrm{C}(\xi)$.

Assume $\mathrm{C}(\xi)$. Then the ring $\Lambda_\theta/(\xi)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a finite product of fields, and ξ' is an invertible element of it. Hence the action of ξ' on any module over this ring is invertible.

Hence by Theorem 7.2.3 (1), we have (1) of Conjecture 7.1.2. The rest of Theorems 7.2.6 and 7.2.8 are deduced from it as is explained in section 7.1.

The following lemma is used in 10.2.3 below.

Lemma 10.2.2. *If $x \in \Lambda_\theta/(\xi)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and if $\xi' x = 0$, then x is nilpotent.*

Proof. The ring $\Lambda_\theta/(\xi)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ is a finite product of rings of the form R/m^n where R is a discrete valuation ring, m is the maximal ideal of R, and $n \geq 1$. The image of ξ' in R/m^n generates the ideal m^{n-1}/m^n. Hence if $\xi' x = 0$, the image of x in R/m^n is contained in m/m^n. Hence x is nilpotent.

□

10.2.3. We prove Theorem 7.2.8 assuming $\mathrm{C}(\mathfrak{h})$. Assume $\mathrm{C}(\mathfrak{h})$.

Let $P = (H_\theta^-/I_\theta H_\theta^-)/(\mathrm{tor})$. Then as in 7.2.10, $P\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \cong \Lambda_\theta/(\xi)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ as a module over $\Lambda_\theta/(\xi)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

We first prove that the composition $f : P \xrightarrow{\varpi} \mathcal{S}_\theta \xrightarrow{\Upsilon} P$ ($= \Upsilon\circ\varpi$ mod torsion) is an isomorphism. It is sufficient to prove that $1-f$ is nilpotent. By Theorem 7.2.3, $\xi'\cdot(1-f) = 0$. Hence $1-f$ is nilpotent on $P\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ by Lemma 10.2.2. This shows that $1-f$ is nilpotent on P.

From this, by the arguments in 7.1.3, we see that $P \to \mathcal{S}_\theta$ and $\mathcal{S}_\theta \to P$ are isomorphisms. This proves Theorem 7.2.8 assuming $\mathrm{C}(\mathfrak{h})$.

10.3 Proofs related to $(1-T^*(p))\{0,\infty\}$ and $\{p, 1-\zeta_{Np^r}\}$

We prove Theorem 7.2.3 (2).

Lemma 10.3.1. *ϖ sends $(1-T^*(p)^{-1})\{0,\infty\}$ to $(\{p,1-\zeta_{Np^r}\})_r$.*

Proof. The element $(1-T^*(p)^{-1})\{0,\infty\}$ of $H = \varprojlim_r H^1_{\text{ét}}(X_1(Np^r))^{\mathrm{ord}}$ coincides with the inverse system $(T^*(p)^{-r}\sum_{a\in(\mathbb{Z}/p^r\mathbb{Z})^\times}\{a/p^r,\infty\})_{r\geq 1} = (T^*(p)^{-r}\sum_{a\in(\mathbb{Z}/p^r\mathbb{Z})^\times}[Na:1]_r)_{r\geq 1}$, and hence it is sent by ϖ to the inverse system $\{\sum_{a\in(\mathbb{Z}/p^r\mathbb{Z})^\times}\{1-\zeta_{p^r}^a, 1-\zeta_{Np^r}\})_r = (\{p,1-\zeta_{Np^r}\})_r$ of $\mathcal{S} = \varprojlim_r H^2(\mathbb{Z}[1/Np,\zeta_{Np^r}],\mathbb{Z}_p(2))^+$.

□

10.3.2. In the following 10.3.3–10.3.11, we prove that Υ sends $(\{p,1-\zeta_{Np^r}\})_r$ to $(1-T^*(p)^{-1})\{0,\infty\}$.

Let $\mathcal{E} = \tilde{H}_{DM,\theta,E}/\mathrm{Ker}\,(H_{\theta,E}\to\mathcal{Q})$ as in 9.6.1, so we have an exact sequence $0\to\mathcal{Q}\to\mathcal{E}\to\mathcal{R}\to 0$. Let $\mathcal{F} = \tilde{H}_{DM,\theta,E}/I_{\theta,E}H_{\theta,E}$. We have an exact sequence $0\to H_\theta/I_\theta H_\theta \to \mathcal{F}\to\mathcal{R}\to 0$.

By using the isomorphism $H^1(\mathbb{Z}[1/Np],\mathcal{R}(2)) \cong \mathcal{R}(1)\otimes_{\mathbb{Z}_p} H^1(\mathbb{Z}[1/Np],\mathbb{Z}_p(1))$, we have an element of $H^1(\mathbb{Z}[1/Np],\mathcal{R}(2))$, which we denote by p, defined as the product of the class of $\{0,\infty\}$ in $\mathcal{R}(1)$ and the Kummer class of p in $H^1(\mathbb{Z}[1/Np],\mathbb{Z}_p(1))$.

Lemma 10.3.3. *The connecting map $H^1(\mathbb{Z}[1/Np], \mathcal{R}(2)) \to H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2))$ of the exact sequence $0 \to \mathcal{Q}(2) \to \mathcal{E}(2) \to \mathcal{R}(2) \to 0$ sends the element p to $(\{p, 1-\zeta_{Np^r}\})_r$.*

Proof. This is reduced to the fact (Theorem 9.6.3) that the connecting map $H^0(\mathbb{Z}[1/Np], \mathcal{R}(1)) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}(1))$ of the exact sequence $0 \to \mathcal{Q}(1) \to \mathcal{E}(1) \to \mathcal{R}(1) \to 0$ sends 1 to $(1-\zeta_{Np^r})_r$. □

10.3.4. Let $x \in H^2(\mathbb{Z}[1/Np], (H_\theta/I_\theta H_\theta)(2))$ be the image of $p \in H^1(\mathbb{Z}[1/Np], \mathcal{R}(2))$ under the connecting map of the exact sequence $0 \to (H_\theta/I_\theta H_\theta)(2) \to \mathcal{F}(2) \to \mathcal{R}(2) \to 0$. By Lemma 10.3.3, $(\{p, 1-\zeta_{Np^r}\})_r \in H^2(\mathbb{Z}[1/Np], \mathcal{Q}(2))$ is the image of x under the canonical projection.

