# QUANTUM COHOMOLOGY OF $\mathrm{Hilb}_n(\mathbb{C}^2)$ AND THE WEIGHTED HOOK WALK ON YOUNG DIAGRAMS

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ABSTRACT. Following the work of Okounkov-Pandharipande [OP1, OP2] and Diaconescu [D], we study the equivariant quantum cohomology $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n)$ of the Hilbert scheme and the relative Donaldson-Thomas theory of $\mathbb{P}^1 \times \mathbb{C}^2$. Using the ADHM construction and a recent work [CDKM], we continue the study the Gromov-Witten invariant of the abelian/nonabelian correspondence, and establish a connection between the $J$-function of the Hilbert scheme and a certain combinatorial identity in two variables. This identity is then generalized to a multivariate identity, which simultaneously generalizes the branching rule for the dimension of irreducible representations of the symmetric group in the staircase shape. We then establish this identity by a weighted generalization of the Greene-Nijenhuis-Wilf hook walk, which is of independent interest.

## 1. INTRODUCTION

In the previous decades, a number of new connections between enumerative algebraic geometry and algebraic combinatorics have been established, which led to further advances in both fields. These works pushed away the traditional boundaries between the fields, and this paper is in the same mold.

Roughly speaking, we study the Gromov-Witten invariants in an important special case by using the machinery of Okounkov-Pandharipande, Diaconescu and recent results of the first named author. The results imply a certain combinatorial identity, which is both new and curious from an algebraic and a combinatorial point of view. The proof of the identity is rather involved, based on the classical "ADHM construction", and uses the full power of the above mentioned delicate algebraic results.

Somewhat surprisingly, there does not seem to be an easy direct proof of this identity, but it follows from a much more general (also new) multivariate identity. Even more surprisingly, the latter identity is also a generalization of the branching rule for the dimension $d_\lambda$ of irreducible representations of the symmetric group (in a special case), which in turn can be interpreted in the language of standard Young tableaux. We then give a completely combinatorial proof of the multivariate identity, using the Greene-Nijenhuis-Wilf hook walk, originally designed to give a new combinatorial/probabilistic proof of the hook length formula for the dimensions $d_\lambda$.

In summary, the results in this paper can be viewed in three different ways:

(1) as a technical algebraic proof of a combinatorial identity;

(2) as a combinatorial proof of a multivariate generalization of both this identity and the classical combinatorial result (the celebrated hook length formula);

(3) as a new algebraic/combinatorial result, which, when combined together with several recent algebraic results, gives a new proof of deep algebraic results (by Okounkov

and Pandharipande computing the equivariant quantum cohomology $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n)$ of the Hilbert scheme).

Given the differences between the field, to ease the burden on the reader, we will split both the rest of the introduction and the rest of the paper into two parts: algebraic and combinatorial.

## 1.1. Algebraic part.

The Hilbert scheme of points in the complex affine plane $\mathrm{Hilb}_n := \mathrm{Hilb}_n(\mathbb{C}^2)$ has been the focus of much study over the last fifteen years, and deep relations between its geometry and representation theory have been discovered. (The literature is much too extensive to be surveyed here; we mention only two of the major works due to Nakajima [N1], and Haiman [H].)

The *equivariant quantum cohomology* $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n)$ of the Hilbert scheme has been recently determined by Okounkov and Pandharipande, and they have also shown that it agrees with the (equivariant) *relative Donaldson-Thomas theory* of $\mathbb{P}^1 \times \mathbb{C}^2$, see [OP1], [OP2].

A different perspective on the study of the relationship between $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n)$ and DT-theory is undertaken in [CDKM]. The main point there is to exploit the fact that the Hilbert scheme is a Geometric Invariant Theory (GIT) quotient via the celebrated "ADHM construction" of Atiyah-Drinfeld-Hitchin-Manin.

On the one hand, this allows one to employ the machinery of the abelian/nonabelian correspondence in Gromov-Witten theory of [CKS], [BCK] to analyze the quantum cohomology of $\mathrm{Hilb}_n$. In particular, one can give a formula (a priori conjectural) for the *J-function* of the Hilbert scheme – a certain generating function for Gromov-Witten invariants of a nonsingular algebraic variety, essentially encoding the same information as the quantum cohomology ring.

On the other hand, the ADHM construction of $\mathrm{Hilb}_n$ is also highly relevant to the Donaldson-Thomas side of the story, due to work of Diaconescu. Namely, in [D] he used it to obtain a gauge-theoretic partial compactification of the space of maps $\mathbb{P}^1 \to \mathrm{Hilb}_n$, his *moduli space of ADHM sheaves* on $\mathbb{P}^1$, and then provided a direct geometric identification of the DT-theory of $\mathbb{P}^1 \times \mathbb{C}^2$ with the intersection theory of this new moduli space.

The main result of [CDKM] is a proof of the above-mentioned formula for the *J*-function of $\mathrm{Hilb}_n$. An outline of the rather intricate argument, which in particular relies on the full force of the results of Okounkov-Pandharipande and Diaconescu, is presented in §2.3 below. As explained there, the following non-trivial identity involving partitions is obtained as a corollary. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ we consider its Young diagram $[\lambda]$ as a collection of boxes with integer coordinates $(i, j)$ with $1 \leq j \leq k$, $1 \leq i \leq \lambda_j$. Let $\alpha, \beta$ be indeterminates. For a box $B = (i, j)$ in $[\lambda]$, we define its *weight* to be

$$w_B := -(i-1)\alpha - (j-1)\beta.$$

We call $B = (i, j)$ a *corner* of $\lambda$ if neither $(i+1, j)$, nor $(i, j+1)$ are in $[\lambda]$ and denote by $\mathcal{C}(\lambda)$ the set of corners in $[\lambda]$.

**Theorem 1.1.** *For each $n \geq 1$ and each partition $\lambda$ of $n$ we have*

$$(1.1) \quad \sum_{B \in \mathcal{C}(\lambda)} (w_B - (\alpha + \beta)) \prod_{C \in [\lambda],\, C \neq B} \frac{(w_B - w_C - \alpha)(w_B - w_C - \beta)}{(w_B - w_C)(w_B - w_C - (\alpha + \beta))} = -n(\alpha + \beta).$$

The goal of the algebraic part of the paper is to derive this theorem.

1.2. **Combinatorial part.** Recall that irreducible representations $\pi_\lambda$ of $S_n$ correspond to partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of $n$, we write $\lambda \vdash n$. We represent partitions as Young diagrams, using the French notation (see Figure 1.2). In a *Young diagram* $[\lambda]$ corresponding to $\lambda$, we define the *hook* $H_{\mathbf{z}} \subset [\lambda]$ to be the set of squares weakly to the right of and above $\mathbf{z} = (i, j) \in [\lambda]$, and the *hook length* $h_{\mathbf{z}} = |H_{\mathbf{z}}| = \lambda_i + \lambda'_j - i - j + 1$ the size of the hook. Here $\lambda'$ is the conjugate partition.
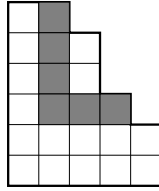


FIGURE 1. Young diagram of $(5, 5, 4, 3, 3, 2)$ and the hook length $h_{23} = 6$.

