THE WEIGHTED HOOK LENGTH FORMULA

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Abstract. Based on the ideas in [CKP], we introduce the weighted analogue of the branching rule for the classical *hook length formula*, and give two proofs of this result. The first proof is completely bijective, and in a special case gives a new short combinatorial proof of the hook length formula. Our second proof is probabilistic, generalizing the (usual) hook walk proof of Green-Nijenhuis-Wilf [GNW1], as well as the q-walk of Kerov [Ker1]. Further applications are also presented.

Introduction

The classical hook length formula gives a short product formula for the dimensions of irreducible representations of the symmetric group, and is a fundamental result in algebraic combinatorics. The formula was discovered by Frame, Robinson and Thrall in [FRT] based on earlier results of Young [You], Frobenius [Fro] and Thrall [Thr]. Since then, it has been reproved, generalized and extended in several different ways, and applied in a number of fields ranging from algebraic geometry to probability, and from group theory to the analysis of algorithms. Still, the hook length formula remains deeply mysterious and its full depth is yet to be completely understood. This paper is a new contribution to the subject, giving a new multivariable extension of the formula, and a new combinatorial proof associated with it.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ be a partition of n, let $[\lambda]$ be the corresponding Young diagram, and let $\mathrm{SYT}(\lambda)$ denote the set of standard Young tableaux of shape λ (full definitions will be given in the next section). The *hook length formula* for the dimension of the irreducible representation π_λ of the symmetric group S_n can be written as follows:

$$\text{(HLF)} \qquad \dim \pi_\lambda \,=\, |\mathrm{SYT}(\lambda)| \,=\, \frac{n!}{\prod_{x\in[\lambda]} h_x}\,,$$

where the first equality is A. Young's combinatorial interpretation, the product on the right is over all squares x in the Young diagram corresponding to partition λ, and h_x are the hook numbers (see below). In fact, Young's original approach to the first equality hints at the direction of the proof of the second equality. More precisely, he proved the following *branching rules*:

$$\text{(BR)} \qquad \dim \pi_\lambda \,=\, \sum_{\mu\to\lambda} \dim \pi_\mu \quad \text{and} \quad |\mathrm{SYT}(\lambda)| \,=\, \sum_{\mu\to\lambda} |\mathrm{SYT}(\mu)|\,,$$

where the summation is over all partitions μ of $n-1$ whose Young diagram fits inside that of λ (the second branching rule is trivial, of course). Induction now implies the first equality in (HLF).

In a similar way, the hook length formula is equivalent to the following *branching rule for the hook lengths*:

$$\text{(BRHL)} \qquad \sum_{\text{corner } (r,s)\,\in\,[\lambda]} \frac{1}{n} \prod_{i=1}^{r-1} \frac{h_{is}}{h_{is}-1} \prod_{j=1}^{s-1} \frac{h_{rj}}{h_{rj}-1} \,=\, 1\,.$$

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Although this formula is very natural, it is difficult to prove directly, so only a handful of proofs employ it (see below and Subsection 6.2).

In an important development, Green, Nijenhuis and Wilf introduced the *hook walk* which proves (BRHL) by a combination of a probabilistic and a short but delicate induction argument [GNW1]. Zeilberger converted this hook walk proof into a bijective proof of (HLF) [Zei], but lamented on the "enormous size of the input and output" and "the recursive nature of the algorithm" (ibid, §3). With time, several variations of the hook walk have been discovered, most notably the q-version of Kerov [Ker1], and its further generalizations and variations (see [CLPS, GH, Ker2]). Still, before this paper, there were no direct combinatorial proofs of (BRHL).

In this paper we introduce and study the following *weighted branching rule for the hook lengths*:

$$\text{(WHL)} \qquad \sum_{\text{corner}\,(r,s)\,\in\,[\lambda]} x_r y_s \prod_{i=1}^{r-1} \left(1 + \frac{x_i}{x_{i+1} + \ldots + x_r + y_{s+1} + \ldots + y_{\lambda_i}}\right) \times \prod_{j=1}^{s-1} \left(1 + \frac{y_j}{x_{r+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_s}\right) = \sum_{(i,j)\in[\lambda]} x_i y_j \,.$$

Here the weights $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ correspond to the rows and columns of the Young diagram, respectively, so the weight of square (i, j) is $x_i y_j$. Note that (WHL) becomes (BRHL) for the unit weights $x_i = y_j = 1$, and can be viewed both as a probabilistic result (when the weights are positive), and as a rational function identity (when the weights are formal commutative variables).

There is an interesting story behind this formula, as a number of its special cases seem to be well known. Most notably, for the staircase shaped diagrams, Vershik discovered the formula and proved it by a technical inductive argument [Ver]. In this case, an elegant Lagrange interpolation argument was later found by Kirillov [Kir] (see also [Ban, Ker2]), while an algebraic application and a hook walk style proof was recently given by the authors in [CKP]. In a different direction, there is a standard (still multiplicative) q-analogue of (HLF), which can be obtained as the branching rule for the Hall-Littlewood polynomials (see [Mac, §3] for the explicit formulas and references).

There are three main tasks in the paper:

(1) give a direct bijective proof of (BRHL),

(2) prove a weighted analogue (WHL), and

(3) give a hook walk proof of (WHL).

Part (1) is done in Section 2 and is completely self-contained. Part (2) is essentially a simple extension of part (1), based on certain properties of the bijection. The bijection in (1) is robust enough to prove several variations on (BRHL), which all have weighted analogues (Section 3). In a special case this gives certain Kirillov's summation formulas and Kerov's q-formulas in [Ker1], which until now had only analytic proofs.

In Section 4 we define two new walks, a "weighted" and a "modified" hook walk. While both can be viewed as extensions of the usual hook walk, we show that the latter reduces to the former. In fact, the modified hook walk is motivated and implicitly studied in our previous paper [CKP]. The complete proof of (WHL) via the weighted hook walk is then given in Section 5. We conclude with historical remarks and final observations in Section 6.

1. Definitions and notations

An integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a *partition* of n, write $\lambda \vdash n$, if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$, and $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$. From now on, let $\ell = \ell(\lambda)$ denote the number of parts, and let $m = \lambda_1$ denote the length of the largest part of λ. Define the *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ by $\lambda'_j = |\{i : \lambda_i \geq j\}|$.

A *Young diagram* $[\lambda]$ corresponding to λ is a collection of squares $(i, j) \in \mathbb{Z}^2$, such that $1 \leq j \leq \lambda_i$. The *hook* $H_{\mathbf{z}} \subset [\lambda]$ is the set of squares weakly to the right and below of $\mathbf{z} = (i, j) \in [\lambda]$, and the *hook length* $h_{\mathbf{z}} = |H_{\mathbf{z}}| = \lambda_i + \lambda'_j - i - j + 1$ is the size of the hook (see Figure 1).