By Proposition 9.4.4, $\Upsilon((\{p, 1-\zeta_{Np^r}\})_r)$ is described as follows. For a prime divisor ℓ of Np, let x_ℓ be the image of x in $H^2(\mathbb{Q}_\ell, (H_\theta/I_\theta H_\theta)(2))$. Let $y_\ell \in H^2(\mathbb{Q}_\ell, \mathcal{P}_\theta(2))$ be the image of x_ℓ under the unique splitting $(H_\theta/I_\theta H_\theta)(2) \to \mathcal{P}(2)$ of the inclusion map $\mathcal{P}(2) \to (H_\theta/I_\theta H_\theta)(2)$ which is compatible with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ (6.3.4). Let $z_\ell \in H_\theta^-/I_\theta H_\theta^-$ be the image of y_ℓ under the canonical isomorphism $H^2(\mathbb{Q}_\ell, \mathcal{P}(2)) \cong \mathcal{P} = H_\theta^-/I_\theta H_\theta^-$. Then $\Upsilon((\{p, 1-\zeta_{Np^r}\})_r) = \sum_{\ell | Np} z_\ell$.

Lemma 10.3.5. *Let ℓ be a prime divisor of Np. Let $s_\ell : \mathcal{Q} \to H_\theta/I_\theta H_\theta$ be the unique splitting of the projection $H_\theta/I_\theta H_\theta \to \mathcal{Q}$ which is compatible with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Let $V_\ell := \mathcal{F}/s_\ell(\mathcal{Q})$. (Note we have an exact sequence $0 \to \mathcal{P} \to V_\ell \to \mathcal{R} \to 0$.)*

(1) $V_p \cong \tilde{H}_\theta/I_\theta H_\theta$ as a representation of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

(2) For a prime divisor ℓ of N, as a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, V_ℓ is unramified.

Proof. (1) is clear.

We prove (2). Let I_ℓ be the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Then the action on I_ℓ on V_ℓ defines a homomorphism $h : I_\ell \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \mathcal{P})$, which sends $\sigma \in I_\ell$ to the homomorphism $x \mod \mathcal{P} \mapsto \sigma(x) - x$ $(x \in V_\ell)$. Since the target is a pro-p group, h factors through a quotient of I_ℓ which is canonically isomorphic to $\mathbb{Z}_p(1)$. The resulting homomorphism $h : \mathbb{Z}_p(1) \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \mathcal{P})$ is compatible with the actions of $\mathrm{Gal}(\bar{\mathbb{F}}_\ell/\mathbb{F}_\ell)$ which acts trivially on $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{R}, \mathcal{P})$. Hence h factors through the quotient $\mathbb{Z}_p(1)/(1-\ell)\mathbb{Z}_p(1)$ of $\mathbb{Z}_p(1)$. But $1-\ell$ is a p-adic unit by our assumption $p \not| \varphi(N)$. Hence $h = 0$. □

10.3.6. Note that y_ℓ coincides with the image of $p \in H^1(\mathbb{Q}_\ell, \mathcal{R}(2))$ under the connecting map of the exact sequence $0 \to \mathcal{P}(2) \to V_\ell(2) \to \mathcal{R}(2) \to 0$.

Lemma 10.3.7. *Let the notation be as in 10.3.4. Then $y_\ell = 0$ for $\ell \neq p$.*

Proof. Since $p \in H^1(\mathbb{Q}_\ell, \mathcal{R}(2))$ belongs to the image $H^1(\mathbb{F}_\ell, \mathcal{R}(2)) \to H^1(\mathbb{Q}_\ell, \mathcal{R}(2))$, and since V_ℓ is unramified, y_ℓ belongs to the image of the composition $H^1(\mathbb{F}_\ell, \mathcal{R}(2)) \to H^2(\mathbb{F}_\ell, \mathcal{P}(2)) \to H^2(\mathbb{Q}_\ell, \mathcal{P}(2))$, where the first arrow is the connecting map of the exact sequence $0 \to \mathcal{P}(2) \to V_\ell(2) \to \mathcal{R}(2) \to 0$. But $H^2(\mathbb{F}_\ell, -) = 0$. □

10.3.8. Let $x' \in H^1(\mathbb{Z}[1/Np], (H_\theta/I_\theta H_\theta)(1))$ be the image of the class of $\{0, \infty\}$ in $H^0(\mathbb{Z}[1/Np], \mathcal{R}(1))$ under the connecting map of the exact sequence $0 \to (H_\theta/I_\theta H_\theta)(1) \to \mathcal{F}(1) \to \mathcal{R}(1) \to 0$. Then $x = \{x', p\}$ where $\{-,-\}$ denotes the cup product. Let x'_p be the image of x' in $H^1(\mathbb{Q}_p, (H_\theta/I_\theta H_\theta)(1))$ and let $y'_p \in H^1(\mathbb{Z}[1/Np].\mathcal{P}(1))$ be the image of x'_p under the unique splitting $(H_\theta/I_\theta H_\theta)(1) \to \mathcal{P}(1)$ of the inclusion map $\mathcal{P}(1) \to (H_\theta/I_\theta H_\theta)(1)$

which is compatible with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then we have $x_p = \{x'_p, p\}$ and hence $y_p = \{y'_p, p\}$.

Furthermore, y'_p coincides with the image of the class of $\{0, \infty\}$ in $H^0(\mathbb{Q}_p, \mathcal{R}(1))$ under the connecting map of the exact sequence $0 \to \mathcal{P}(1) \to V_p(1) \to \mathcal{R}(1) \to 0$.

Lemma 10.3.9. *Let* $\nu \in H^1(\mathbb{Q}_p, \mathbb{Z}_p) = \mathrm{Hom}_{cont}(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p)$ *be the unique unramified element such that* $\nu(Fr_p) = 1$. *Then* y'_p *is the product* $\nu \cdot (1 - T^*(p))\{0, \infty\}$ *of* ν *and* $(1 - T^*(p))\{0, \infty\} \in H^0(\mathbb{Q}_p, \mathcal{P}(1))$,

Proof. This follows from the fact that V_p is a quotient of $H_{\mathrm{quo},\theta}(1)$ and hence Fr_p acts on V_p as $T^*(p)$ by 1.8.1. □

Lemma 10.3.10. z_p *coincides with the class of* $(1 - T^*(p))\{0, \infty\}$.