A *standard Young tableau $A$* of shape $\lambda$, where $\lambda \vdash n$ is a partition, is a bijection $f : [\lambda] \to [n] = \{1, \ldots, n\}$, with labels increasing in rows and columns. Denote by $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of shape $\lambda$, and recall that $\dim \pi_\lambda = \mathrm{SYT}(\lambda)$. The classical *hook length formula* for the number of standard Young tableaux, says:

$$(HLF) \qquad |\mathrm{SYT}(\lambda)| = \frac{n!}{\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}},$$

Recall the *branching rule*

$$\dim \pi_\lambda = \sum_{\mu \to \lambda} \dim \pi_\mu,$$

where $\mu \to \lambda$ means $[\mu] = [\lambda] - \mathbf{z}$ for some $\mathbf{z} \in \mathcal{C}(\lambda)$. In the language of standard Young tableaux, this can be written as:

$$(BR) \qquad |\mathrm{SYT}(\lambda)| = \sum_{\mu \to \lambda} |\mathrm{SYT}(\mu)|.$$

Substituting $(HLF)$ into $(BR)$ and rearranging the terms, we obtain the following *branching rule for the hook lengths*:

$$(BRHL) \qquad n = \sum_{(r,s) \in \mathcal{C}(\lambda)} \prod_{i=1}^{r-1} \left(1 + \frac{1}{h_{is} - 1}\right) \prod_{j=1}^{s-1} \left(1 + \frac{1}{h_{rj} - 1}\right).$$

Denote by $\rho_\ell = (\ell, \ell - 1, \ldots, 2, 1)$ the *staircase shape*. The main combinatorial result of this paper is the following identity.

**Theorem 1.2.** *Fix $\ell \geq 1$, and let $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_\ell$ be formal variables. Then:*

$$\sum_{k=1}^{\ell} x_k y_k \prod_{p=1}^{k-1} \left(1 + \frac{y_p}{x_p + \ldots + x_{k-1} + y_{p+1} + \ldots + y_k}\right) \prod_{q=k+1}^{\ell} \left(1 + \frac{x_q}{x_k + \ldots + x_{q-1} + y_{k+1} + \ldots + y_q}\right) = \sum_{1 \leq p \leq q \leq \ell} x_q y_p.$$

The identity in Theorem $(1.2)$ is a direct generalization of $(BRHL)$ for the case of the staircase shape $\rho_\ell$. To see this, set $x_1 = \ldots = x_\ell = y_1 = \ldots = y_\ell = 1$ and observe that the product terms are equal to $1 + 1/(h_{\mathbf{z}} - 1)$.

In fact, this identity implies $(1.1)$ as well. Roughly speaking, one needs to collapse a partition $\lambda$ with $\ell$ corners to a staircase shape $\rho_\ell$ and then set all $x_i = \alpha$ and all $y_j = \beta$. The detailed proof of this connection is given in Section 3.

Let us say a few words about the different proofs of the hook length formula. The formula was originally discovered by Frame, Robinson and Thrall in [FRT] based on earlier formula of Thrall [Thr]. This formula has a number of combinatorial proofs, including purely bijective (see [FZ, NPS, Pak, Rem, Zei]). The analytic proofs by induction in [Ban, GN, Ker2, Kir, Ver] give yet another useful approach to the problem.

In [GNW1], an inductive proof was established based on an elegant probabilistic argument, and this is the proof we adapt for the proof of Theorem 1.2. The idea is to start a so called "hook walk" from a random square in the Young diagram, and prove that it ends at a corner $\mathbf{z} \in \mathcal{C}(\lambda)$ with probability precisely $\mathrm{SYT}(\lambda - \mathbf{z})/\mathrm{SYT}(\lambda)$. Then the summation of these probabilities gives 1 and implies the branching rule.

We generalize this walk to a "weighted hook walk" by fixing positive weights $x_i$ and $y_j$ for the transition probabilities (see Section 4). Together with an inductive argument, this gives a complete proof of Theorem 1.2. In the forthcoming papers [CKP] and [Kon] we further study the weighted hook walk, and extend it to general Young diagrams, shifted Young diagrams, and find other variations.

Let us mention that the hook walk inspired a number of further developments, including the "complementary hook walk", $q$-hook walk, Garsia-Haiman's $(q, t)$-hook walk, Kerov's segment sampling algorithm, etc. (see [CLPS, GH, GNW2, Ker1, Ker2]).

## 2. THE GEOMETRIC ORIGIN OF THE IDENTITY

In this section we present in some detail how $(1.1)$ emerges from the approach to the Gromov-Witten theory of $\mathrm{Hilb}_n$ in [CDKM].

2.1. **Reminder on** $\mathrm{Hilb}_n$**.** The Hilbert scheme of points parametrizes subschemes of length $n$ in the affine plane $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x, y])$. Alternatively, it parametrizes ideals $I$ in the polynomial ring $\mathbb{C}[x, y]$ of colength $n$, i.e., for which

$$\dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n.$$

It is a smooth, quasiprojective variety and comes with a tautological rank $n$ vector bundle $\mathcal{V}$, whose fiber over a point $[I] \in \mathrm{Hilb}_n$ is equal to $\mathbb{C}[x, y]/I$.

Let $\mathbf{S} = (\mathbb{C}^*)^2$ denote the two-dimensional algebraic torus. It acts naturally on $\mathbb{C}^2$ by dilations in the directions of the coordinate axes and the corresponding (dual)

action on the functions $\mathbb{C}[x, y]$ on $\mathbb{C}^2$ is then given by

$$(s_1, s_2) \cdot x = s_1^{-1}x, \quad (s_1, s_2) \cdot y = s_2^{-1}y, \quad (s_1, s_2) \in \mathbf{S}.$$

This induces an action on $\mathrm{Hilb}_n$ in the obvious way. It is easy to see that there are finitely many fixed points for the $\mathbf{S}$-action, indexed by all partitions of $n$. Indeed, a point $[I]$ is fixed if and only if $I$ is a *monomial* ideal, i.e., it is generated by monomials. If we represent monomials by integer points in the first quadrant, placing $x^i y^j$ at the point $(i+1, j+1)$, then together with each generator $x^i y^j$ of $I$, all monomials North-East of it will also be in $I$. It follows that the monomials in $I$ form the complement (in the first quadrant) of the Young diagram of a partition $\lambda$ (in "French notation"), whose boxes correspond to a $\mathbb{C}$-basis of $\mathbb{C}[x, y]/I$. We will denote by $I_\lambda$ the monomial ideal corresponding to the partition $\lambda$, and by $p_\lambda = [I_\lambda]$ the corresponding $\mathbf{S}$-fixed point in $\mathrm{Hilb}_n$.