We say that $(i_1, j_1) \prec (i_2, j_2)$ if $i_1 \leq i_2$, $j_1 \leq j_2$, and $(i_1, j_1) \neq (i_2, j_2)$. A *standard Young tableau* A of shape λ is a bijective map $f : [\lambda] \to [n] = \{1, \ldots, n\}$, such that $f(i_1, j_1) < f(i_2, j_2)$ for all $(i_1, j_1) \prec (i_2, j_2)$. We denote the set of standard Young tableaux of shape λ by $\mathrm{SYT}(\lambda)$. For example, for $\lambda = (3, 2, 2) \vdash 7$, the *hook length formula* (HLF) in the introduction gives:

$$|\mathrm{SYT}(3, 2, 2)| \,=\, \frac{7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1} \,=\, 21.$$

Throughout the paper, we draw a Young diagram with the first coordinate increasing downwards, and the second coordinate increasing from left to right. We then label the rows of the diagram with variables $x_1, x_2, \ldots$, and the columns with variables $y_1, y_2, \ldots$ (see Figure 1). Thus, if the reader prefers the French notation (and standard Descartes coordinates), then a 90° counterclockwise rotation of a diagram is preferable to the mirror reflection as suggested in [Mac].

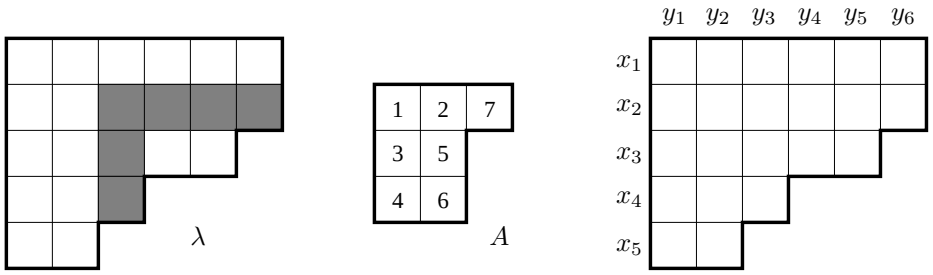

FIGURE 1. Young diagram $[\lambda]$, $\lambda = (6, 6, 5, 3, 2)$, and a hook H_{23} with hook length $h_{23} = 6$; a standard Young tableau A of shape $(3, 2, 2)$; a labeling of rows and columns of λ.

A *corner* of the Young diagram $[\lambda]$ is a square $(i, j) \in [\lambda]$ such that $(i+1, j) \notin [\lambda]$, $(i, j+1) \notin [\lambda]$. Clearly, $(i, j) \in [\lambda]$ is a corner if and only if $h_{ij} = 1$. By $\mathcal{C}[\lambda]$ we denote the set of corners of $[\lambda]$. For example, the diagram $[3, 2, 2]$ has two corners, $(1, 3)$ and $(3, 2)$.

As in the introduction, we write $\mu \to \lambda$ for all $|\mu| = |\lambda| - 1$ such that $[\mu] \subset [\lambda]$. Alternatively, this is equivalent to saying that $[\mu] = [\lambda] \smallsetminus \mathbf{z}$, for some corner $\mathbf{z} \in \mathcal{C}[\lambda]$. Now the *branching rule* (BR) for the standard Young tableaux follows immediately by removing the corner containing n.

2. A new bijective proof of the hook length formula

2.1. **The algebraic setup.** We start by formalizing the induction approach outlined in the introduction. First, observe that to obtain the hook length formula (HLF) by induction it suffices to prove the following identity:

$$\frac{n!}{\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}} \,=\, \sum_{\mu \to \lambda} \frac{(n-1)!}{\prod_{\mathbf{u} \in [\mu]} h_{\mathbf{u}}}\,. \tag{1}$$

Indeed, by the branching rule (BR) for the standard Young tableaux, this immediately gives the induction step:

$$|\mathrm{SYT}(\lambda)| \,=\, \sum_{\mu\to\lambda} |\mathrm{SYT}(\mu)| \,=\, \sum_{\mu\to\lambda} \frac{(n-1)!}{\prod_{\mathbf{u}\in[\mu]} h_{\mathbf{u}}} \,=\, \frac{n!}{\prod_{\mathbf{z}\in[\lambda]} h_{\mathbf{z}}}\,,$$

which proves the (HLF). Rewriting (1), we obtain:

$$1 \,=\, \sum_{\mu\to\lambda} \frac{(n-1)!}{n!}\frac{\prod_{\mathbf{z}\in\lambda} h_{\mathbf{z}}}{\prod_{\mathbf{u}\in[\mu]} h_{\mathbf{u}}} \,=\, \sum_{(r,s)\in\mathcal{C}[\lambda]} \frac{1}{n}\prod_{i=1}^{r-1}\frac{h_{is}}{h_{is}-1}\prod_{j=1}^{s-1}\frac{h_{rj}}{h_{rj}-1}\,. \tag{2}$$

Multiplying both sides of (2) by the common denominator, we get the following equivalent identity:

$$n\cdot\prod_{\mathbf{z}\in[\lambda]\smallsetminus\mathcal{C}[\lambda]}(h_{\mathbf{z}}-1) \,=\, \sum_{(r,s)\in\mathcal{C}[\lambda]}\prod_{i=1}^{r-1} h_{is}\prod_{j=1}^{s-1} h_{rj}\prod_{\mathbf{z}\in\mathcal{D}_{rs}[\lambda]}(h_{\mathbf{z}}-1)\,, \tag{3}$$

where the last product is over the set

$$\mathcal{D}_{rs}[\lambda] \,=\, \{(i,j)\in[\lambda]\smallsetminus\mathcal{C}[\lambda],\ \text{such that } i\neq r,\, j\neq s\}.$$

Below we prove the following multivariable extension of this identity:

$$\begin{aligned}
&\left[\sum_{(p,q)\in[\lambda]} x_p y_q\right]\cdot\left[\prod_{(i,j)\in[\lambda]\smallsetminus\mathcal{C}[\lambda]}\Big(x_{i+1}+\ldots+x_{\lambda'_j}+\,y_{j+1}+\ldots+y_{\lambda_i}\Big)\right]\\
&\quad=\sum_{(r,s)\in\mathcal{C}[\lambda]} x_r y_s\left[\prod_{(i,j)\in\mathcal{D}_{rs}[\lambda]}\Big(x_{i+1}+\ldots+x_{\lambda'_j}+\,y_{j+1}+\ldots+y_{\lambda_i}\Big)\right]\\
&\quad\times\left[\prod_{i=1}^{r-1}(x_i+\ldots+x_r+\,y_{s+1}+\ldots+y_{\lambda_i})\right]\cdot\left[\prod_{j=1}^{s-1}\Big(y_j+\ldots+y_s+\,x_{r+1}+\ldots+x_{\lambda'_j}\Big)\right]
\end{aligned} \tag{4}$$

Clearly, when $x_1=x_2=\ldots=y_1=y_2=\ldots=1$, we obtain (3).[1] Note also that both sides are homogenous polynomials of degree $d_\lambda=|\lambda|+2-\big|\mathcal{C}[\lambda]\big|$.

2.2. **The bijection.** Now we present a bijective proof of (4), by interpreting both sides as certain sets of arrangements of labels (see Section 1).