Proof. By 10.3.9, $y_p = \{y'_p, p\} \in H^2(\mathbb{Q}_p, \mathcal{P}(2))$ is the product $\{\nu, p\} \cdot (1 - T^*(p))\{0, \infty\}$ of $\{\nu, p\} \in H^2(\mathbb{Q}_p, \mathbb{Z}_p(1))$ and $(1 - T^*(p))\{0, \infty\} \in H^0(\mathbb{Q}_p, \mathcal{P}(1))$. The canonical isomorphism $H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$ sends $\{\nu, p\}$ to 1. Hence we have the result. □

10.3.11. Now we prove that Υ sends $(\{p, 1 - \zeta_{Np^r}\})_r$ to the class of $(1 - T^*(p))\{0, \infty\}$. By 10.3.4, we have $\Upsilon((\{p, 1 - \zeta_{Np^r}\})_r) = \sum_{\ell | Np} z_\ell$. By Lemma 10.3.7, $z_\ell = 0$ if $\ell \neq p$. By Lemma 10.3.10, z_p coincides with the class of $(1 - T^*(p))\{0, \infty\}$.

10.3.12. We prove Theorem 7.2.6 and Theorem 7.2.8 assuming $\mathrm{C}(T^*(p))$.

Recall that the composition $H^-_\theta/I_\theta H^-_\theta \to \mathcal{S}_\theta \to H^-_\theta/I_\theta H^-_\theta$ sends the class β of $(1 - T^*(p))\{0, \infty\}$ to β. Hence under the condition $\mathrm{C}(T^*(p))$, the composition $H^-_\theta/I_\theta H^-_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathcal{S}_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^-_\theta/I_\theta H^-_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the identity map. This proves Conjecture 7.1.2 (2). As is explained in section 7.1, this proves Theorems 7.2.6 and 7.2.8 assuming $\mathrm{C}(T^*(p))$.

11 Some relation to Iwasawa theory of modular forms

We expect that interesting relations exist between conjectures of Sharifi and the Eisenstein component of the Iwasawa theory of modular forms. Such direction is studied in Sharifi [49] section 6. We continue his study here. See also a recent paper [52] of Sharifi for results in this direction.

In this section, we assume $N = 1$.

11.1 Results related Iwasawa theory of modular forms

11.1.1. To the knowledge of the authors, in all known examples, the following (a) and (b) are satisfied:

(a) As an $\mathfrak{h}_\theta/I_\theta$-module, $H^-_\theta/I_\theta H^-_\theta$ is generated by $(1 - T^*(p))\{0, \infty\}$.

(b) $\mathrm{rank}_{\mathcal{O}_\theta}(\Lambda_\theta/(\xi)) \leq 1$. In this subsection, we relate these conditions to some Iwasawa theoretic conditions.

The following Theorem 11.1.2 was proved by Sharifi [49] under a slightly stronger assumption.

Theorem 11.1.2. *(Recall that we assume $N = 1$.) Let the condition (a) be as above. We also consider the following conditions:*

(a)' $(\{p, 1 - \zeta_{p^r}\})_r$ generates $X_{p^\infty,\chi}$ as a Λ_χ-module.
(a)" $H^2(\mathbb{Z}[1/p], \tilde{H}_{DM}(2)_{\theta,E}) = 0$.

(1) We have implications $(a) \Rightarrow (a)'' \Rightarrow (a)'$.

(2) Assume either the Eisenstein component $\mathfrak{h}_{\theta,E}$ of $\mathfrak{h}_\theta$ or the Eisenstein component $\mathfrak{H}_{\theta,E}$ of $\mathfrak{H}_\theta$ is Gorenstein. Then all the conditions $(a), (a)', (a)''$ are equivalent.

The proof will be given in section 11.3.

11.1.3. Consider the Selmer groups of Greenberg type

$$\mathrm{Sel} := \mathrm{Ker}\,(H^1(\mathbb{Z}[1/p, \zeta_{p^\infty}], H^\vee) \longrightarrow H^1(\mathbb{Q}_p(\zeta_{p^\infty}), H^\vee_{\mathrm{sub}})).$$

Here $(\ )^\vee = \mathrm{Hom}_{\,\mathrm{cont}}(\ \ , \mathbb{Q}_p/\mathbb{Z}_p)$, $\mathbb{Z}[1/p, \zeta_{p^\infty}] = \cup_n \mathbb{Z}[1/p, \zeta_{p^n}]$ and $\mathbb{Q}_p(\zeta_{p^\infty}) = \cup_n \mathbb{Q}_p(\zeta_{p^n})$. Consider the dual Selmer group

$$\mathfrak{X} := \mathrm{Hom}\,(\mathrm{Sel}, \mathbb{Q}_p/\mathbb{Z}_p)$$

which we regard as an $\mathfrak{h}[[\mathbb{Z}_p^\times]]$-module on which the group element $[a]$ ($a \in \mathbb{Z}_p^\times$) acts as the element $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$. By [22], $\mathfrak{X}$ is a torsion $\mathfrak{h}[[\mathbb{Z}_p^\times]]$-module, that is,

$$\mathfrak{X} \otimes_{\mathfrak{h}[[\mathbb{Z}_p^\times]]} Q(\mathfrak{h}[[\mathbb{Z}_p^\times]]) = 0.$$

We have $\mathfrak{h}[[\mathbb{Z}_p^\times]] \cong \prod_{i \in \mathbb{Z}/(p-1)\mathbb{Z}} \mathfrak{h}[[\mathbb{Z}_p^\times]]_{(\omega^i)}$, where $[a]$ for $a \in (\mathbb{Z}/p\mathbb{Z})^\times \subset \mathbb{Z}_p^\times$ is sent to $w^i(a)$ in the i-th component. For any $\mathfrak{h}[[\mathbb{Z}_p^\times]]$-module M, let $M = \bigoplus_{i \in \mathbb{Z}/(p-1)\mathbb{Z}} M_{(\omega^i)}$ be the corresponding decomposition.

11.1.4. Let $\mathbb{H} = H(1)\hat{\otimes}_{\mathbb{Z}_p[\{\pm 1\}]}\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, where -1 in $\{\pm 1\}$ acts on $H(1)$ by the complex conjugation and acts on $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ by $[-1]$. We have $\mathbb{H}_{(\omega^i)} = H(1)^\pm[[\mathbb{Z}_p^\times]]_{(\omega^i)}$, where $\pm = (-1)^i$.

We regard the p-adic L-function $\mathcal{L}$ of Mazur and Kitagawa in two variables as an element of $\mathbb{H}$. Let

$$(\mathcal{L}) = \mathfrak{h}[[\mathbb{Z}_p^\times]]\mathcal{L} \subset \mathbb{H}.$$

The quotient $\mathbb{H}/(\mathcal{L})$ is a torsion $\mathfrak{h}[[\mathbb{Z}_p^\times]]$-module.