We collect next some facts about the equivariant cohomology ring $H_{\mathbf{S}}^*(\mathrm{Hilb}_n)$. First, recall that the $\mathbf{S}$-equivariant cohomology with complex coefficients of a point (viewed as an $\mathbf{S}$-space with trivial action) is a polynomial ring in two variables, which we denote by $\mathbb{C}[\alpha, \beta]$. The equivariant cohomology of any $\mathbf{S}$-space is an algebra over $\mathbb{C}[\alpha, \beta]$ and for an equivariant map of $\mathbf{S}$-spaces $f : Y \to X$, the pull-back

$$f^* : H_{\mathbf{S}}^*(X) \longrightarrow H_{\mathbf{S}}^*(Y)$$

is a morphism of $\mathbb{C}[\alpha, \beta]$-algebras. In particular, the inclusions

$$i_\lambda : \{p_\lambda\} \longrightarrow \mathrm{Hilb}_n$$

give "restriction to the fixed points" maps

$$i_\lambda^* : H_{\mathbf{S}}^*(\mathrm{Hilb}_n) \longrightarrow H_{\mathbf{S}}^*(\{p_\lambda\}).$$

Since $\mathrm{Hilb}_n$ is "equivariantly formal" for the $\mathbf{S}$-action, it is a general fact that an equivariant cohomology class is uniquely determined by its restrictions to the fixed points.

Furthermore, the bundle $\mathcal{V}$ has a natural linearization (i.e., a lifting of the action to the total space) and we will view it as an $\mathbf{S}$-equivariant bundle via this linearization. It is well-known that the equivariant Chern classes $c_1^{\mathbf{S}}(\mathcal{V}), \ldots, c_n^{\mathbf{S}}(\mathcal{V})$ generate $H_{\mathbf{S}}^*(\mathrm{Hilb}_n)$ as a ring. We denote by $H_1, \ldots H_n$ the equivariant Chern roots of $\mathcal{V}$, so that $c_i^{\mathbf{S}}(\mathcal{V})$ is the $i^{\mathrm{th}}$ elementary symmetric polynomial in the $H_j$'s. Let $\lambda$ be a partition of $n$. The pull-back of $\mathcal{V}$ to the fixed point $p_\lambda$ is just the $\mathbf{S}$-representation space $\mathbb{C}[x, y]/I_\lambda$. The weights of the $\mathbf{S}$-action on it are

$$\{w_B = -(i-1)\alpha - (j-1)\beta \mid B = (i, j) \text{ a box in the Young diagram of } \lambda\}$$

(recall that we are using the French convention for Young diagrams). It follows that if an equivariant cohomology class on $\mathrm{Hilb}_n$ is represented by a symmetric function $f(H_1, \ldots, H_n)$ in the Chern roots, then its restriction to $p_\lambda$ is obtained by evaluating $f$ at the weights $w_B$.

Finally, we recall a version of the Atiyah-Bott localization theorem:

The injective map

$$(i_\lambda)_{\lambda \vdash n} : H_{\mathbf{S}}^*(\mathrm{Hilb}_n) \longrightarrow \oplus_\lambda H_{\mathbf{S}}^*(\{p_\lambda\})$$

becomes an isomorphism after tensoring with the field of fractions $\mathbb{C}(\alpha, \beta)$ over $\mathbb{C}[\alpha, \beta]$.

2.2. **The ADHM construction.** We first recall briefly here the description of $\mathrm{Hilb}_n$ as a GIT quotient due to [ADHM]; complete details can be found in Chapters 1-3 of Nakajima's book [N2], to which the reader is referred.

Fix a complex vector space $V$ of dimension $n$ and put

$$X := \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(V, V) \oplus \mathrm{Hom}(\mathbb{C}, V) \oplus \mathrm{Hom}(V, \mathbb{C}).$$

We view the vector space $X$ as an affine algebraic variety in the usual way. The group $\mathbf{G} := GL(V) \cong GL_n(\mathbb{C})$ acts on $X$ by

$$g \cdot (A, B, i, j) = (gAg^{-1}, gBg^{-1}, gi, jg^{-1}).$$

We consider the linearization of this action (on the trivial bundle $X \times \mathbb{C}$) given by the character

$$\chi : \mathbf{G} \longrightarrow \mathbb{C}^*, \quad \chi(g) = \det(g),$$

so that we have the GIT quotient $X /\!/_\chi \mathbf{G}$. This linearization will be fixed from now on and we drop it from the notation. The affine subvariety $Z$ of $X$ given by the equation $[A, B] + ij = 0$ is $\mathbf{G}$-invariant, hence we have an induced quotient $Z /\!/ \mathbf{G}$. In fact, if we consider $V$ as the natural representation of $\mathbf{G}$ given by $g \cdot v = gv$, then we get an induced bundle $\mathcal{V}_{X /\!/ \mathbf{G}}$ on $X /\!/ \mathbf{G}$ such that $[A, B] + ij$ descends to a section $\sigma$ of $\mathrm{Hom}(\mathcal{V}_{X /\!/ \mathbf{G}}, \mathcal{V}_{X /\!/ \mathbf{G}})$ whose zero locus is precisely $Z /\!/ \mathbf{G}$. Now the ADHM description is as follows:

**Theorem 2.1.** $Z /\!/ \mathbf{G}$ *is identified with* $\mathrm{Hilb}_n$.

Note that for a point $[I] \in \mathrm{Hilb}_n$ we have two natural commuting endomorphisms $A$ and $B$ of $\mathbb{C}[x, y]/I \cong V$ given by the multiplication maps with $x$, respectively $y$, as well as a canonical choice of a vector (corresponding to $i$) in $\mathbb{C}[x, y]/I$ given by (the image of) $1 \in \mathbb{C}[x, y]$. The fact that $A$ and $B$ commute implies that we must take the covector (corresponding to $j$) to be equal to zero. It turns out that this is precisely the identification in the theorem.

Additionally, we note that under the above identification, the vector bundle $\mathcal{V}$ on $\mathrm{Hilb}_n$ is just the restriction of $\mathcal{V}_{X /\!/ \mathbf{G}}$ to $Z /\!/ \mathbf{G}$.

Last, the above construction can in fact be made $\mathbf{S}$-equivariantly. Namely, we let $\mathbf{S}$ act on $X$ by

$$(s_1, s_2) \cdot (A, B, i, j) = (s_1 A, s_2 B, i, s_1 s_2 j).$$

Note that $Z$ is an invariant $\mathbf{S}$-subvariety. Furthermore, the action commutes with the action of $\mathbf{G}$, so it descends to the quotients $X /\!/ \mathbf{G}$ and $Z /\!/ \mathbf{G}$. Under the identification of Theorem 2.1.1, we get precisely the $\mathbf{S}$-action on $\mathrm{Hilb}_n$ described earlier.