For the l.h.s. of (4), we are given:

- special labels x_p, y_q, corresponding to the first summation $\sum_{(p,q)\in[\lambda]} x_p y_q$;
- a label x_k for some $i<k\le\lambda'_j$, or y_l for some $j<l\le\lambda_i$, in every non-corner square (i,j).

Denote by F the resulting arrangement of d_λ labels (see Figure 2, first diagram), and by $\mathcal{F}_\lambda$ the set of such labeling arrangements F.

For the r.h.s. of (4), we are given

- special labels x_r, y_s, corresponding to the corner (r,s);
- a label x_k for some $i<k\le\lambda'_j$, or y_l for some $j<l\le\lambda_i$, in every non-corner square (i,j), $i\neq r$, $j\neq s$;
- a label x_k for some $i\le k\le\lambda'_j$, or y_l for some $s<l\le\lambda_i$, in every non-corner square (i,s);
- a label x_k for some $r<k\le\lambda'_j$, or y_l for some $j\le l\le\lambda_i$, in every non-corner square (r,j).

[1] In fact, equation (4) immediately implies (WHL), but more on this in the next section.

Denote by G the resulting arrangement of d_λ labels (see Figure 2, last diagram), and by $\mathcal{G}_\lambda$ the set of such labeling arrangements G. The bijection $\varphi : F \mapsto G$ is now defined by rearranging the labels.

Direct bijection $\varphi : \mathcal{F}_\lambda \to \mathcal{G}_\lambda$.
We can interpret the special labels x_p, y_q as the starting square (p, q). Furthermore, we can interpret all other labels as arrows pointing to a square in the hook. More specifically, if the label in square (i, j) is x_k, the arrow points to (k, j), and if the label is y_l, the arrow points to (i, l).

Let the arrow from square (p, q) point to a square (p', q') in the hook $H_{pq} \setminus \{(p, q)\}$, the arrow from (p', q') point to a square $(p'', q'') \in H_{p'q'} \setminus \{(p', q')\}$, etc. Iterating this, we eventually obtain a *hook walk* W which reaches a corner $(r, s) \in \mathcal{C}[\lambda]$ (see Figure 2, second diagram).

Shade row r and column s. Now we shift the labels in the hook walk and in its projection onto the shaded row and column. If the hook walk has a horizontal step from (i, j) to (i, j'), move the label in (i, j) right and down from (i, j) to (r, j'), and the label from (r, j) up to (i, j). If the hook walk has a vertical step from (i, j) to (i', j), move the label from (i, j) down and right to (i', s), and the label from (i, s) left to (i, j). Finally, move the label x_p to (p, s), the label y_q to (r, q), the label x_r to $(r, 0)$, and the label y_s to $(0, s)$. See Figure 2, third diagram. We denote by G the resulting arrangement of labels (Figure 2, fourth diagram).

We now have labels in all non-corner squares, and special labels x_r and y_s corresponding to the corner (r, s). We claim that $G \in \mathcal{G}_\lambda$. Indeed, if there is a horizontal step in the hook walk from (i, j) to (i, j'), that means that the label in (i, j) is $y_{j'}$, and then the new label in (r, j') is $y_{j'}$; since the label in that square should be x_k for some $r < k \le \lambda'_j$, or y_l for some $j' \le l \le \lambda_i$, this is acceptable. Also, the new label in (i, j) is the old label from (r, j), so it is either x_k for $k > r \ge i$ or y_l for $l > j$; both are acceptable. The case when the step is vertical is analogous.

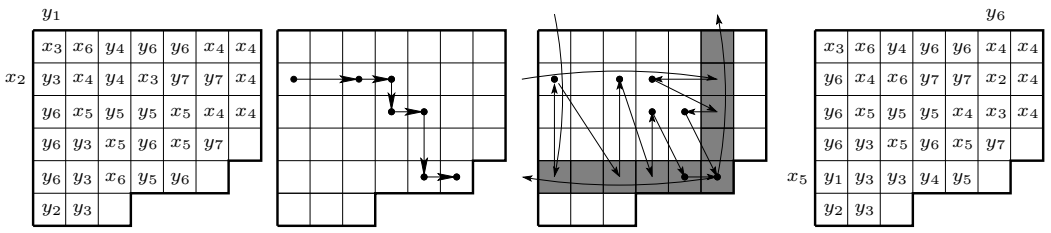

FIGURE 2. An example of an arrangement corresponding to the left-hand side of WBR for $\lambda = 777763$; hook walk; shift of labels; final arrangement.

Lemma 1. *The map $\varphi : \mathcal{F}_\lambda \to \mathcal{G}_\lambda$ defined above is a bijection.*

The lemma follows from the construction of the inverse map.

Inverse bijection $\varphi^{-1} : \mathcal{G}_\lambda \to \mathcal{F}_\lambda$.
Start with G and shade the row and column of $[\lambda]$ corresponding to the two special labels x_r and y_s, where (r, s) is the given corner. Recall from the construction of φ that the projections of W onto the shaded row are the squares (r, j) with label y_j, and the projections of W onto the shaded column are the squares (i, s) with label x_i. Clearly, the smallest such i and j give the special labels x_p, y_q (if no such i and/or j exists, take $p = r$ and/or $q = s$). Suppose that the label in square (p, q) is x_k for $k > p$. If $k \le r$, then x_k is an acceptable label for the square (p, s) (and not for (r, q)). If $k > r$, then it is an acceptable label for (r, q) (and not for (p, s)). On the other hand, if the label in (p, q) is y_l for $l > q$, then y_l is an acceptable label for (r, q) if $l \le s$ and an acceptable label for (p, s) if $l > s$. Therefore, the label at (p, q) determines in which direction from (p, q) the step of the walk W is made.

Now find the next square in that direction whose projections onto shaded row and column are in the projections of W, and repeat the procedure. At the end we obtain the whole walk W. Then simply undo the shifting of labels described in the construction of φ.

A straightforward check shows that this is indeed the initial label arrangement F. This implies the lemma and completes the proof of (4) and of the hook length formula (HLF). $\square$

3. Weighted branching rule for the hook lengths

3.1. **Main theorem.** The main result of this paper can be summarized in one theorem:

Theorem 2. *Fix a partition λ. For commutative variables x_i, y_j, write*

$$\prod\nolimits_{rs} = \; x_r\, y_s \prod_{i=1}^{r-1} \Big(1 + \tfrac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\Big) \cdot \prod_{j=1}^{s-1} \Big(1 + \tfrac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\Big).$$

Then we have the following rational function identities:

(a) $\sum_{(r,s)} \prod_{rs} = \sum_{(p,q)\in[\lambda]} x_p y_q$

(b) $\sum_{(r,s)} \frac{1}{x_{r+1}+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_s} \cdot \prod_{rs} = \sum_{p=1}^{\ell(\lambda)} x_p$

(c) $\sum_{(r,s)} \frac{1}{x_1+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_1}} \cdot \prod_{rs} = \sum_{q=1}^{\lambda_1} y_q$

