THE COLLAPSING WALLS THEOREM

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Abstract. Let $P \subset \mathbb{R}^3$ be a pyramid with the base a convex polygon Q. We show that when other faces are *collapsed* (rotated around the edges onto the plane spanned by Q), they cover the whole Q.

1. Introduction

Let P be a convex pyramid in R^3 over the base Q, which is a convex polygon in a horizontal plane. Think of the other faces F of P as the "walls" of a wooden box, and that each wall F is hinged to the base Q along the edge. Suppose now that the walls are "collapsed", i.e. rotated around the edges towards the base onto the horizontal plane. The question is: *do they cover the whole base* Q*?*

Figure 1. An impossible configuration of four collapsing walls of a pyramid leaving a hole in the base.

At first, this may seem obvious, but in fact the problem is already non-trivial even in the case of four-sided pyramids, which can possibly have some obtuse dihedral angles (see Figure 1). Formally, we have the following result:

Collapsing Walls Theorem. *Let $P \subset \mathbb{R}^3$ be a pyramid over a convex polygon Q. For a face F of P, denote by e_F the edge between F and the base: $e_F = F \cap Q$, and let A_F denotes the result of rotation of F around e_F in the direction of P, onto the plane which contains Q. Then*

$$Q \subseteq \cup_F A_F ,$$

where the union is over all faces F of P, different from Q.

For example, suppose pyramid P in the theorem has a very large height, so that all walls are nearly vertical. The theorem then implies that every point $z \in Q$ has an orthogonal projection into the interior of some edge e of Q. This is a classical result

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with a number of far-reaching generalizations (see [Pak, §9]). Thus, the collapsing walls theorem can be viewed as yet another generalization of this result (cf. Section 3).

2. Proof of the theorem

Consider $\mathbb{R}^3$ endowed with the standard Cartesian coordinates (x_1, x_2, x_3). Without loss of generality assume that the plane H spanned by Q is horizontal, i.e. given by $x_3 = 0$, and that P is contained in the half-space $x_3 \geq 0$. Denote by $F_1, \ldots, F_m$ the faces of P different from Q, by H_i the planes spanned by F_i, and by $e_i = F_i \cap Q$ the edges of Q, for all $1 \leq i \leq m$.

Denote by Φ_i the rotation about e_i of H_i onto H (the rotation is performed in the direction dictated by P, so that throughout the rotation H_i intersects the interior of P). Similarly, let $A_i = \Phi_i(F_i)$ is the rotation of the face F of P onto Q, $1 \leq i \leq m$. We need to show that every point in Q lies in $\cup_{i=1}^m A_i$. Without loss of generality we can take this point to be the origin O.

Further, denote by $L_i = H_i \cap H$ the line through e_i. Let r_i be the distance from the origin to L_i, and let α_i be the dihedral angle of P at e_i, i.e the angle between H and H_i which contains P.

Suppose now F_1 is a face such that

$$\tau_i = r_i \cdot \tan \frac{\alpha_i}{2} \quad \text{is minimized at} \quad \tau_1 .$$

We will show that the origin O is contained in A_1. In other words, we prove that if $O \notin A_1$, then $\tau_i < \tau_1$ for some $i > 1$.

Let $z \in H_1$ such that the rotation of z onto Q is the origin: $\Phi_1(z) = O$. It suffices to show that $z \in F_1$. Let $\boldsymbol{v} = (v_1, v_2, 0)$ be the unit vector that is a normal to L_1 in the horizontal plane. It is easy to see that

$$\overrightarrow{Oz} = \bigl(r_1(1 - \cos\alpha_1)v_1, r_1(1 - \cos\alpha_1)v_2, r_1 \sin\alpha_1\bigr).$$

To prove the theorem, assume to the contrary that $z \notin F_1$. Then there exists a face of P, say F_2, such that H_2 separates z from the origin. Denote by y the closest point to z on L_2, and by α' the angle between the line (zy) and the horizontal plane H, where the angle is taken with the half-plane of H which contains Q (and thus the origin). In this notation, the above condition implies that $\alpha' > \alpha_2$.