We now explain what the general abelian/nonabelian correspondence of [CKS], applied to the ADHM construction, says about the genus zero Gromov-Witten theory of $\mathrm{Hilb}_n$. Let $\mathbf{T} \cong (\mathbb{C}^*)^n$ be the maximal torus of diagonal matrices in $\mathbf{G} = GL(V)$. The linearized action of $\mathbf{G}$ on $X$ induces a linearized action of $\mathbf{T}$ and we have the GIT quotients $X /\!/ \mathbf{T} = X /\!/_\chi \mathbf{T}$ (which is a toric variety) and $Z /\!/ \mathbf{T} = Z /\!/_\chi \mathbf{T}$. Note that when viewed as a $T$-representation, $V$ decomposes completely as a direct sum of one-dimensional irreducibles, and therefore the induced bundle $\mathcal{V}_{X /\!/ \mathbf{T}}$ on $X /\!/ \mathbf{T}$ is a decomposable bundle, i.e., we have

$$\mathcal{V}_{X /\!/ \mathbf{T}} = \oplus_{i=1}^n M_i,$$

with $M_i$ line bundles on $X//\mathbf{T}$. We abuse notation slightly and denote the $\mathbf{S}$-equivariant first Chern classes of these line bundles also by $H_i = c_1^{\mathbf{S}}(M_i), \quad i = 1, ..., n$.

The bundle $\mathrm{Hom}(\mathcal{V}_{X//\mathbf{T}}, \mathcal{V}_{X//\mathbf{T}})$ is also decomposable,

$$\mathrm{Hom}(\mathcal{V}_{X//\mathbf{T}}, \mathcal{V}_{X//\mathbf{T}}) = \oplus_{1 \leq i,j \leq n}(M_i \otimes M_j^{-1}),$$

hence $Z//\mathbf{T}$ is a complete intersection subvariety in the toric variety $X//\mathbf{T}$.

### 2.3. Gromov-Witten theory and the abelian/nonabelian correspondence.

One way to encode part of the genus zero Gromov-Witten theory of a nonsingular projective algebraic variety $Y$ is via a generating formal function called the (small) *J-function* of $Y$, whose coefficients are certain Gromov-Witten invariants.

By results of Givental and others, one has an explicit procedure to obtain the $J$-function of a complete intersection $Y$ in a toric variety $W$ as follows: The $J$-function lies in

$$H^*(Y)[[q_1, ..., q_{b_2(Y)}]]\left[\left[\frac{1}{z}\right]\right].$$

Here, $b_2(Y) = \dim_{\mathbb{Q}} H_2(Y, \mathbb{Q})$ is the second Betti number. It is a general property of these $J$-functions that, as power series in $1/z$, they have expansion of the form

$$(2.1) \qquad\qquad 1 + J^{(2)}\frac{1}{z^2} + J^{(3)}\frac{1}{z^3} + ...$$

(with coefficients cohomology-valued formal power series in the $q$-variables). One writes down, in closed form, another formal function $I$, of hypergeometric type, in the same variables $q_i$ and $z$, directly from the combinatorial data defining the toric variety as a GIT quotient of a vector space by a torus and the equations defining $Y$ in $W$. If the function $I$ happens to have the same asymptotic expansion $(2.1)$ in $z$ as the $J$-function of $Y$, then Givental's theory says that $I = J$. In general, however, $I$ will have expansion

$$I^{(-m)}z^m + \cdots + I^{(-1)}z + I^{(0)} + I^{(1)}\frac{1}{z} + I^{(2)}\frac{1}{z^2} + \cdots$$

and the coefficients $I^{(-m)}, \ldots, I^{(1)}$ provide a change of $q$-variables (the so-called "mirror map") which transforms $I$ into $J$. Finally, the story applies in the case of equivariant Gromov-Witten theory of complete intersections in quasi-projective toric varieties. Due to non-compactness, in this case the $I$ and $J$-functions will still be vector-valued functions of $q$ and $z$, but the cohomology $H^*(Y)$ needs to be replaced by the *localized* equivariant cohomology.

The abelian/nonabelian correspondence of [CKS] provides a conjectural extension of the above procedure to the case of a variety $Y$ which can be realized as the zero locus of a regular section of a homogeneous vector bundle on a GIT quotient $X//\mathbf{G}$ of a vector space $X$ by a reductive group $\mathbf{G}$. Namely, $Y$ corresponds to a complete intersection in the abelian quotient $X//\mathbf{T}$, which is a toric variety. A canonical modification (also of hypergeometric type) of the Givental $I$-function of this complete intersection, followed by a specialization of the $q$-variables gives a function $I_Y$. The conjecture then says that the $J$-function of $Y$ can be obtained from $I_Y$ as in the toric case, by a transformation determined by the $1/z$ asymptotic expansion of $I_Y$.

As described in §2.2, this is precisely the situation we have for $\mathrm{Hilb}_n = Z//\mathbf{G}$. The $\mathbf{S}$-equivariant $J$-function $J_{\mathrm{Hilb}_n}$ lies in

$$\left(H^*_{\mathbf{S}}(\mathrm{Hilb}_n) \otimes_{\mathbb{C}[\alpha,\beta]} \mathbb{C}(\alpha,\beta)\right)[[q]]\left[\left[\frac{1}{z}\right]\right].$$

We write down next the explicit formula for the function $I_{\mathrm{Hilb}_n}$. Introduce the notation

$$\Delta_{\vec{d}}(P, w) := \frac{\prod_{k=-\infty}^{P\cdot\vec{d}}(P + w + kz)}{\prod_{k=-\infty}^{0}(P + w + kz)},$$

where $\vec{d} = (d_1, \ldots, d_n)$ is an $n$-tuple of *nonnegative* integers, $P = a_1 H_1 + \ldots + a_n H_n$ is a linear combination with integer coefficients of the $H_i$'s, while $P \cdot \vec{d} = \sum_{i=1}^n a_i d_i$. Then the recipe provided by the abelian/nonabelian correspondence gives

$$(2.2) \qquad I_{\mathrm{Hilb}_n}(q, z) := 1 + \sum_{d \geq 1} q^d \sum_{\vec{d}:d_1+\cdots+d_n=d} I_{\vec{d}}$$

with

$$(2.3) \qquad I_{\vec{d}} = \prod_{i\neq j} \frac{\Delta_{\vec{d}}(H_i - H_j, \alpha+\beta)\Delta_{\vec{d}}(H_i - H_j, 0)}{\Delta_{\vec{d}}(H_i - H_j, \alpha)\Delta_{\vec{d}}(H_i - H_j, \beta)} \prod_{i=1}^n \frac{1}{\Delta_{\vec{d}}(H_i, 0)\Delta_{\vec{d}}(-H_i, \alpha+\beta)}.$$

Recall that $H_1, \ldots, H_n$ are the Chern roots of the tautological bundle $\mathcal{V}$ on $\mathrm{Hilb}_n$. It is straightforward to check that the $I$-function above is invariant under permutations of the $H_i$'s, hence it can be viewed as an expression in the Chern classes of $\mathcal{V}$ and so it does indeed give an $H^*_{\mathbf{S}}(\mathrm{Hilb}_n) \otimes_{\mathbb{C}[\alpha,\beta]} \mathbb{C}(\alpha,\beta)$-valued function of $q$ and $z$.