(d) $\sum_{(r,s)} \frac{1}{(x_{r+1}+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_s)(x_1+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_1})} \cdot \prod_{rs} = 1$

Proof. It is clear that we get part (a) from equation (4) by dividing both sides by the expression $\prod_{(i,j)\in[\lambda]\smallsetminus\mathcal{C}[\lambda]} (x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i})$. Identity (b) is equivalent to

$$\begin{aligned} &\left[\sum_{p=1}^{\ell(\lambda)} x_p\right] \cdot \left[\prod_{(i,j)\in[\lambda]\smallsetminus\mathcal{C}[\lambda]} \Big(x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i}\Big)\right] \\ &= \sum_{(r,s)\in\mathcal{C}[\lambda]} x_r \left[\prod_{(i,j)\in\mathcal{D}_{rs}[\lambda]} \Big(x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i}\Big)\right] \\ &\quad \times \left[\prod_{i=1}^{r-1} (x_i + \ldots + x_r + y_{s+1} + \ldots + y_{\lambda_i})\right] \cdot \left[\prod_{j=2}^{s} \Big(y_j + \ldots + y_s + x_{r+1} + \ldots + x_{\lambda'_j}\Big)\right] \end{aligned} \tag{5}$$

Let us show that by analogy with (4), this identity can be proved by using the bijection φ. The left-hand side of (5) corresponds to arrangements as in the left-hand side of (4) with an additional label x_p. Similarly, the right hand side of (5) corresponds to arrangements as in the right-hand side of (4), except the square $(r, 1)$ does not get a label. Start the hook walk in square $(p, 1)$ and proceed as in the proof of (4). Now observe that the bijection φ gives the bijection between these sets of label arrangements. We omit the easy details.

Identity (c) follows from (b) by conjugation, and (d) can be rewritten in the following form:

$$\begin{aligned} &\left[\prod_{(i,j)\in[\lambda]\smallsetminus\mathcal{C}[\lambda]} \Big(x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i}\Big)\right] \\ &= \sum_{(r,s)\in\mathcal{C}[\lambda]} \left[\prod_{(i,j)\in\mathcal{D}_{rs}[\lambda]} \Big(x_{i+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_{\lambda_i}\Big)\right] \\ &\quad \times \left[\prod_{i=2}^{r} (x_i + \ldots + x_r + y_{s+1} + \ldots + y_{\lambda_i})\right] \cdot \left[\prod_{j=2}^{s} \Big(y_j + \ldots + y_s + x_{r+1} + \ldots + x_{\lambda'_j}\Big)\right] \end{aligned} \tag{6}$$

We prove (6) in a similar way. Start the walk in square $(1,1)$ and proceed as above. Observe that in this case, we do not get a label in squares $(r,1)$ and $(1,s)$. The bijection φ, restricted to this set of label arrangements, proves the equality. We omit the easy details. □

3.2. **The q-version.** In [Ker1], Kerov proved the following identities.[2]

Corollary 3 (Kerov)**.** *Fix a pair of sequences of reals $X_1,\dots,X_d$ and $Y_0,\dots,Y_d$ such that $Y_0 < X_1 < Y_1 < X_2 < \ldots < X_d < Y_d$. Define*

$$\pi_k(q) = \prod_{i=1}^{k-1} \frac{q^{Y_i}-q^{X_k}}{q^{X_i}-q^{X_k}} \prod_{i=k+1}^{d} \frac{q^{X_k}-q^{Y_{i-1}}}{q^{X_k}-q^{X_i}}, \qquad 1 \le k \le d$$

$$Z = \sum_{i=1}^{d} q^{X_i} - \sum_{i=1}^{d-1} q^{Y_i}, \qquad S = \sum_{1\le i\le j\le d} (q^{Y_{i-1}}-q^{X_i})(q^{X_j}-q^{Y_j}).$$

Then:

(a) $\displaystyle\sum_k \pi_k(q) = 1$

(b) $\displaystyle\sum_k \frac{q^{Y_0}-q^{X_k}}{q^{Y_0}-Z}\,\pi_k(q) = 1$

(c) $\displaystyle\sum_k \frac{q^{X_k}-q^{Y_d}}{Z-q^{Y_d}}\,\pi_k(q) = 1$

(d) $\displaystyle\sum_k \frac{(q^{Y_0}-q^{X_k})(q^{X_k}-q^{Y_d})}{S}\,\pi_k(q) = 1$

Proof. The formulas follow by setting

$$x_i = q^{X_i} - q^{Y_{i-1}} \qquad y_j = q^{Y_{d+1-j}} - q^{X_{s+1-j}}$$

and taking equations (a)–(d) from Theorem 2 for the staircase partition $\lambda = (d, d-1, \dots, 1)$.

In (a), let $r = k$, $s = d+1-k$, $\lambda_i = d+1-i$, $\lambda'_j = d+1-j$. We have

$$x_{i+1} + \ldots + x_r = \left(q^{X_{i+1}} - q^{Y_i}\right) + \ldots + \left(q^{X_k} - q^{Y_{k-1}}\right)$$

and

$$y_{s+1} + \ldots + y_{\lambda_i} = \left(q^{Y_i} - q^{X_i}\right) + \ldots + \left(q^{Y_{k-1}} - q^{X_{k-1}}\right).$$

That means that

$$\prod_{i=1}^{r-1} \left(1 + \tfrac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\right) = \prod_{i=1}^{k-1} \left(1 + \tfrac{q^{X_i}-q^{Y_{i-1}}}{q^{X_k}-q^{X_i}}\right) = \prod_{i=1}^{k-1} \tfrac{q^{Y_{i-1}}-q^{X_k}}{q^{X_i}-q^{X_k}}.$$

Similarly,

$$\prod_{j=1}^{s-1} \left(1 + \tfrac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\right) = \prod_{j=1}^{d-k} \tfrac{q^{Y_{d+1-j}}-q^{X_k}}{q^{X_{d+1-j}}-q^{X_k}} = \prod_{i=k+1}^{d} \tfrac{q^{X_k}-q^{Y_i}}{q^{X_k}-q^{X_i}}.$$

We also have

$$\tfrac{x_r y_s}{(x_{r+1}+\ldots+x_{\ell(\lambda)}+y_1+\ldots+y_s)(x_1+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_1})} = \tfrac{\left(q^{X_k}-q^{Y_{k-1}}\right)\left(q^{Y_k}-q^{X_k}\right)}{\left(q^{Y_d}-q^{X_k}\right)\left(q^{X_k}-q^{Y_0}\right)}.$$

Together with the identity (d) in Theorem 2, this implies

$$1 = \sum_{k=1}^{d} \tfrac{\left(q^{Y_{k-1}}-q^{X_k}\right)\left(q^{X_k}-q^{Y_k}\right)}{\left(q^{X_k}-q^{Y_d}\right)\left(q^{Y_0}-q^{X_k}\right)} \prod_{i=1}^{k-1} \tfrac{q^{Y_{i-1}}-q^{X_k}}{q^{X_i}-q^{X_k}} \prod_{i=k+1}^{d} \tfrac{q^{X_k}-q^{Y_i}}{q^{X_k}-q^{X_i}} =$$