Theorem 11.1.5. *(Recall that we assume $N = 1$.) Assume that (a) is satisfied. Then we have an isomorphism of $\mathfrak{h}[[\mathbb{Z}_p^\times]]_{\theta,E,(\omega^{-1})}$-modules*

$$\mathfrak{X}_{\theta,E,(\omega^{-1})} \cong (\mathbb{H}/(\mathcal{L}))_{\theta,E,(\omega^{-1})}.$$

The proof will be given in section 11.3.

Remark 11.1.6. By Theorem 8.2.2 (2), $(\mathbb{H}/(\mathcal{L}))_{\theta,E,(\omega^{-1})} = 0$ if and only if ξ' is invertible in $\Lambda_\theta/(\xi)$. This shows that if the conditions (a) and (b) are satisfied (as in all known examples), we have $H^2(\mathbb{Z}[1/p], \tilde{H}_{DM,\theta,E}(2)) = 0$ and $\mathfrak{X}_{\theta,E,(\omega^{-1})} = 0$.

11.2 Relation with the main conjecture for modular forms

11.2.1. The Iwasawa main conjecture for modular forms is proved by Skinner-Urban [55] under some mild assumption. See Ochiai [37] for an earlier work on the study of the Iwasawa main conjecture for Hida family.

Since $\mathfrak{h}$ is not necessarily a regular ring, it is not evident how to formulate the Iwasawa main conjecture for the Galois representation H over $\mathfrak{h}$. We introduce an Iwasawa main conjecture 11.2.9 for a component $\mathfrak{h}_m$ of $\mathfrak{h}$ which is Gorenstein, which is implied by the Iwasawa main conjecture in [13]. Theorem 11.1.5 is closely related to Conjecture 11.2.9 (see Corollary 11.3.10).

11.2.2. Let R be a local ring and let M be an R-module such that $M \otimes_R Q(R) = 0$, and assume that M has a finite resolution by free R-modules of finite rank. Then we have the class

$$[M] \in Q(R)^\times / R^\times$$

defined as follows. Take a resolution

$$0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to M \to 0,$$

where L_i are free R-modules of finite rank. We have an exact sequence

$$0 \to L_n \otimes_R Q(R) \to L_{n-1} \otimes_R Q(R) \to \cdots \to L_0 \otimes_R Q(R) \to 0.$$

Let K_i be the kernel of $L_i \otimes_R Q(R) \to L_{i-1} \otimes_R Q(R)$, and take a splitting $L_i \otimes_R Q(R) \cong K_i \oplus K_{i-1}$ of the exact sequence $0 \to K_i \to L_i \otimes_R Q(R) \to K_i \to 0$. These splittings give isomorphisms

$$\bigoplus_{i:\text{odd}} L_i \otimes_R Q(R) \cong \bigoplus_i K_i \cong \bigoplus_{i:\text{even}} L_i \otimes_R Q(R).$$

Take an R-basis $(e_j)_j$ of $\oplus_{i:\text{odd}} L_i$ and an R-basis $(f_j)_j$ of $\oplus_{i:\text{even}} L_i$, and let $A \in GL_n(Q(R))$ ($n = \sum_{i:\text{odd}} \operatorname{rank} L_i = \sum_{i:\text{even}} \operatorname{rank} L_i$) be the matrix which expresses the images of e_j in $\oplus_{i:\text{even}} L_i \otimes_R Q(R)$ by (f_j). Then $[M]$ is defined to be $\det(A) \in Q(R)^\times/R^\times$. This is independent of the choices of the resolutions and the splittings. If we have an exact sequence $0 \to M' \to M \to M'' \to 0$ of such R-modules, then $[M] = [M'][M'']$. For example, $[R/aR] = a \mod R^\times$ for a non-zero-divisor $a \in R$.

11.2.3. Define the Selmer complex ([35]) Sc to be the mapping cone of $C(\mathbb{Z}[1/p, \zeta_{p^\infty}], H^\vee) \to C(\mathbb{Q}_p(\zeta_{p^\infty}), (H_{\text{sub}})^\vee)$, where $C(\mathbb{Z}[1/p, \zeta_{p^\infty}], H^\vee)$ (resp. $C(\mathbb{Q}_p(\zeta_{p^\infty}), (H_{\text{sub}})^\vee)$) denotes the standard complex to compute the continuous cohomology of the profinite group $\pi_1(\operatorname{Spec}(\mathbb{Z}[1/p, \zeta_{p^\infty}])$ (resp. $\operatorname{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))$) with coefficients in $H^\vee$ (resp. $(H_{\text{sub}})^\vee$). We have a long exact sequence

$$\cdots \to H^{i-1}(\mathrm{Sc}) \to H^i(\mathbb{Z}[1/p, \zeta_{p^\infty}], H^\vee) \to H^i(\mathbb{Q}_p(\zeta_{p^\infty}), (H_{\text{sub}})^\vee) \to H^i(\mathrm{Sc}) \to \dots$$

By Poitou-Tate duality (9.4.1), we have a long exact sequence

$$\cdots \to \operatorname{Hom}(H^{2-i}(\mathrm{Sc}), \mathbb{Q}_p/\mathbb{Z}_p) \to \varprojlim_n H^i(\mathbb{Z}[1/p, \zeta_{p^n}], H(1)) \to \varprojlim_n H^i(\mathbb{Q}_p(\zeta_{p^n}), H_{\text{quo}}(1))$$

$$\to \operatorname{Hom}(H^{1-i}(\mathrm{Sc}), \mathbb{Q}_p/\mathbb{Z}_p) \to \dots.$$

Lemma 11.2.4. *The canonical map* $\varprojlim_n H^1(\mathbb{Z}[1/p,\zeta_{p^n}], H(1)) \to \varprojlim_n H^1(\mathbb{Q}_p(\zeta_{p^n}), H_{\mathrm{quo}}(1))$ *is injective.*

Proof. This follows from 3.1.3 and 3.1.4. □

11.2.5. Fix a prime ideal $\mathfrak{p}$ of $\mathfrak{h}[[\mathbb{Z}_p^\times]]$.

Lemma 11.2.6. *(1)* $H^i(\mathrm{Sc}) = 0$ *if* $i \neq 0, -1$.

(2) If $\mathrm{Hom}\,(H^0(\mathbb{Z}[1/p,\zeta_{p^\infty}], H^\vee), \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{p} = 0$*, then* $H^{-1}(\mathrm{Sc})_\mathfrak{p} = 0$.