We are interested in the $1/z$-expansion of $I_{\mathrm{Hilb}_n}$ and one calculates directly that we have

$$I_{\mathrm{Hilb}_n} = 1 + I^{(1)}\frac{1}{z} + I^{(2)}\frac{1}{z^2} + \ldots$$

with

$$I^{(1)} = \left(\sum_{d\geq 1} \frac{(-1)^d}{d} q^d\right) \sum_{i=1}^n (H_i - (\alpha+\beta)) \prod_{j\neq i} \frac{(H_i - H_j - \alpha)(H_i - H_j - \beta)}{(H_i - H_j)(H_i - H_j - (\alpha+\beta))} =$$

$$= \ln(1+q) \sum_{i=1}^n (H_i - (\alpha+\beta)) \prod_{j\neq i} \frac{(H_i - H_j - \alpha)(H_i - H_j - \beta)}{(H_i - H_j)(H_i - H_j - (\alpha+\beta))}.$$

Since only the coefficient of $1/z$ in this expansion is "wrong", the mirror map in this case turns out to be very simple. Denote by $\gamma_n \in H^*_{\mathbf{S}}(\mathrm{Hilb}_n) \otimes_{\mathbb{C}[\alpha,\beta]} \mathbb{C}(\alpha,\beta)$ the class in localized equivariant cohomology from the expression for $I^{(1)}$ above,

$$(2.4) \qquad \gamma_n := \sum_{i=1}^n (H_i - (\alpha+\beta)) \prod_{j\neq i} \frac{(H_i - H_j - \alpha)(H_i - H_j - \beta)}{(H_i - H_j)(H_i - H_j - (\alpha+\beta))}.$$

The function

$$\exp(-I^{(1)}/z) I_{\mathrm{Hilb}_n}(q, z) = (1+q)^{(-\gamma_n/z)} I_{\mathrm{Hilb}_n}(q, z)$$

has the "correct" expansion (2.1) in $1/z$ and the abelian/nonabelian correspondence can be formulated as

**Conjecture 2.2.** *The* **S***-equivariant J-function of the Hilbert scheme of n points in* $\mathbb{C}^2$ *is given by the formula*

$$J_{\mathrm{Hilb}_n}(q,z) = (1+q)^{(-\gamma_n/z)} I_{\mathrm{Hilb}_n}(q,z),$$

*with* $I_{\mathrm{Hilb}_n}$ *as in (2.2)–(2.3).*

In [CDKM], this conjecture is proved in the following form:

**Theorem 2.3.** *The* **S***-equivariant J-function of the Hilbert scheme of n points in* $\mathbb{C}^2$ *is*

$$J_{\mathrm{Hilb}_n}(q,z) = (1+q)^{(n(\alpha+\beta)/z)} I_{\mathrm{Hilb}_n}(q,z),$$

*with* $I_{\mathrm{Hilb}_n}$ *as in (2.2)–(2.3).*

The proof given there is quite involved. It uses first the results in [OP1], [OP2], together with a geometric argument relating relative and absolute Donaldson-Thomas invariants of $\mathbb{P}^1 \times \mathbb{C}^2$, to show that

$$J_{\mathrm{Hilb}_n}(q,z) = (1+q)^{(n(\alpha+\beta)/z)} F_{DT}(q,z),$$

where $F_{DT}(q,z)$ is a generating function for absolute Donaldson-Thomas invariants. Next, Diaconescu's paper [D] constructs a "moduli space of ADHM sheaves on $\mathbb{P}^1$", and identifies it geometrically, via a relative Beilinson transform, with the Donaldson-Thomas moduli space. This implies an equality $F_{DT} = F_{ADHM}$, where $F_{ADHM}$ is a generating function for invariants providing "virtual" counts of ADHM sheaves on $\mathbb{P}^1$. Finally, a lengthy localization computation shows directly that $F_{ADHM} = I_{\mathrm{Hilb}_n}$ and finishes the proof.

The identity we are concerned with in this paper is the consequence of matching the formulation of Theorem 2.3 with that of Conjecture 2.2. Note that the theorem says in particular that

$$(2.5) \qquad\qquad \gamma_n = -n(\alpha+\beta)$$

in $H^*_{\mathbf{S}}(\mathrm{Hilb}_n) \otimes_{\mathbb{C}[\alpha,\beta]} \mathbb{C}(\alpha,\beta)$, and therefore it does imply the conjecture. Indeed, the coefficient of $1/z$ in the right-hand side of the equality in Theorem 2.3 is

$$(\gamma_n + n(\alpha+\beta)) \ln(1+q),$$

while the coefficient in the left-hand side vanishes by general properties of the $J$-function. However, when viewed in its own right as an equality in the equivariant cohomology of the Hilbert scheme, the identity (2.5) appears quite nontrivial. As explained in §2.1, one way to verify it directly is to check that it holds after restriction to each fixed point. Let $\lambda$ be a partition of $n$ and let $p_\lambda$ be the corresponding fixed point. Recall from §2.1 that the restriction to $p_\lambda$ of an equivariant cohomology class given as a symmetric function in $H_1, \dots, H_n$ is obtained by replacing the $H_i$'s by the weights of the boxes in the Young diagram of $\lambda$. From this and (2.4) we get

$$i^*_{p_\lambda}(\gamma_n) = \sum_{B \text{ a box in } \lambda} (w_B - (\alpha+\beta)) \prod_{C \neq B} \frac{(w_B - w_C - \alpha)(w_B - w_C - \beta)}{(w_B - w_C)(w_B - w_C - (\alpha+\beta))}.$$

On the other hand, we have obviously

$$i^*_{p_\lambda}(-n(\alpha+\beta)) = -n(\alpha+\beta),$$

so (2.5) is equivalent to

$$\sum_{B \text{ a box in } \lambda} (w_B - (\alpha + \beta)) \prod_{C \neq B} \frac{(w_B - w_C - \alpha)(w_B - w_C - \beta)}{(w_B - w_C)(w_B - w_C - (\alpha + \beta))} = -n(\alpha + \beta).$$

The above identity is easily converted into (1.1) from the Introduction. Namely, if $B$ is not a corner of $\lambda$, then the product

$$\prod_{C \neq B, B+(1,1)} \frac{(w_B - w_C - \alpha)(w_B - w_C - \beta)}{(w_B - w_C)(w_B - w_C - (\alpha + \beta))}$$

vanishes: either it has no poles and at least one zero, or a simple pole and two zeros. Furthermore, if $B$ is a corner, there is no need for the restriction $C \neq B + (1,1)$.