[2]Let us note that this is a corrected version of the theorem as the original contained a typo.

$$= \sum_{k=1}^{d} \prod_{i=1}^{k-1} \frac{q^{Y_i}-q^{X_k}}{q^{X_i}-q^{X_k}} \prod_{i=k+1}^{d} \frac{q^{X_k}-q^{Y_{i-1}}}{q^{X_k}-q^{X_i}} = \sum_{k=1}^{d} \pi_k(q),$$

as desired. The proof of identities (b)–(d) follows the same lines. □

4. Weighted and modified hook walks

4.1. **Weighted hook walk.** Fix a partition λ and positive weights $x_1, \ldots, x_{\lambda'_1}, y_1, \ldots, y_{\lambda_1}$. Consider the following combinatorial random process. Select the starting square $(i,j) \in [\lambda]$ with probability proportional to $x_i y_j$. At each step, move from square (i,j) to a random square in $H_{ij} \setminus \{(i,j)\}$ so that the probability of moving to the square (k,j), $i < k \le \lambda'_j$, is proportional to x_k, and the probability of moving to the square (i,l), $j < l \le \lambda_i$, is proportional to y_l. When we reach a corner, the process ends. We call this a *weighted hook walk*.

Theorem 4. *The probability that the weighted hook walk stops in the corner (r,s) of λ is equal to*

$$\frac{x_r y_s}{\sum_{(p,q)\in[\lambda]} x_p y_q} \prod_{i=1}^{r-1} \Big(1 + \frac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\Big) \prod_{j=1}^{s-1} \left(1 + \frac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\right)$$

Note that the sum of these products over all $(r,s) \in \mathcal{C}[\lambda]$ is equal to the ratio of the left-hand side and the right-hand side of Theorem 2, part (a). Since the sum of these probabilities over all corners is equal to 1, we conclude that Theorem 4 implies (WHL). We prove Theorem 4 in the next section by an inductive argument. From above, this gives an alternative proof of (WHL).

4.2. **Modified weighted hook walk.** Take a square (i,j) in $[\lambda]$, and find the corner (r_1,s_1) with the smallest r_1 satisfying $r_1 \ge i$, and the corner (r_2,s_2) with the smallest s_2 satisfying $s_2 \ge j$. The *modified hook* is the set $\{(k,j)\colon r_1 < k \le \lambda'_j\} \cup \{(i,l)\colon s_2 < l \le \lambda_i\}$. An example is given in Figure 3.

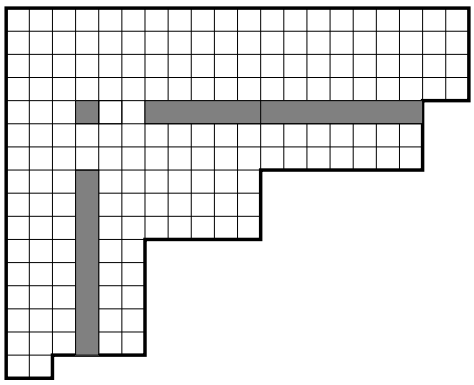

Figure 3. The square $(5,4)$ of the diagram and its modified hook of length 20 in the partition $(20,20,20,20,18,18,18,11,11,11,6,6,6,6,6,2)$.

Recall that we have positive weights $x_1, \ldots, x_{\lambda'_1}, y_1, \ldots, y_{\lambda_1}$. Select the starting square $(i,j) \in [\lambda]$ with probability proportional to $x_i y_j$. At each step, move from square (i,j) to a random square in the modified hook so that the probability of moving to the square (k,j) is proportional to x_k, and the probability of moving to the square (i,l) is proportional to y_l. When we reach a corner, the process ends. We call this a *modified weighted hook walk*.

If λ has c corners, there are c different parts of λ, and also c different parts of λ'. Take the ordered set partition $(U_1, \ldots, U_c)$ of the set $\{1,2,\ldots,\lambda'_1\}$ so that i and j are in the same subset if and only $\lambda_i = \lambda_j$, and so that the elements of the set U_k are smaller than the elements of the

set U_l if $k < l$. Then define X_k as the sum of the elements of U_k. Similarly, take the ordered set partition $(V_1, \ldots, V_c)$ of the set $\{1, 2, \ldots, \lambda_1\}$ so that i and j are in the same subset if and only $\lambda'_i = \lambda'_j$, and so that the elements of the set V_k are smaller than the elements of the set V_l if $k < l$. Then define Y_k as the sum of the elements of U_k.

In the example given in Figure 4, we have $X_1 = x_1 + x_2 + x_3 + x_4, X_2 = x_5 + x_6 + x_7, X_3 = x_8 + x_9 + x_{10}, X_4 = x_{11} + x_{12} + x_{13} + x_{14} + x_{15} +, X_5 = x_{16}, Y_1 = y_1 + y_2, Y_2 = y_3 + y_4 + y_5 + y_6, Y_3 = y_7 + y_8 + y_9 + y_{10} + y_{11}, Y_4 = y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18}$, and $Y_5 = y_{19} + y_{20}$.

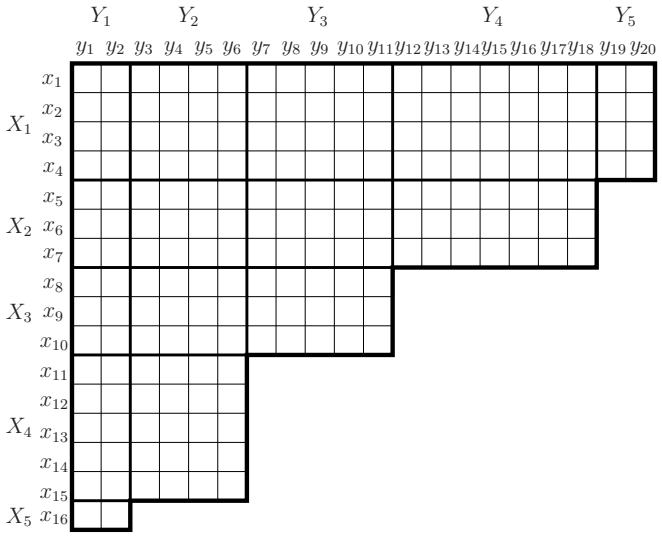

FIGURE 4. The partition $(20, 20, 20, 20, 18, 18, 18, 11, 11, 11, 6, 6, 6, 6, 6, 2)$ and corresponding sums $X_1, \ldots, X_5, Y_1, \ldots, Y_5$.

Let us number the corners so that the top right corner is the first and the bottom left corner is the last.

Theorem 5. *The probability that a modified weighted hook walk ends in corner r is equal to*

$$\frac{X_r Y_s}{\sum_{(p,q)\in[\lambda]} x_p y_q} \prod_{i=1}^{r-1} \left(1 + \frac{X_i}{X_{i+1}+\ldots+X_r+Y_{s+1}+\ldots+Y_{c+1-i}}\right) \prod_{j=1}^{s-1} \left(1 + \frac{Y_j}{X_{r+1}+\ldots+X_{c+1-j}+Y_{j+1}+\ldots+Y_s}\right),$$

where $s = c + 1 - r$.