(3) If $\mathrm{Hom}\,(H^0(\mathbb{Q}_p(\zeta_{p^\infty}), (H_{\mathrm{sub}})^\vee), \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{p} = 0$*, then* $\mathrm{Hom}\,(H^0(\mathrm{Sc}), \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{p} = \mathfrak{X}_\mathfrak{p}$.

Proof. (1) follows from the second long exact sequence in 11.2.3 and from Lemma 11.2.4.
(2) and (3) follow from the first long exact sequence in 11.2.3. □

11.2.7. It can be shown that the assumption of (2) and the assumption of (3) in Lemma 11.2.6 are satisfied if the prime ideal $\mathfrak{p}$ is of height one.

Lemma 11.2.8. *Assume that there is a maximal ideal* $\mathfrak{m}$ *of* $\mathfrak{h}$ *such that* $\mathfrak{h}\cap\mathfrak{p}\subset\mathfrak{m}$ *and such that* $\mathfrak{h}_\mathfrak{m}$ *is Gorenstein, and assume that the assumptions of (2) and (3) in Lemma 11.2.6 are satisfied. Then the* $\mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p}$*-module* $\mathfrak{X}_\mathfrak{p}$ *is of finite projective dimension.*

Proof. By the Gorenstein property, $H_\mathfrak{m}$ is free of rank 2 over $\mathfrak{h}_\mathfrak{m}$. From this ([13] Proposition 1.6.5), we have that $R\mathrm{Hom}\,(\mathrm{Sc}, \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{p}$ is a perfect complex (that is, it is represented by a bounded complex of finitely generated free modules) over $\mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p}$. By Lemma 11.2.6, $R\mathrm{Hom}\,(\mathrm{Sc}, \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{p} \cong \mathfrak{X}_\mathfrak{p}$ in the derived category. □

Conjecture 11.2.9. *Assume that the assumptions in Lemma 11.2.8 are satisfied. Then we have*

$$[\mathfrak{X}_\mathfrak{p}] = [(\mathbb{H}/(\mathcal{L}))_\mathfrak{p}]$$

in $Q(\mathfrak{h}[[\mathbb{Z}_p^\times]])^\times/(\mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p})^\times$.

Remark 11.2.10. By the assumption of Gorenstein property, $\mathbb{H}_\mathfrak{p}$ is free of rank 1 over $\mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p}$. If h denotes an isomorphism $\mathbb{H}_\mathfrak{p} \xrightarrow{\cong} \mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p}$ of $\mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p}$-modules, then $[(\mathbb{H}/(\mathcal{L}))_\mathfrak{p}]$ is nothing but the class of $h(\mathcal{L}) \in Q(\mathfrak{h}[[\mathbb{Z}_p^\times]])^\times$ modulo $(\mathfrak{h}[[\mathbb{Z}_p^\times]]_\mathfrak{p})^\times$.

11.2.11. We expect that Conjecture 11.2.9 can be proved by using the theory of Skinner-Urban in [55]. For the Eisenstein component of this conjecture, one problem may be that in [55], the residue Galois representation is assumed to be irreducible, but this is not satisfied for the Eisenstein component.

11.2.12. We explain that the conjecture 11.2.9 is a consequence of the Iwasawa main conjecture in [13].

Assume that there is a maximal ideal $\mathfrak{m}$ of $\mathfrak{h}$ such that $\mathfrak{h}\cap\mathfrak{p}\subset\mathfrak{m}$ and such that $\mathfrak{h}_\mathfrak{m}$ is Gorenstein. Let $\gamma^\pm$ be a basis of the of the free $\mathfrak{h}_\mathfrak{m}$-module $H_\mathfrak{m}^\pm$ of rank 1, and let $\gamma = \gamma^+ + \gamma^-$. Let ω be a basis of the free $\mathfrak{h}_\mathfrak{m}$-module $S_{\Lambda,\mathfrak{m}}$ of rank 1. Then $\mathcal{M}(\gamma)/\omega \in \mathfrak{h}_\mathfrak{m}[[\mathbb{Z}_p^\times]]$ is the p-adic L-function associated to the pair $(H_{\mathrm{sub},\mathfrak{m}}, H_\mathfrak{m})$ of the representation $H_{\mathrm{sub},\mathfrak{m}}$

of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and the representation $H_{\mathfrak{m}}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over $\mathfrak{h}_{\mathfrak{m}}$ (defined with respect to $(\gamma^{\pm},\omega)$). The Iwasawa main conjecture in section 4 of [13] is that

$$(1)\quad [R\mathrm{Hom}\,(\mathrm{Sc},\mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{m}}] = [(S_\Lambda[[\mathbb{Z}_p^\times]]/(\mathcal{M}(\gamma)))_{\mathfrak{m}}] \quad \text{in} \;\; Q(\mathfrak{h}_{\mathfrak{m}})^\times/\mathfrak{h}_{\mathfrak{m}}^\times.$$

Here $[R\mathrm{Hom}\,(\mathrm{Sc},\mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{m}}]$ is the class of the perfect complex $R\mathrm{Hom}\,(\mathrm{Sc},\mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{m}}$ whose cohomology groups are torsion modules over $\mathfrak{h}_{\mathfrak{m}}[[\mathbb{Z}_p^\times]]$, and $(\mathcal{M}(\gamma)) := \mathfrak{h}[[\mathbb{Z}_p^\times]]\mathcal{M}(\gamma) \subset S_\Lambda[[\mathbb{Z}_p^\times]]$. By 4.4.3 and 1.6.6 (3), $((-,\gamma))_\Lambda$ induces an isomorphism $\mathbb{H} \xrightarrow{\cong} S_\Lambda$ such that the induced isomorphism $\mathbb{H}[[\mathbb{Z}_p^\times[[\xrightarrow{\cong} S_\Lambda[[\mathbb{Z}_p^\times]]$ sends $\mathcal{L}$ to $\mathcal{M}(\gamma)$.

If the assumptions of (2) and (3) in Lemma 11.2.6 are satisfied, then $[R\mathrm{Hom}\,(\mathrm{Sc},\mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{p}}] = [\mathfrak{X}_{\mathfrak{p}}]$. Hence the above conjecture (1) implies 11.2.9.

11.3 Proofs of Theorem 11.1.2 and Theorem 11.1.5

11.3.1. We prove (1) of Theorem 11.1.2.

If (a) is satisfied, then by Theorem 7.2.3 (2) and by Theorem 7.2.8 (1), (a)' is satisfied.