## 3. Equivalent formulation of the identity

An *inner corner* of a partition is a square $(i, j)$ in the Young diagram so that $(i+1, j)$ and $(i, j+1)$ are in the Young diagram but $(i+1, j+1)$ is not. Additionally, we count $(\lambda_1, 0)$ and $(0, \ell(\lambda))$ as inner corners, where $\lambda_1$ is the first part of $\lambda$ and $\ell(\lambda)$ is the number of parts of $\lambda$. For example, the inner corners of the partition $(6, 4, 4, 4, 2, 1, 1)$ are $(6, 0), (4, 1), (2, 4), (1, 5), (0, 7)$ and are shown in Figure 3.
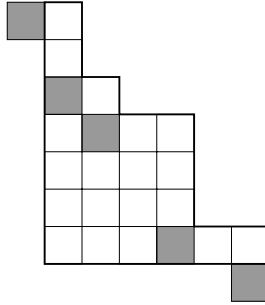


FIGURE 2. Inner corners of a diagram.

Fix a corner $B = (i, j)$ of the partition $\lambda$, and denote

$$(w_B - (\alpha + \beta)) \prod_{C \neq B} \frac{(w_B - w_C - \alpha)(w_B - w_C - \beta)}{(w_B - w_C)(w_B - w_C - (\alpha + \beta))}$$

by $f_B$. A square $C = (i', j') \neq (i, j)$ of $\lambda$ contributes

$$\frac{((i' - i - 1)\alpha + (j' - j)\beta)((i' - i)\alpha + (j' - j - 1)\beta)}{((i' - i - 1)\alpha + (j' - j - 1)\beta)((i' - i)\alpha + (j' - j)\beta)}$$

to $f_B$. Now fix $(i', j') \neq (i, j)$. We have the following cases:

(1) $(i', j')$ is a square in $\lambda$ for which $(i' + 1, j')$, $(i', j' + 1)$ and $(i' + 1, j' + 1)$ are also squares of $\lambda$, and $(i' + 1, j' + 1) \neq (i, j)$. In this case, $(i' - i)\alpha + (j' - j)\beta$ appears twice in the numerator of $f_B$ (for $C = (i' + 1, j')$ and $C = (i', j' + 1)$) and twice in the denominator of $f_B$ (for $C = (i', j')$ and $C = (i' + 1, j' + 1)$). Therefore, $(i' - i)\alpha + (j' - j)\beta$ is not a factor of $f_B$.

(2) $(i', j') \neq (i-1, j)$ is a square in $\lambda$ for which $(i'+1, j')$ is also a square of $\lambda$, but $(i', j'+1)$ is not. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the numerator of $f_B$ (for $C = (i'+1, j')$) and once in the denominator of $f_B$ (for $C = (i', j')$). Therefore, $(i'-i)\alpha + (j'-j)\beta$ is not a factor of $f_B$.

(3) $(i', j') \neq (i, j-1)$ is a square in $\lambda$ for which $(i', j'+1)$ is also a square of $\lambda$, but $(i'+1, j')$ is not. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the numerator of $f_B$ (for $C = (i', j'+1)$) and once in the denominator of $f_B$ (for $C = (i', j')$). Therefore, $(i'-i)\alpha + (j'-j)\beta$ is not a factor of $f_B$.

(4) $(i', j')$ is not a square of $\lambda$, but $(i'+1, j')$ and $(i'+1, j'+1)$ are (we must have $i' = 0$). In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the numerator of $f_B$ (for $C = (i'+1, j')$) and once in the denominator of $f_B$ (for $C = (i'+1, j'+1)$). Therefore, $(i'-i)\alpha + (j'-j)\beta$ is not a factor of $f_B$.

(5) $(i', j')$ is not a square of $\lambda$, but $(i', j'+1)$ and $(i'+1, j'+1)$ are (we must have $j' = 0$). In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the numerator of $f_B$ (for $C = (i', j'+1)$) and once in the denominator of $f_B$ (for $C = (i'+1, j'+1)$). Therefore, $(i'-i)\alpha + (j'-j)\beta$ is not a factor of $f_B$.

(6) $(i', j')$ is a square in $\lambda$ for which $(i'+1, j')$, $(i', j'+1)$ are also squares of $\lambda$, but $(i'+1, j'+1)$ is not. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears twice in the numerator of $f_B$ (for $C = (i'+1, j')$ and $C = (i', j'+1)$) and once in the denominator of $f_B$ (for $C = (i', j')$). Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta$ in the numerator.

(7) $(i', j')$ and $(i'+1, j'+1)$ are not squares in $\lambda$, but $(i'+1, j')$ is. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the numerator of $f_B$ (for $C = (i'+1, j')$) and never in the denominator of $f_B$. Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta$ in the numerator.

(8) $(i', j')$ and $(i'+1, j'+1)$ are not squares in $\lambda$, but $(i', j'+1)$ is. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the numerator of $f_B$ (for $C = (i', j'+1)$) and never in the denominator of $f_B$. Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta$ in the numerator.

(9) $(i', j')$ is a square of $\lambda$, but $(i'+1, j')$ and $(i', j'+1)$ are not. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears once in the denominator of $f_B$ (for $C = (i'+1, j'+1)$) and never in the numerator of $f_B$. Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta$ in the denominator.

(10) $(i', j') = (i-1, j)$. In this case, $(i'-i)\alpha + (j'-j)\beta$ does not appear in the numerator of $f_B$, and it appears once in the denominator of $f_B$ (for $C = (i', j')$). Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta = -\alpha$ in the denominator.

(11) $(i', j') = (i, j-1)$. In this case, $(i'-i)\alpha + (j'-j)\beta$ does not appear in the numerator of $f_B$, and it appears once in the denominator of $f_B$ (for $C = (i', j')$). Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta = -\beta$ in the denominator.

(12) $(i', j') = (i-1, j-1)$. In this case, $(i'-i)\alpha + (j'-j)\beta$ appears twice in the numerator of $f_B$ (for $C = (i'+1, j')$ and $C = (i', j'+1)$) and once in the denominator of $f_B$ (for $C = (i', j')$). Therefore, $f_B$ contains one copy of $(i'-i)\alpha + (j'-j)\beta) = -\alpha - \beta$ in the numerator.

(13) $(i', j') = (0, 0)$. In this case, $(i' - i)\alpha + (j' - j)\beta$ appears once *with a minus sign* in the numerator of $f_B$ (in $w_s + \alpha + \beta$) and once in the denominator of $f_B$ (for $C = (1, 1)$). Therefore, $(i' - i)\alpha + (j' - j)\beta$ is not a factor of $f_B$, and the minus cancels out with the minus from $-\alpha - \beta$ from 12.