Proof. Observe that the modified weighted hook walk is equivalent to the (ordinary) weighted hook walk on the staircase shape $(c, c-1, \ldots, 1)$, where the k-th row is weighted by the sumX_k, and the l-th column is weighted by the sum Y_l. The formula then follows from Theorem 4 and the equality $\sum_{p+q\le c+1} X_p Y_q = \sum_{(p,q)\in[\lambda]} x_p y_q$. □

5. The hook walk proof

What follows is an adaptation of the Greene-Nijenhuis-Wilf proof [GNW1]. Assume that the random process is $(i_1, j_1) \to (i_2, j_2) \to \ldots \to (r, s)$. Then let $I = \{i_1, i_2, \ldots, r\}$ and $J = \{j_1, j_2, \ldots, s\}$ be its vertical and horizontal projections.

Lemma 6. *The probability that the vertical and horizontal projections are I and J, conditional on starting at (i_1, j_1), is*

$$\frac{\prod_{i\in I\setminus\{i_1\}} x_i}{\prod_{i\in I\setminus\{r\}}(x_{i+1} + \ldots + x_r + y_{s+1} + \ldots + y_{\lambda_i})} \cdot \frac{\prod_{j\in J\setminus\{j_1\}} y_j}{\prod_{j\in J\setminus\{s\}}(x_{r+1} + \ldots + x_{\lambda'_j} + y_{j+1} + \ldots + y_s)}.$$

The lemma implies Theorem 4. Indeed, if we denote by S the starting corner and by F the final corner of the hook walk, then

$$\mathbf{P}\big(F=(r,s)\big)=\sum_{(i_1,j_1)\in[\lambda]}\mathbf{P}\big(S=(i_1,j_1)\big)\cdot\mathbf{P}\big(F=(r,s)|S=(i_1,j_1)\big)=$$

$$\sum_{i_1,j_1}\frac{x_{i_1}y_{j_1}}{\sum_{(p,q)\in[\lambda]}x_py_q}\left[\sum\frac{\prod_{i\in I\setminus\{i_1\}}x_i}{\prod_{i\in I\setminus\{r\}}(x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i})}\cdot\frac{\prod_{j\in J\setminus\{j_1\}}y_j}{\prod_{j\in J\setminus\{s\}}(x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s)}\right],$$

where the last sum is over I,J satisfying $i_1=\min I$, $r=\max I$, $j_1=\min J$, $s=\max J$. Since

$$x_{i_1}\cdot\prod_{i\in I\setminus\{i_1\}}x_i=x_r\cdot\prod_{i\in I\setminus\{r\}}x_i\qquad\text{and}\qquad y_{j_1}\cdot\prod_{j\in J\setminus\{j_1\}}y_j=y_s\cdot\prod_{j\in J\setminus\{s\}}y_j,$$

this is equal to

$$\frac{x_ry_s}{\sum_{(p,q)\in[\lambda]}x_py_q}\cdot\left[\sum\prod_{i\in I\setminus\{r\}}\frac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\cdot\prod_{j\in J\setminus\{s\}}\frac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\right],$$

where the sum is over all I,J with $r=\max I$, $s=\max J$. It is clear that this last product equals

$$\prod_{i=1}^{r-1}\left(1+\frac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\right)\times\prod_{j=1}^{s-1}\left(1+\frac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\right).$$

Proof of Lemma 6. The proof is by induction on $|I|+|J|$. Denote the claimed probability by $\prod$. If $I=\{r\}$ and $J=\{s\}$, the probability is indeed 1. For $|I|+|J|>2$, we have

$$\mathbf{P}\big(I,J|S=(i_1,j_1)\big)$$

$$=\frac{x_{i_2}}{x_{i_1+1}+\ldots+x_{\lambda'_{j_1}}+y_{j_1+1}+\ldots+y_{\lambda_{i_1}}}\cdot\mathbf{P}\big(I\setminus\{i_1\},J|S=(i_2,j_1)\big)$$

$$+\frac{y_{j_2}}{x_{i_1+1}+\ldots+x_{\lambda'_{j_1}}+y_{j_1+1}+\ldots+y_{\lambda_{i_1}}}\cdot\mathbf{P}\big(I,J\setminus\{j_1\}|S=(i_1,j_2)\big).$$

By the induction hypothesis,

$$\mathbf{P}\big(I\setminus\{i_1\},J|S=(i_2,j_1)\big)=\frac{x_{i_1+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_{i_1}}}{x_{i_2}}\prod,$$

$$\mathbf{P}\big(I,J\setminus\{j_1\}|S=(i_1,j_2)\big)=\frac{x_{r+1}+\ldots+x_{\lambda'_{j_1}}+y_{j_1+1}+\ldots+y_s}{y_{j_2}}\prod.$$

Because $(x_{i_1+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_{i_1}})+(x_{r+1}+\ldots+x_{\lambda'_{j_1}}+y_{j_1+1}+\ldots+y_s)=x_{i_1+1}+\ldots+x_{\lambda'_{j_1}}+y_{j_1+1}+\ldots+y_{\lambda_{i_1}}$, it follows that $\mathbf{P}\big(I,J|S=(i_1,j_1)\big)=\prod$, which completes the proof. □

6. Final remarks

6.1. As Knuth wrote in 1973, "Since the hook-lengths formula is such a simple result, it deserves a simple proof ..." (see p. 63 of the first edition of [Knu], cited also in [Zei]). Unfortunately, the desired simple proofs have been sorely lacking. It is our hope that Section 2 can be viewed as one such proof.

6.2. Surveying the history of the hook length formula is a difficult task, even if one is restricted to purely combinatorial proofs. This is further complicated by the ambiguity of the notions, since it is often unclear whether a given technique is bijective or even combinatorial. Below we give a brief outline of some important developments, possibly omitting a number of interesting and related papers.[3]

The first breakthrough in the understanding of the role of hooks was made by Hillman and Grassl in [HG], where they proved the (special case of) Stanley hook-content formula by an elegant bijection. It is well known that this formula implies the hook length formula via the P-partition theory [Sta, §4] (see also [Pak]). This approach was further developed in [BD, Gan2, KP, Kra1, Kra2, Kra3]. Let us mention also papers [Gan1, Pak], where the connection to the Robinson-Schensted-Knuth correspondence (see e.g. [Sta, §7]) was established, and a recent follow up [BFP] with further variations and algorithmic applications.

The next direction came in [GNW1], where an inductive proof was established based on an elegant probabilistic argument. This in turn inspired a number of further developments, including [GH, GNW2, Ker1, Ker2], and most recently [CLPS, CKP]. In fact, the underlying hook length identities leading to the proof have been also studied directly, without the probabilistic interpretation; we refer to [Ver] and later developments [Ban, GN, Ker2, Kir]. Needless to say, our two proofs can be viewed as direct descendants of these two interrelated approaches.