Without loss of generality we may assume that line L_2 is given by equations $x_2 = r_2$ and $x_3 = 0$. Then

$$y = \bigl(r_1(1 - \cos\alpha_1)v_1, r_2, 0\bigr),$$

and

$$\cos\alpha' = \cos\widehat{Oyz} = \frac{r_2 - r_1(1 - \cos\alpha_1)v_2}{\sqrt{r_1^2 \sin^2\alpha_1 + (r_2 - r_1(1 - \cos\alpha_1)v_2)^2}}.$$

Note that the function $x/\sqrt{a^2 + x^2}$ is monotone increasing as a function of x, and that $v_2 \leq 1$. We get

$$\cos\alpha' \geq \frac{r_2 - r_1(1 - \cos\alpha_1)}{\sqrt{r_1^2 \sin^2\alpha_1 + (r_2 - r_1(1 - \cos\alpha_1))^2}}.$$

Applying $\cos\alpha' < \cos\alpha_2$, we conclude:

$$\frac{r_2 - r_1(1-\cos\alpha_1)}{\sqrt{r_1^2\sin^2\alpha_1 + (r_2 - r_1(1-\cos\alpha_1))^2}} < \cos\alpha_2\,. \tag{1}$$

Recall the assumption that $\tau_1 \le \tau_2$. This gives $r_1 \tan\frac{\alpha_1}{2} \le r_2 \tan\frac{\alpha_2}{2}$, or

$$\frac{r_2}{r_1} \ge \frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\alpha_2}{2}}\,. \tag{2}$$

The rest of this section is dedicated to showing that both (1) and (2) are impossible. This gives a contradiction with our assumptions and proves the claim. We split the proof into two cases depending on whether the dihedral angle α_2 is acute or obtuse. In each case we repeatedly rewrite (1) and (2), eventually leading to a contradiction.

Case 1 (obtuse angles)**.** Suppose $\frac{\pi}{2} < \alpha_2 < \pi$. In this case $\cos\alpha_2 < 0$, and (1) is equivalent to

$$1 + \frac{r_1^2\sin^2\alpha_1}{(r_2 - r_1(1-\cos\alpha_1))^2} < \frac{1}{\cos^2\alpha_2}\,, \tag{3}$$

and

$$\frac{r_1\sin\alpha_1}{r_2 - r_1(1-\cos\alpha_1)} > \tan\alpha_2\,. \tag{4}$$

This can be further rewritten as:

$$\frac{r_2}{r_1} < 1 - \cos\alpha_1 + \frac{\sin\alpha_1}{\tan\alpha_2}\,. \tag{5}$$

Now (5) and (2) together imply

$$\frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\alpha_2}{2}} < 1 - \cos\alpha_1 + \frac{\sin\alpha_1}{\tan\alpha_2}\,,$$

which is impossible. Indeed, suppose for some $0 < a, b < \pi$, we have

$$\frac{\tan\frac{a}{2}}{\tan\frac{b}{2}} < 1 - \cos a + \frac{\sin a}{\tan b}\,. \tag{6}$$

Dividing both sides by $(\tan\frac{a}{2})$, after some easy manipulations, we conclude that (6) is equivalent to

$$\frac{1}{\tan\frac{b}{2}} < \sin a + \frac{1+\cos a}{\tan b}\,, \tag{7}$$

which in turn is equivalent to

$$\left(\frac{1}{\tan\frac{b}{2}} - \frac{1}{\tan b}\right)\sin b < \cos(a-b)\,. \tag{8}$$

Since the left hand side of (8) is equal to 1, we get a contradiction and complete the proof in Case 1.