Note that the squares of type 9 are the corners of $\lambda$ except for $B$, plus the squares $B - (1, 0)$ and $B - (0, 1)$, and the squares of type 6, 7 and 8 are the inner corners of $\lambda$. See Figure 3 for an example for $\lambda = (7, 5, 4, 4, 1, 1)$ and $s = (4, 4)$.

| 7 | 9 |
|---|---|

| 4 | 3 |
|---|---|

| 4 | 6 | 2 | 10 | ■ |
|---|---|---|---|---|

| 4 | 1 | 1 | 12 | 11 |

| 4 | 1 | 1 | 1 | 6 | 9 |

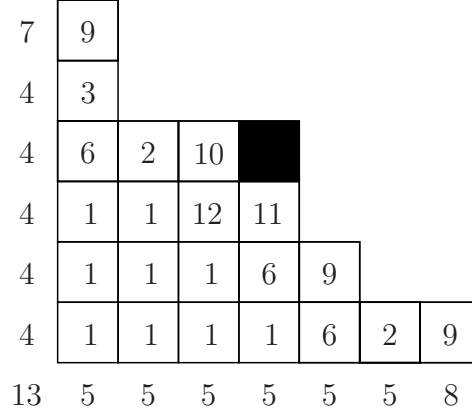| 4 | 1 | 1 | 1 | 1 | 6 | 2 | 9 |

| 13 | 5 | 5 | 5 | 5 | 5 | 5 | 8 |

FIGURE 3. Calculating $f_B$.

There are some special cases (for example, when $\lambda = (1)$ and $(i, j) = (1, 1)$, $(i', j') = (0, 0)$ is considered in the last two cases), but we leave it as an exercise for the reader that Theorem 1.1 is still equivalent to Theorem 1.2 below.

If $\lambda$ has $\ell$ corners, there are $\ell$ different parts of $\lambda$. Let $x_\ell$ denote the smallest part, $x_{\ell-1} + x_\ell$ the second smallest etc., and $x_1 + x_2 + \ldots + x_\ell$ the largest part. Furthermore, suppose that there are $y_\ell$ copies of the smallest part, $y_{\ell-1}$ copies of the second smallest part etc., and $y_1$ copies of the largest part. For example, for $\lambda = (7, 5, 4, 4, 1, 1)$, we have $x_1 = 2, x_2 = 1, x_3 = 3, x_4 = 1$ and $y_1 = 1, y_2 = 1, y_3 = 2, y_4 = 2$. The following figure makes clear the geometric meaning of these numbers.
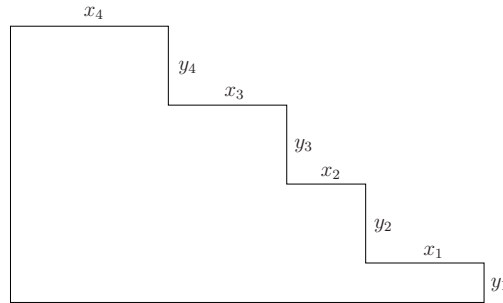


FIGURE 4. A partition and corresponding $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$.

Every collection $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$ of positive integers corresponds to a partition $\lambda$ with $\ell$ corners. Clearly, $[\lambda] = \sum_{1 \le p \le q \le \ell} x_q y_p$.

Number the corners $1, \ldots, \ell$ so that they are increasing in the north-west direction. Number the inner corners $1, \ldots, \ell + 1$ in the same direction. If $B = (i, j)$ is corner $k$ and $C = (i', j')$ is corner $p$, $k > p$, then $(i' - i)\alpha + (j' - j)\beta$ is equal to $(x_p + \ldots + x_{k-1})\alpha - (y_{p+1} + \ldots + y_k)\beta$. If, however, $C = (i', j')$ is corner $q$, $k < q$, then $(i' - i)\alpha + (j' - j)\beta = -(x_k + \ldots + x_{q-1})\alpha + (y_{k+1} + \ldots + y_q)\beta$. Similarly, if $B = (i, j)$ is corner $k$ and $C = (i', j')$ is inner corner $p$, $k \leq p$, then $(i' - i)\alpha + (j' - j)\beta$ is equal to $(x_p + \ldots + x_{k-1})\alpha - (y_p + \ldots + y_k)\beta$, and if $C = (i', j')$ is inner corner $q + 1$, $k \leq q$, then $(i' - i)\alpha + (j' - j)\beta = -(x_k + \ldots + x_q)\alpha + (y_{k+1} + \ldots + y_{q-1})\beta$.

Replace $\beta$ by $-\beta$. After sign cancellations, the cancellation of $\alpha + \beta$ on both sides, and cancellation of $x_k\alpha$ from inner corner $k + 1$ (respectively, $y_k\beta$ from inner corner $k$) with $\alpha$ from $(i', j') = (i - 1, j)$ (respectively, $\beta$ from $(i', j') = (i, j - 1)$) we see that Theorem 1.1 is equivalent to the following statement:

$$\sum_{k=1}^{\ell} x_k y_k \frac{\prod_{p=1}^{k-1}((x_p + \ldots + x_{k-1})\alpha + (y_p + \ldots + y_k)\beta) \cdot \prod_{q=k+1}^{\ell}((x_k + \ldots + x_q)\alpha + (y_{k+1} + \ldots + y_q)\beta)}{\prod_{p=1}^{k-1}((x_p + \ldots + x_{k-1})\alpha + (y_{p+1} + \ldots + y_k)\beta) \cdot \prod_{q=k+1}^{\ell}((x_k + \ldots + x_{q-1})\alpha + (y_{k+1} + \ldots + y_q)\beta)} = \sum_{1 \leq p \leq q \leq \ell} x_q y_p.$$

The top (respectively, bottom) left product corresponds to inner corners (respectively, corners) $p$, $1 \leq p < k$; the top (respectively, bottom) right product corresponds to inner corners (respectively, corners) $q + 1$ (respectively, $q$), $k < q \leq \ell$.

Note that each factor in the numerator on the left-hand side is equal to a factor in the denominator, plus $x_q\alpha$ or $y_p\beta$. Also note that we can replace $x_i\alpha$ by $x_i$ and $y_i\beta$ by $y_i$ and we get an equivalent identity, in which $\alpha$ and $\beta$ do not appear. Also, since this is obviously equivalent to a polynomial identity, an identity for positive integer values of $x_i, y_i$ is equivalent to such an identity for positive real numbers $x_i, y_i$. (In fact, it is equivalent to the same identity for any commuting variables, but the probabilistic spirit of our proof will require that the variables are positive reals.)

This is the reformulation of Theorem 1.1 that we prove in Section 4.

## 4. A PROBABILISTIC PROOF OF THE THEOREM

In this section, we present a weighted version of the Greene-Nijenhuis-Wilf proof of the hook length formula that proves our theorem. The proof here deals only with the staircase shape. A probabilistic proof of the analogous result for arbitrary shapes, as well as a direct bijective proof of this result, is presented in the companion paper [CKP].

We are given $\ell$ and positive real numbers $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$. Select the starting square of the diagram of the staircase shape $\rho_\ell$ so that the probability of selecting the square $(i, j)$ is proportional to $x_{\ell+1-i} y_j$. In each step, move from square $(i, j)$ to a square in the hook of $(i, j)$ (i.e. either strictly to the right or strictly up) so that the probability of moving to square $(k, j)$ is proportional to $x_{\ell+1-k}$, and the probability of moving to square $(i, l)$ is proportional to $y_l$. When we reach a corner, the process ends. We call this a *weighted hook walk on the staircase shape*.

Our goal is to prove the following.