As we mentioned in the introduction, an important breakthrough was made by Zeilberger, who found a "direct bijectation" of the GNW hook walk proof [Zei]. In fact, his proof has several similar bijective steps as our proof, but differs in both in technical details and the general scheme, being an involved bijection of (HLF) rather than (BRHL).

Historically, the first bijective proof of the hook length formula is due to Remmel [Rem] (see also [RW]). Essentially, he uses the standard algebraic proof of Young (of the Frobenius-Young product formula for $\dim \pi_\lambda$) and the Frame-Robinson-Thrall argument, and replaces each step with a bijective version (sometimes by employing new bijections and at one key step he uses the Gessel-Viennot involution on intersecting paths [GV]). He then repeatedly applies the celebrated Garsia-Milne involution principle to obtain an ingenious but completely intractable bijection (a related approach was later outlined in [GV] as well).

Finally, there are two direct bijective proofs of the hook length formula: [FZ] and [NPS], both of which are highly non-trivial, with the second using a variation on the jeu-de-taquin algorithm (see [Sta, §7]). We refer to [Sag2] for a nice and careful presentation of the NPS bijection, and to [Knu] for an elegant concise version.

6.3. There are several directions in which our results can be potentially extended. First, it would be interesting to obtain the analogues of our results for the *shifted* Young diagrams and Young tableaux, for which there is an analogue of the hook length formula due to Thrall [Thr] (see also [Sag2]). We refer to [Ban, Fis, Kra1, Sri] for other proofs of the HLF in this case, and, notably, to [Sag1] for the shifted hook walk proof. We intend to return to this problem in the future. Let us mention that a weighted version of the branching rule for trees is completely straightforward.

Extending to semi-standard and skew tableaux is another possibility, in which case one would be looking for a weighted analogue of Stanley's hook-content formula [Sta] (see also [Mac]).

In a different direction, the weighted analogue of the "complementary hook walk" in [GNW2] was discovered recently by the second author [Kon]. The paper [GNW2] is based on the observation that the Burnside identity

$$\sum_{\lambda \vdash n} |\mathrm{SYT}(\lambda)|^2 = n!$$

[3]We apologize in advance to the authors of the papers we do not mention; the literature is simply too big to be fully surveyed here.

is equivalent to the identity

$$\prod_{\mathbf{z}\in[\lambda]} (h_{\mathbf{z}}+1) \,=\, \sum_{(r,s)\in\mathcal{C}'[\lambda]} \prod_{i=1}^{r-1} h_{is} \prod_{j=1}^{s-1} h_{rj} \prod_{\mathbf{z}\in\mathcal{D}'_{rs}[\lambda]} (h_{\mathbf{z}}+1)\,,$$

where $\mathcal{C}'[\lambda]$ is the set of squares (r,s) that can be added to the diagram of λ so that the result is still a diagram of a partition (in other words, $\mathcal{C}'[\lambda]$ are the corners of the complementary partition), and

$$\mathcal{D}'_{rs}[\lambda] \,=\, \{(i,j)\in[\lambda],\ \text{such that } i\neq r,\ j\neq s\}.$$

In [Kon], the following complementary weighted branching rule is proved:

$$\begin{aligned}
&\prod_{(i,j)\in[\lambda]} \Big(x_i+\ldots+x_{\lambda'_j}+\,y_j+\ldots+y_{\lambda_i}\Big)\\
&=\sum_{(r,s)\in\mathcal{C}'[\lambda]} \left[\prod_{(i,j)\in\mathcal{D}'_{rs}[\lambda]} \Big(x_i+\ldots+x_{\lambda'_j}+\,y_j+\ldots+y_{\lambda_i}\Big)\right]\\
&\times\,\left[\prod_{i=1}^{r-1} (x_{i+1}+\ldots+x_r+\,y_s+\ldots+y_{\lambda_i})\right]\cdot\left[\prod_{j=1}^{s-1} \Big(x_r+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s\Big)\right]
\end{aligned}$$

Let us note that although the (q,t)-hook walk defined in [GH] has several similarities, in full generality it is not a special case of the weighted hook walk. While this might seem puzzling, let us emphasize that the walks come from algebraic constructions of a completely different nature. In many ways, it is much more puzzling that the algebraic part of [CKP] is related to the branching rule at all.

Finally, let us mention several new extensions of the hook length formula recently introduced by Guo-Niu Han in [Han1, Han2]. There is also a hook walk style proof of the main identity in [CLPS], which suggests a possibility of a "weighted" generalization.

6.4. As we mentioned in the introduction, this paper extends the results in our previous paper [CKP], where we gave a combinatorial proof of the following delicate result in the enumerative algebraic geometry. Denote by $w_{\mathbf{z}} = i\alpha+j\beta$ the weight of a square $\mathbf{z}=(i,j)\in\lambda$ in a Young diagram λ. Then:

$$\sum_{\mathbf{z}\in[\lambda]} w_{\mathbf{z}} \cdot \prod_{\mathbf{u}\in[\lambda^{\mathbf{z}}]} \frac{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha)(w_{\mathbf{u}}-w_{\mathbf{z}}-\beta)}{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha-\beta)(w_{\mathbf{u}}-w_{\mathbf{z}})} \,=\, n\,(\alpha+\beta)\,,$$

where the product is over all squares in $[\lambda^{\mathbf{z}}]$, defined as the Young diagram $[\lambda]$ without squares $\mathbf{z}=(i,j)$ and $(i+1,j+1)$, at which the denominator vanishes. We refer to [CKP] for an explicit substitution which allows us to derive this formula from (WHL).

In a similar direction, we can obtain formulas corresponding to identities (b)–(d) in Theorem 2. We present them here without a proof. Denote by $m=\lambda_1$ and $\ell=\lambda'_1$ the lengths of the first row and the first column of $[\lambda]$, respectively. Then $w_{m0}=\lambda_1\alpha$, $w_{0\ell}=\lambda'_1\beta$, and we have:

$$\sum_{\mathbf{z}\in[\lambda]} \frac{w_{\mathbf{z}}}{w_{\mathbf{z}}-w_{m0}} \cdot \prod_{t\in[\lambda^{\mathbf{z}}]} \frac{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha)(w_{\mathbf{u}}-w_{\mathbf{z}}-\beta)}{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha-\beta)(w_{\mathbf{u}}-w_{\mathbf{z}})} \,=\, m\left(1+\frac{\alpha}{\beta}\right),$$

$$\sum_{\mathbf{z}\in[\lambda]} \frac{w_{\mathbf{z}}}{w_{\mathbf{z}}-w_{0\ell}} \cdot \prod_{\mathbf{u}\in[\lambda^{\mathbf{z}}]} \frac{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha)(w_{\mathbf{u}}-w_{\mathbf{z}}-\beta)}{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha-\beta)(w_{\mathbf{u}}-w_{\mathbf{z}})} \,=\, \ell\left(1+\frac{\beta}{\alpha}\right),$$

$$\sum_{\mathbf{z}\in\lambda} \frac{w_{\mathbf{z}}}{(w_{\mathbf{z}}-w_{m0})(w_{\mathbf{z}}-w_{0\ell})} \cdot \prod_{\mathbf{u}\in\lambda^{\mathbf{z}}} \frac{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha)(w_{\mathbf{u}}-w_{\mathbf{z}}-\beta)}{(w_{\mathbf{u}}-w_{\mathbf{z}}-\alpha-\beta)(w_{\mathbf{u}}-w_{\mathbf{z}})} \,=\, \frac{1}{\alpha}+\frac{1}{\beta}\,.$$

It would be interesting to understand the role of these formulas in the algebraic context.