The exact sequence $0 \to \mathcal{Q}(2) \to \mathcal{E}(2) \to \mathcal{P}(2) \to 0$ induces an exact sequence

$$H^1(\mathbb{Z}[1/p],\mathcal{R}(2)) \to H^2(\mathbb{Z}[1/p],\mathcal{Q}(2)) \to H^2(\mathbb{Z}[1/p],\mathcal{E}(2)) \to H^2(\mathbb{Z}[1/p],\mathcal{R}(2)) \to 0.$$

We have $H^2(\mathbb{Z}[1/p],\mathcal{R}(2)) = 0$ by 6.3.7, and we have an isomorphism $H^2(\mathbb{Z}[1/p],\mathcal{Q}(2)) \cong \mathcal{S}_\theta$ (9.1.1 (2)). By 10.3.3, the connecting map $\partial : \mathfrak{h}_\theta/I_\theta \cong H^1(\mathbb{Z}[1/p],\mathcal{R}(2)) \to H^2(\mathbb{Z}[1/p],\mathcal{Q}(2)) \cong \mathcal{S}_\theta$ is the $\mathfrak{h}_\theta$-homomorphism

$$\mathfrak{h}_\theta/I_\theta \to \mathcal{S}_\theta\;;\; 1 \mapsto (\{p,1-\zeta_{p^r}\})_r.$$

Hence $H^2(\mathbb{Z}[1/p],\mathcal{E}(2)) = 0$ if and only if the condition (a)' is satisfied. If the condition (a)" is satisfied, then $H^2(\mathbb{Z}[1/p],\mathcal{E}(2)) = 0$ and hence (a)' is satisfied.

Assume (a) is satisfied. Then by Nakayama's lemma, the Eisenstein component of H_θ^- is generated by $(1-T^*(p))\{0,\infty\}$ as an $\mathfrak{h}_\theta$-module. Hence $\tilde{H}_{DM,\theta}/I_\theta\tilde{H}_{DM,\theta} \cong \mathcal{E}$, and hence $H^2(\mathbb{Z}[1/p],\mathcal{E}(2)) \cong H^2(\mathbb{Z}[1/p],\tilde{H}_{DM,\theta}(2))/I_\theta H^2(\mathbb{Z}[1/p],\tilde{H}_{DM,\theta}(2))$. Since (a) is satisfied, (a)' is also satisfied as we have seen above, and hence $H^2(\mathbb{Z}[1/p],\mathcal{E}(2)) = 0$. By Nakayama's lemma, we have $H^2(\mathbb{Z}[1/p],\tilde{H}_{DM,\theta,E}(2)) = 0$.

11.3.2. We prove (2) of Theorem 11.1.2. It is sufficient to prove (a)' $\Rightarrow$ (a).

Assume first $\mathfrak{h}_{\theta,E}$ is Gorenstein. Then $H_\theta^-/I_\theta H_\theta^-$ is generated by one element as an $\mathfrak{h}_\theta$-module (7.2.12). Since the composition $X_{p^\infty,\chi} \xrightarrow{\Upsilon} H_\theta^-/I_\theta H^- \xrightarrow{\varpi} X_{p^\infty,\chi}$ sends the generator $(\{p,1-\zeta_{p^r}\})_r$ to itself by Theorem 7.2.3 (2), we have that $H_\theta^-/I_\theta H_\theta^- = A \oplus B$ where $A = \mathrm{Image}(\Upsilon)$ and $B = \mathrm{Ker}\,(\varpi)$. Since $H_\theta^-/I_\theta H_\theta^-$ is generated by one element, we have $B = 0$ and hence Υ is surjective. Hence by Theorem 7.2.3 (2), the condition (a) is satisfied.

Assume next $\mathfrak{H}_{\theta,E}$ is Gorenstein. Then by Ohta [42], the map $\Upsilon : X_{p^\infty,\chi}^- \to H_\theta^-/I_\theta H_\theta^-$ is an isomorphism. Hence by Theorem 7.2.3 (2), we see that the condition (a) is satisfied.

We give some preliminary lemmas for the proof of Theorem 11.1.5.

Lemma 11.3.3. *Let $i \in \mathbb{Z}/(p-1)\mathbb{Z}$.*

(1) $\mathrm{Hom}\,(H^0(\mathbb{Q}_p(\zeta_{p^\infty}),H_{\mathrm{sub}}^\vee),\mathbb{Q}_p/\mathbb{Z}_p)_{\theta,E,(\omega^i)}$ *is zero if $\theta \neq \omega^{-i}$.*

(2) $\mathrm{Hom}\,(H^0(\mathbb{Z}[1/p,\zeta_{p^\infty}]),H^\vee),\mathbb{Q}_p/\mathbb{Z}_p)_{\theta,E,(\omega^i)}$ *is zero if $\theta \neq \omega^{-i}$ and if $\Upsilon : X_{p^\infty,\chi} \longrightarrow H_\theta^-/I_\theta H_\theta^-$ is surjective.*

Proof. (1) follows from 1.7.14 (4).

We prove (2). Let $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$. Consider the exact sequence $0 \to \mathcal{P} \to H_\theta/I_\theta H_\theta \to \mathcal{Q} \to 0$. By Proposition 6.3.2, we have $(\mathcal{Q}_G)_{(\omega^i)} = 0$ if $\theta \neq \omega^{-i}$. Here $(-)_G$ denotes the G-coinvariant. By the surjectivity of Υ, the image of $\mathcal{P}$ in $(H_\theta/I_\theta H_\theta)_G$ is zero. Hence $((H_\theta/I_\theta H_\theta)_G)_{(\omega^i)} = 0$ if $\theta \neq \omega^{-i}$. By Nakayama's lemma, this proves $(H_G)_{(\omega^i)} = 0$ if $\theta \neq \omega^{-i}$. □

Lemma 11.3.4. *$H^1(\mathbb{Z}[1/p], H_\theta(2))$ has no Λ-torsion.*

Proof. This follows from Proposition 3.3.6 by Assumption 4 (section 6.1). □

Let β be the element $(1 - T^*(p))\{0, \infty\}$ of H_θ^-.