**Proposition 4.1.** *The probability of this random process ending in corner $k$ is equal to*

$$\frac{x_k y_k}{\sum_{1 \le p \le q \le \ell} x_q y_p} \prod_{p=1}^{k-1} \left(1 + \frac{y_p}{x_p + \ldots + x_{k-1} + y_{p+1} + \ldots + y_k}\right) \prod_{q=k+1}^{\ell} \left(1 + \frac{x_q}{x_k + \ldots + x_{q-1} + y_{k+1} + \ldots + y_q}\right).$$

Since any weighted hook walk must end in a corner, this will prove Theorem 1.2.

Assume that the random process is $(i_1, j_1) \to (i_2, j_2) \to \ldots \to (\ell + 1 - k, k)$. Let $I = \{i_1, i_2, \ldots, \ell + 1 - k\}$ and $J = \{j_1, j_2, \ldots, k\}$ be its vertical and horizontal projections.

**Lemma 4.2.** *The probability that the vertical and horizontal projections are $I$ and $J$, conditional on starting at $(i_1, j_1)$, is*

$$\frac{\prod_{i \in I \setminus \{i_1\}} x_{\ell+1-i}}{\prod_{i \in I \setminus \{\ell+1-k\}}(x_k + \ldots + x_{\ell-i} + y_{k+1} + \ldots + y_{\ell+1-i})} \cdot \frac{\prod_{j \in J \setminus \{j_1\}} y_j}{\prod_{j \in J \setminus \{k\}}(x_j + \ldots + x_{k-1} + y_{j+1} + \ldots + y_k)}.$$

*Proof.* The proof is by induction on $|I| + |J|$. Denote the claimed probability by $\prod$. If $I = \{\ell + 1 - k\}$ and $J = \{k\}$, the probability is indeed 1. For $|I| + |J| > 2$, we have

$$\mathbf{P}\big(I, J | S = (i_1, j_1)\big) =$$

$$= \frac{x_{\ell+1-i_2}}{x_{j_1} + \ldots + x_{\ell-i_1} + y_{j_1+1} + \ldots + y_{\ell+1-i_1}} \cdot \mathbf{P}\big(I \setminus \{i_1\}, J | S = (i_2, j_1)\big) +$$

$$+ \frac{y_{j_2}}{x_{j_1} + \ldots + x_{\ell-i_1} + y_{j_1+1} + \ldots + y_{\ell+1-i_1}} \cdot \mathbf{P}\big(I, J \setminus \{j_1\} | S = (i_1, j_2)\big).$$

By the induction hypothesis,

$$\mathbf{P}\big(I \setminus \{i_1\}, J | S = (i_2, j_1)\big) = \frac{x_k + \ldots + x_{\ell-i_1} + y_{k+1} + \ldots + y_{\ell+1-i_1}}{x_{\ell+1-i_2}} \prod,$$

$$\mathbf{P}\big(I, J \setminus \{j_1\} | S = (i_1, j_2)\big) = \frac{x_{j_1} + \ldots + x_{k-1} + y_{j_1+1} + \ldots + y_k}{y_{j_2}} \prod.$$

Because $(x_k + \ldots + x_{\ell-i_1} + y_{k+1} + \ldots + y_{\ell+1-i_1}) + (x_{j_1} + \ldots + x_{k-1} + y_{j_1+1} + \ldots + y_k) = x_{j_1} + \ldots + x_{\ell-i_1} + y_{j_1+1} + \ldots + y_{\ell+1-i_1}$, it follows that $\mathbf{P}\big(I, J | S = (i_1, j_1)\big) = \prod$, which completes the proof. $\qquad\square$

The lemma proves Proposition 4.1. Indeed, if we denote by $S$ the starting corner and by $F$ the final corner of the hook walk, then

$$\mathbf{P}\big(F = (\ell + 1 - k, k)\big) = \sum_{i_1 + j_1 \le \ell+1} \mathbf{P}\big(S = (i_1, j_1)\big) \cdot \mathbf{P}\big(F = (\ell + 1 - k, k) | S = (i_1, j_1)\big) =$$

$$\sum_{i_1, j_1} \frac{x_{\ell+1-i_1} y_{j_1}}{\sum_{1 \le p \le q \le \ell} x_q y_p} \left[\sum \frac{\prod_{i \in I \setminus \{i_1\}} x_{\ell+1-i}}{\prod_{i \in I \setminus \{\ell+1-k\}}(x_k + \ldots + x_{\ell-i} + y_{k+1} + \ldots + y_{\ell+1-i})} \frac{\prod_{j \in J \setminus \{j_1\}} y_j}{\prod_{j \in J \setminus \{k\}}(x_j + \ldots + x_{k-1} + y_{j+1} + \ldots + y_k)}\right]$$

where the last sum is over $I, J$ satisfying $i_1 = \min I$, $\ell + 1 - k = \max I$, $j_1 = \min J$, $k = \max J$. Since

$$x_{\ell+1-i_1} \cdot \prod_{i \in I \setminus \{i_1\}} x_{\ell+1-i} = x_k \cdot \prod_{i \in I \setminus \{k\}} x_{\ell+1-i} \qquad \text{and} \qquad y_{j_1} \cdot \prod_{j \in J \setminus \{j_1\}} y_j = y_k \cdot \prod_{j \in J \setminus \{k\}} y_j,$$

this is equal to

$$\frac{x_k y_k}{\sum_{1 \le p \le q \le \ell} x_q y_p} \cdot \left[ \sum \prod_{i \in I \setminus \{\ell+1-k\}} \frac{x_{\ell+1-i}}{x_k + \ldots + x_{\ell-i} + y_{k+1} + \ldots + y_{\ell+1-i}} \cdot \prod_{j \in J \setminus \{k\}} \frac{y_j}{x_j + \ldots + x_{k-1} + y_{j+1} + \ldots + y_k} \right],$$

where the sum is over all $I, J$ with $\ell + 1 - k = \max I$, $k = \max J$. It is clear that this last sum equals

$$\prod_{i=1}^{\ell+2-k} \left( 1 + \frac{x_{\ell+1-i}}{x_k + \ldots + x_{\ell-i} + y_{k+1} + \ldots + y_{\ell+1-i}} \right) \times \prod_{j=1}^{k-1} \left( 1 + \frac{y_j}{x_j + \ldots + x_{k-1} + y_{j+1} + \ldots + y_k} \right).$$

## 5. Final remarks

**5.1.** In the forthcoming papers [CKP, Kon] we explore several generalizations and variations on the weighted hook walk. We also present a bijective proof of Theorem 1.2, which gives a new combinatorial proof of the hook length formula $(HLF)$, different from those in [FZ, NPS, Rem, Zei].

**5.2.** We have yet to fully understand the nature of Theorem 1.1, both from the algebraic and combinatorial point of view. For example, there is no obvious algebraic connection between the identity in the theorem and the branching rule for irreducible representations of $S_n$. In a different direction, what role to the weights $x_i, y_j$ play in the Gromov-Witten theory?

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