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References

[Ban] J. Bandlow, An elementary proof of the hook formula, *Electron. J. Combin.* **15** (2008), no. 1, RP 45, 14 pp.

[BD] J. Bandlow and M. D'Adderio, A new proof of a theorem of Littlewood, *European J. Combin.* **30** (2009), no. 2, 415–424.

[BFP] O. Bodini, É. Fusy and C. Pivoteau, Random sampling of plane partitions, *Combinatorics, Probability and Computing* **19** (2010), no. 2, 201–226 `arXiv:0712.0111`.

[CLPS] K. Carde, J. Loubert, A. Potechin and A. Sanborn, Proof of Han's hook expansion conjecture, `arXiv:0808.0928`.

[CKP] I. Ciocan-Fontanine, M. Konvalinka and I. Pak, Quantum cohomology of $\mathrm{Hilb}_n(\mathbb{C}^2)$ and the weighted hook walk on Young diagrams, preprint, 2009.

[Fis] I. Fischer, A bijective proof of the hook-length formula for shifted standard tableaux, preprint, `arXiv:math/0112261`.

[FRT] J. S. Frame, G. de B. Robinson and R. M. Thrall, The hook graphs of the symmetric group, *Canad. J. Math.* **6** (1954), 316–325.

[FZ] D. S. Franzblau and D. Zeilberger, A bijective proof of the hook-length formula, *J. Algorithms* **3** (1982), 317–343.

[Fro] F. G. Frobenius, *Gesammelte Abhandlungen*, Bände I, II, III, Springer-Verlag, Berlin-New York (1968)

[Gan1] E. Gansner, Matrix correspondences of plane partition, *Pacific J. Math.* **92** (1981), 295–315.

[Gan2] E. Gansner, The Hillman-Grassl correspondence and the enumeration of reverse plane partitions, *J. Combin. Theory, Ser. A* **30** (1981), 71–89.

[GH] A. M. Garsia and M. Haiman, A random q, t-hook walk and a sum of Pieri coefficients, *J. Combin. Theory, Ser. A* **82** (1998), 74–111.

[GV] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.

[GN] K. Glass and C.-K. Ng, A simple proof of the hook length formula, *Amer. Math. Monthly* **111** (2004), 700–704.

[GNW1] C. Greene, A. Nijenhuis and H. S. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. in Math.* **31** (1979), 104–109.

[GNW2] C. Greene, A. Nijenhuis and H. S. Wilf, Another probabilistic method in the theory of Young tableaux, *J. Combin. Theory, Ser. A* **37** (1984), 127–135.

[Han1] G.-N. Han, The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications, `arXiv:0805.1398`.

[Han2] G.-N. Han, Discovering hook length formulas by an expansion technique, *Electronic J. Combin.* **15** (2008), RP 133, 41 pp.

[HG] A. P. Hillman and R. M. Grassl, Reverse plane partitions and tableau hook numbers, *J. Combin. Theory, Ser. A* **21** (1976), 216–221.

[Ker1] S. Kerov, A q-analog of the hook walk algorithm for random Young tableaux, *J. Algebraic Combin.* **2** (1993), 383–396.

[Ker2] S. V. Kerov, Transition probabilities of continual Young diagrams and the Markov moment problem, *Funct. Anal. Appl.* **27** (1993), no. 2, 104–117.

[Kir] A. N. Kirillov, The Lagrange identity and the hook formula, *J. Soviet Math.* **59** (1992), 1078–1084

[KP] A. A. Kirillov and I. Pak, Covariants of the symmetric group and its analogues in A. Weil algebras, *Funct. Anal. Appl.* **24** (1990), 172–176.

[Knu] D. E. Knuth, *The Art of Computer Programming* (Second ed.), Vol. 3, Addison-Wesley, Reading MA, 1998.

[Kon] M. Konvalinka, The weighted hook-length formula II: Complementary formulas, preprint

[Kra1] C. Krattenthaler, Bijective proofs of the hook formulas for the number of the standard Young tableaux, ordinary and shifted, *Electron. J. Comb.* **2** (1995), #R13

[Kra2] C. Krattenthaler, An involution principle-free bijective proof of Stanley's hook-content formula, *Discrete Math. Theor. Comput. Sci.* **3** (1998/99), no. 1, 11–32.

[Kra3] C. Krattenthaler, Another involution principle-free bijective proof of Stanley's hook-content formula, *J. Combin. Theory, Ser. A* **88** (1999), 66–92.

[Mac] I. G. Macdonald, *Symmetric functions and Hall polynomials* (Second edition), Oxford University Press, New York, 1995.

[NPS] J.-C. Novelli, I. Pak and A. V. Stoyanovskii, A direct combinatorial proof of the hook-length formula *Discrete Math. and Theor. Comp. Sci.* **1** (1997), 53–67.

[Pak] I. Pak, Hook length formula and geometric combinatorics, *Sém. Lothar. Combin.* **46** (2001/02), Art. B46f, 13 pp.

[Rem] J. B. Remmel, Bijective proofs of formulae for the number of standard Young tableaux, *Linear and Multilinear Algebra* **11** (1982), 45–100.

[RW] J. B. Remmel and R. Whitney, A bijective proof of the generating function for the number of reverse plane partitions via lattice paths, *Linear and Multilinear Algebra* **16** (1984), 75–91.

[Sag1] B. Sagan, On selecting a random shifted Young tableau. *J. Algorithms* **1** (1980), 213–234.

[Sag2] B. Sagan, *The symmetric group* (Second edition), Springer, New York, 2001.

[Sri] R. Srinivasan, On a theorem of Thrall in combinatorial analysis, *Amer. Math. Monthly* **70** (1963), 41–44.

[Sta] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, 2, Cambridge University Press, 1997, 1999.

[Thr] R. M. Thrall, A combinatorial problem, *Michigan Math. J.* **1** (1952), 81–88.

[Ver] A. M. Vershik, The hook formula and related identities, *J. Soviet Math.* **59** (1992), 1029–1040.

[You] A. Young, *The collected papers of Alfred Young (1873–1940)*, Mathematical Expositions, No. 21. University of Toronto Press, Toronto, Ont., Buffalo, N. Y. (1977)

[Zei] D. Zeilberger, A short hook-lengths bijection inspired by the Greene–Nijenhuis–Wilf proof, *Discrete Math.* **51** (1984), 101–108.