Lemma 11.3.5. *In $H^1(\mathbb{Z}[1/p], H_\theta(2)) \otimes_{\Lambda_\theta} Q(\Lambda_\theta)$, $z^\sharp_{p^\infty}(\beta)$ belongs to $H^1(\mathbb{Z}[1/p], H_\theta(2))$.*

Proof. Recall that

$$z^\sharp_{p^\infty}(\beta) = (\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} T^*(p)^{-r}\{g_{0,a/p^r}, g_{0,1/p^r}\})_r.$$

For integers c, d such that $(c, 6Np) = (d, 6Np) = 1$, we have

$$\sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \{{}_c g_{0,a/p^r}, {}_d g_{0,1/p^r}\} = (c^2 - 1)(d^2 - \langle d \rangle) \sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} \{g_{0,a/p^r}, g_{0,1/p^r}\}.$$

Since $p > 3$, $c^2 - 1$ is a p-adic unit for some c. Since $\theta \neq \omega^2$ by Assumption 4 (section 6.1), $d^2 - \theta(d)$ is a p-adic unit for some d. This proves Lemma 11.3.5. □

Lemma 11.3.6. *The image of $z_{1,p^\infty,p^\infty}(\{0,\infty\}) \in \varprojlim_n H^1(\mathbb{Z}[1/p, \zeta_{p^n}], H_\theta(2))$ in $H^1(\mathbb{Z}[1/p], H_\theta(2))$ coincides with $z^\sharp_{p^\infty}(\beta)$.*

Proof. By Proposition 2.2.2, the image of $z_{1,p^\infty,p^\infty}(\beta)$ in $H^1(\mathbb{Z}[1/p], H_\theta(2))$ coincides with $(1 - T^*(p))z^\sharp_{p^\infty}(\beta)$. This shows that the image of $z_{1,p^\infty,p^\infty}(\{0,\infty\})$ and $z^\sharp_{p^\infty}(\beta)$ become the same after we apply $1 - T^*(p)$. By Lemma 11.3.4, they coincide already before we apply $1 - T^*(p)$. □

Let F_n be the unique subextension of $\mathbb{Q}$ in $\mathbb{Q}(\zeta_{p^n})$ of degree p^n, let $F_\infty = \cup_n F_n$, and let $\Gamma = \mathrm{Gal}(F_\infty/\mathbb{Q})$. So $\Gamma \cong \mathbb{Z}_p$.

Lemma 11.3.7. *Assume (a) in 11.1.1.*

(1) We have $H^1(\mathbb{Z}[1/p], H_{\theta,E}(2)) = H^1(\mathbb{Z}[1/p], \tilde{H}_{DM,\theta,E}(2))$.

(2) As an $\mathfrak{h}$-module, $H^1(\mathbb{Z}[1/p], H_{\theta,E}(2))$ is generated by $z^\sharp_{p^\infty}(\beta)$.

(3) $z_{1,p^\infty,p^\infty}(\{0,\infty\})$ generates $\varprojlim_n H^1(O_{F_n[1/p]}, \tilde{H}_{DM,\theta,E}(2))$ as an $\mathfrak{h}[[\Gamma]]$-module.

Proof. By the assumption (a), the $\mathfrak{h}$-module $\tilde{H}^-_{DM,\theta,E}(2)$ is free of rank 1 and generated by $\{0,\infty\}$. Hence we have a spectral sequence

$$E_2^{ij} = \mathrm{Tor}_{-i}^{\mathfrak{h}_\theta}(H^j(\mathbb{Z}[1/p], \tilde{H}_{DM,\theta}(2)), \mathfrak{h}_\theta/I_\theta) \Rightarrow H^{i+j}(\mathbb{Z}[1/p], (\tilde{H}_{DM,\theta}/I_\theta\tilde{H}_{DM,\theta})(2)).$$

We have

$$\tilde{H}_{DM,\theta}/I_\theta\tilde{H}_{DM,\theta} = \mathcal{E}.$$

In the exact sequence

$$0 \to H^1(\mathbb{Z}[1/p], \mathcal{Q}(2)) \to H^1(\mathbb{Z}[1/p], \mathcal{E}(2)) \to H^1(\mathbb{Z}[1/p], \mathcal{R}(2)) \to H^2(\mathbb{Z}[1/p], \mathcal{Q}(2)),$$

the map $H^1(\mathbb{Z}[1/p], \mathcal{R}(2)) \to H^2(\mathbb{Z}[1/p], \mathcal{Q}(2))$ is bijective because it is

$$\Lambda_\theta\,/(\xi) \to X^-_{p^\infty}\ ;\ 1 \mapsto (\{p, 1-\zeta_{p^r}\})_r.$$

Hence

$$H^1(\mathbb{Z}[1/p], \mathcal{Q}(2)) \xrightarrow{\cong} H^1(\mathbb{Z}[1/p], \mathcal{E}(2)).$$

Since the image of $z^\sharp_{p^\infty}(\beta) \in H^1(\mathbb{Z}[1/p], \mathcal{Q}(2)) \cong \mathcal{S}_\theta$ is $(\{p, 1-\zeta_{p^r}\})_r$, this shows that the image of $z^\sharp_{p^\infty}(\beta)$ generates $H^1(\mathbb{Z}[1/p], \mathcal{E}(2))$. By Nakayama's lemma, Lemma 11.3.7 is proved.

(3) follows from (1) and (2) by Nakayama's lemma and by Lemma 11.3.6. □

Lemma 11.3.8. *We have an isomorphism*

$$Col\ :\ \varprojlim_n H^1(\mathbb{Q}_pF_n, \tilde{H}_{DM}(2)) \xrightarrow{\cong} M_{\Lambda,DM}[[\Gamma]].$$

Proof. (1) By 4.2.7, it is sufficient to prove $\tilde{H}^{Fr_p=1}_{\mathrm{quo},DM} = 0$. We have $\tilde{H}_{\mathrm{quo},DM} \subset H_{\mathrm{quo}} \otimes_\Lambda Q(\Lambda)$ and $(H_{\mathrm{quo}} \otimes_\Lambda Q(\Lambda))^{Fr_p=1} = 0$ by 3.3.3.

□

11.3.9. We prove Theorem 11.1.5. Define a variant $\mathfrak{X}'$ of $\mathfrak{X}$ and a variant Sc$'$ of Sc by using $\tilde{H}_{DM}$ instead of H in the definitions. We obtain long exact sequences as in 11.2.3 for Sc$'$ replacing H in 11.2.3 by $\tilde{H}_{DM}$.