Case 2 (right and acute angles)**.** Suppose now that $0 < \alpha_2 \le \frac{\pi}{2}$. Then $\cos\alpha_2 \ge 0$, and $0 < \tan\frac{\alpha_2}{2} \le 1$. Let us first show that the numerator of (1) is nonnegative, i.e. that $r_2 \ge r_1(1-\cos\alpha_1)$. From the contrary assumption we have $r_2/r_1 < (1-\cos\alpha_1)$. Together with (2), this implies:

$$1-\cos\alpha_1 \,>\, \frac{r_2}{r_1} \,\ge\, \frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\alpha_2}{2}} \,\ge\, \tan\frac{\alpha_1}{2}\,,$$

which is impossible for all $0 < \alpha_1 < \pi$.

From above, we can exclude the right angle case $\alpha_2 = \frac{\pi}{2}$, for else the l.h.s. of (1) is nonnegative, while r.h.s. is equal to zero. Thus, $\cos\alpha_2 > 0$. Therefore, the inequality (1) in this case can be rewritten as

$$1 + \frac{r_1^2\sin^2\alpha_1}{(r_2 - r_1(1-\cos\alpha_1))^2} \,>\, \frac{1}{\cos^2\alpha_2}\,, \tag{9}$$

and

$$\frac{r_1\sin\alpha_1}{r_2 - r_1(1-\cos\alpha_1)} \,>\, \tan\alpha_2\,. \tag{10}$$

Note now that (10) coincides with (4). Since (6) holds for all $0 < a, b < \pi$, we obtain the contradiction verbatim the proof in Case 1. This completes the analysis of Case 2 and finishes the proof of the theorem. □

3. Final remarks

3.1. The collapsing walls theorem extends verbatim to higher dimensions. Moreover, it also extends to every polytope $P \subset \mathbb{R}^d$, as follows. Fix one facet Q of P and assume all other facets F of P are rotated around the affine subspace $H_F \cap H$ onto the hyperplane H containing Q, then they cover the whole facet Q. Here H_F denotes the hyperplane that contains the facet F. We refer to [PP], where this result is proved in full generality, and is used to show that a smaller polyhedron can always be sequentially cut out of a bigger polyhedron, in any dimension.

3.2. Let us note that when the walls of a pyramid are collapsed *outside*, rather than *onto* the base, they are pairwise non-intersecting (see Figure 2). We leave this easy exercise to the reader.

3.3. Continuing with the example of "vertical walls" as given in the introduction right after the theorem, recall that for the center of mass $z = \mathrm{cm}(Q)$, there are at least two such edges onto which orthogonal projection of z lies in the interior (see e.g. [Pak, §9]).[1] It would be interesting to see if this result extends to the setting of the theorem (of course, the notion of the center of mass would have to be modified appropriately). Let us note here that the center of mass result is closely related to the four vertex theorem [Tab], and fails in higher dimension [CGG].

[1] One can give a construction with there is only one such edge, if the center of mass is replaced by a general point in Q (see [CGG] and [Pak, §9]).

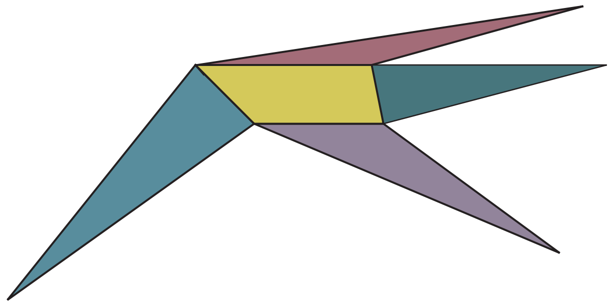

FIGURE 2. Walls of a pyramid collapsing outside the base do not intersect.

3.4. The proof of the theorem is based on an implicit subdivision of Q given by the smallest of the linear functions τ_i at every point $z \in Q$. Recall that τ_i is a weighted distance to the edge e_i. Thus this subdivision is in fact a weighted analogue of the dual Voronoi subdivision in the plane (see [Aur, For]). As a consequence, computing this subdivision can be done efficiently, both theoretically and practically.

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