By Lemma 11.3.3 (1) and by the Sc$'$-version of the long exact sequences in 11.2.3, we have an exact sequence

$$\varprojlim_n H^1(O_{F_n}[1/p], \tilde{H}_{DM,\theta,E}(2)) \to \varprojlim H^1(\mathbb{Q}_pF_n, \tilde{H}_{\mathrm{quo,DM},\theta,\mathrm{E}}(2)) \to \mathfrak{X}'_{\theta,E,(\omega^{-1})}(1)$$

$$\to \varprojlim H^2(O_{F_n}[1/p], \tilde{H}_{DM,\theta,E}(2)).$$

By Theorem 11.1.2 and by Nakayama's lemma, we have $\varprojlim_n H^2(O_{F_n}[1/p], \tilde{H}_{DM,\theta,E}(2)) = 0$. Hence by (3) of Lemma 11.3.7, this exact sequence shows that

$$\varprojlim H^1(\mathbb{Q}_pF_n, \tilde{H}_{\mathrm{quo,DM},\theta,\mathrm{E}}(2))/(z) \cong \mathfrak{X}'_{,\theta,E,(\omega^{-1})}(1),$$

where (z) denotes the $\mathfrak{h}[[\Gamma]]$-submodule of $\varprojlim_n H^1(\mathbb{Q}_pF_n, \tilde{H}_{\mathrm{quo},\theta,\mathrm{E}}(2))$ generated by $z_{1,p^\infty,p^\infty}(\{0,\infty\})$.

By simple comparison, we have $\mathfrak{X}'_{(\omega^{-1})} = \mathfrak{X}_{(\omega^{-1})}$ (use $H^i(O_{F_\infty}[1/p], \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for $i = 1, 2$.) By (2) of Lemma 11.3.8,

$$\varprojlim_n H^1(\mathbb{Q}_pF_n, \tilde{H}_{\mathrm{quo},\theta,\mathrm{E}}(2))/(z) \cong M_{\Lambda,\theta,E}[[\Gamma]]/(\mathcal{M}(\{0,\infty\})) \cong (\mathbb{H}/(\mathcal{L}))_{\theta,E,(\omega^{-1})}.$$

Corollary 11.3.10. *Assume that the condition (a) in section 11.1 is satisfied. Then for any prime ideal* $\mathfrak{p}$ *of* $\mathfrak{h}[[\mathbb{Z}_p^\times]]$ *which is the inverse image of a prime ideal of* $\mathfrak{h}[[\mathbb{Z}_p^\times]]_{\theta,E,(\omega^{-1})}$*, the assumptions of Lemma 11.2.8 are satisfied and Conjecture 11.2.9 is true.*

Proof. The condition (a) tells that $\mathfrak{H}_{\theta,E}$ and $\mathfrak{h}_{\theta,E}$ are Gorenstein. Hence by 7.1.4, $\Upsilon : X_{p^\infty,\chi} \to H_\theta/I_\theta H_\theta$ is surjective. By this and by Lemma 11.3.3, the assumptions of Lemma 11.2.8 is satisfied. Hence Theorem 11.1.5 tells that Conjecture 11.2.9 is true in this case. □

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List of notation

Numbers
p a prime number ≥ 5, 0.26
ζ_n, 0.26
N an integer ≥ 1 which is prime to p, 1.5.1, 6.1.1

Special field
$K = \cup_r \mathbb{Q}(\zeta_{Np^r})$, 6.1.2

Important homomorphisms
Υ, 6.4.3
ϖ, Theorem 5.2.3

Iwasawa modules
X_{Np^∞}, 6.4.1
$\mathcal{S}$, 5.2.6

Operators
Hecke operator $T(n)$, dual Hecke operator $T^*(n)$, 1.2.3, 1.2.4
diamond operators, $\langle a \rangle$ 1.2.9, $\langle \sigma \rangle$ 1.2.9 (2)
w_M (Atkin-Lehner operator), 1.4.2 , 1.4.4, 1.5.9
φ (Frobenius operator), 1.7.5

$\mathcal{H}$ upper half plane, 1.1.5

Modular curves
$X(m, M)$, $Y(m, M)$, $X_1(M)$, $Y_1(M)$, 1.1
$X_1'(M)$, 1.4
ψ_ℓ (a morphism), 1.2.3

Cohomology of modular curves
$H^m_{\text{ét}}(C)$ for a scheme C over $\mathbb{Q}$, 1.2.8
H, $\tilde{H}$, 1.5.1
$\tilde{H}_{DM}$, 6.2.1
$\tilde{H}_c$, 1.8
H_{sub}, H_{quo}, $\tilde{H}_{\text{quo}}$, 1.7.2
$(-)^{\text{ord}}$, 1.2.10
$(-)_E$ (Eisenstein component), 1.9.2, 6.2.3
$\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, 6.3.1

$\{\alpha, \beta\}$ $(\alpha, \beta \in P^1(\mathbb{Q}))$, 2.3.3
$[u : v]_r$, 2.4.1

Spaces of modular forms
$S_2(M)$, $M_2(M)$, 1.1.7, $S_k(M), M_k(M)$, 1.5.6
S_Λ, M_Λ, 1.5.11

Pairings
$(\, , \,)$, $(\, , \,)_\Lambda$, $((\, , \,))_\Lambda$, section 1.6

p-adic L-functions in two variables

$\mathcal{M}$, 4.3, ${}^{s}\mathcal{M}$, 4.3 ($\mathcal{M} = {}^{1}\mathcal{M}$), $\mathcal{L}$, 4.4
$\mathcal{M}_{s=0}$ 4.3.7, $\mathcal{L}_{s=0}$, 8.2.1

Characters
ω, 1.8.2,
κ, 1.2.9, (2)
χ, 6.1.4
θ, 6.1.4

Rings
Hecke algebras
$\mathfrak{h}(M)_{\mathbb{Z}}$, $\mathfrak{H}(M)_{\mathbb{Z}}$, 1.2.6
$\mathfrak{h} = \varprojlim \mathfrak{h}(Np^r)^{\mathrm{ord}}$, 1.5.1
The version of $\mathfrak{h}$ for the Hecke algebra of modular forms is $\mathfrak{H}$, 1.5.1
$\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times]] = \varprojlim_n \mathbb{Z}_p[(\mathbb{Z}/Np^n\mathbb{Z})^\times]$, 3.1.1
$Q(R)$, 1.5.3
Λ_ψ, O_ψ for a character ψ of $(\mathbb{Z}/Np\mathbb{Z})^\times$, 6.1.3

I Eisenstein ideal, 1.9.1

Elements in Galois groups
Frobenius automorphism Fr_p, Fr_ℓ, 1.7.4, 1.2.9 (1)
σ_a, 2.4.3

Other important notation

$\hat{\otimes}$, 1.7.3

$D(T)$, 1.7.4

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