Combinatorial evaluations of the Tutte polynomial

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Introduction

The Tutte polynomial is one of the most important and most useful invariants of a graph. It was discovered as a two variable generalization of the chromatic polynomial [15, 16], and has been studied in literally hundreds of papers, in part due to its connections to various fields ranging from Enumerative Combinatorics to Knot Theory, from Statistical Physics to Computer Science. We refer the reader to [3] for a nice introduction to the subject and to [18] for a well written and extensive survey of modern theory and applications.

The main subject of this paper is *combinatorial evaluations* of the Tutte polynomial, by which we mean combinatorial interpretations of its values. The reader may recall classical evaluations in terms of the number of proper (vertex) colorings, spanning trees, spanning subgraphs, acyclic orientations, etc. We show that when the graph is planar, certain other values of the Tutte polynomial have a number of combinatorial interpretations specific to the plane embedding. We give new combinatorial evaluations in terms of two different edge colorings, claw coverings, and, for particular graphs on a square grid, in terms of Wang tilings and T-tetromino tilings.

It is natural to ask which of our constructions extend to nonplanar graphs, in a situation when the graph is embedded into a surface of higher genus. We discover that natural generalizations of our results give combinatorial evaluations not of the Tutte polynomial, but of a specialization of a recently introduced Bollobás-Riordan polynomial [4, 5]. We should mention here that there are certain limitations of this approach; many, but not all of our results extend to that case.

This paper is a sequel to our previous work [10], but can be read independently as it does not use any of our previous results. In fact, the paper is completely self-contained. Basically, in [10] we found an unexpected connection between the number of T-tetromino tilings of rectangular shape regions and the evaluation of the Tutte polynomial at $(3, 3)$. Here we undertake a thorough investigation of this phenomenon, and explain it by a series of bijections and combinatorial evaluations. One can view this paper as a generalization of the results in [10] presented in a different language and with a philosophy opposite to that in [10]. There we were studying and enumerating combinatorial objects whose number turned out to be the evaluation of the Tutte polynomial, while here we find combinatorial objects whose number is the evaluation of the Tutte polynomial.

Let us mention a few words about the proofs. Once found, combinatorial evaluations are relatively easy to prove due to the simple recursive nature of the Tutte polynomial. Thus, whenever possible, we present a bijective proof of the results, going from one combinatorial interpretation to another. This approach makes the proofs clear and natural, highlights the generalizations, and shows the limits to which the results can be extended. Some of our bijections have an independent value, and in special cases can lead to combinatorial results of a different type [10].

The structure of the paper is as follows. We start with basic definitions and notation in section 1. In the next six sections we present various combinatorial evaluations of the Tutte and Bollobás-Riordan polynomials. For convenience, in these sections we only state the theorems and displace the proofs to section 8. We conclude with final remarks in the last section 9.

Notation: We denote all graphs by Roman capital letters: G, H, U, F. We use $\mathbb{N}$ to denote $\{1, 2, \ldots\}$.

1 Basic definitions

Throughout the paper we consider only finite unoriented graphs; loops and multiple edges are allowed. A *planar graph* is a graph which allows embedding into a plane with edges given by simple nonintersecting curves. In contrast, a *plane graph* is a graph given together with such an embedding into a plane. To specify an embedding we need only a circular order on edges adjacent to the same vertex; if the order is the same we think of two such plane graphs as isomorphic (see Figure 1). For example, every tree is planar, but may correspond to many different plane trees.

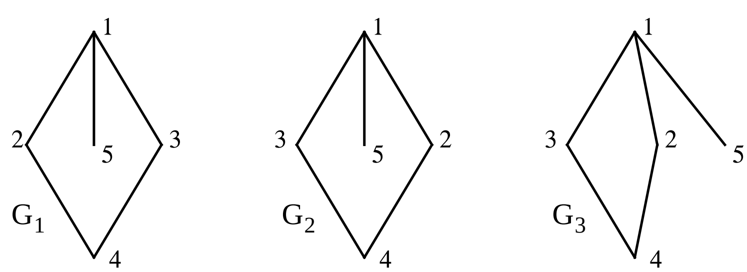

Figure 1: Graphs G_1, G_3 are isomorphic as plane graphs, but not to G_2.

Let G be a graph, and let $v(G)$ and $e(G)$ denote the number of vertices and edges of G, respectively. Let $c(G)$ denote the number of connected components of G. Let $r(G) = v(G) - c(G)$ be the *rank* of G, and let $n(G) = e(G) - r(G)$ be the *nullity* of G. The *Tutte* *polynomial* $T(G;x,y)$ is defined as follows:

$$(\lozenge) \qquad T(G;x,y) = \sum_{J\subset G} (x-1)^{r(G)-r(J)}(y-1)^{n(J)},$$

where the sum is over all spanning subgraphs $J \subset G$.

If e is an edge of G, let $G\backslash e$ be the graph formed by deleting e from G. Similarly, let G/e be the graph formed by contracting e in G, i.e. obtained by identifying the endpoints of e into a single vertex. Finally, by $G_1 \sqcup G_2$ we denote a disjoint union of two graphs. The Tutte polynomial satisfies the following recursive formulas:

- $T(G;x,y) = 1$ if G has no edges,
- $T(G;x,y) = y \cdot T(G\backslash e;x,y)$ if e is a loop,
- $T(G;x,y) = x \cdot T(G/e;x,y)$ if e is a bridge,
- $T(G;x,y) = T(G_1;x,y) \cdot T(G_2;x,y)$ if $G = G_1 \sqcup G_2$,
- $T(G;x,y) = T(G\backslash e;x,y) + T(G/e;x,y)$ if e is neither a loop nor a bridge.

These formulas can be used to give an alternative definition of the Tutte polynomial (see [3, 18]). There is another way to define the Tutte polynomial in terms of spanning trees with a given number of internal and external activities [16, 3]. We will not need this definition for our purposes.

For every plane graph G, denote by G^* its dual graph, which is also a plane graph. We have

$$(\circledast) \qquad T(G;x,y) = T(G^*;y,x).$$

Let G be a connected plane graph. Define the *medial graph* $H = H(G)$ as follows. Let the vertices of H correspond to edges of G and let two vertices of H be connected by an edge if the corresponding edges in G are subsequent in the cyclic order of edges around some vertex in G (see Figure 2). Notice that the medial graph of any plane graph is always 4-regular and planar. Also, observe that the medial graph of the dual graph G^* coincides with the medial graph of G: $H(G) \simeq H(G^*)$.

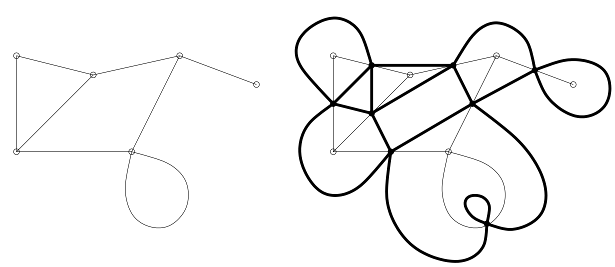

Figure 2: A graph G, and its medial graph H (bold edges).

2 Coloring edges of the medial graph

Consider a coloring of the edges of H with k colors subject to the following rules:

- each vertex is incident to an even number of edges of each color,
- there does not exist a vertex v and distinct colors c_1, c_2 such that the edges incident to v are colored c_1, c_2, c_1, c_2 in cyclic order.

Let $\mathcal{C}_k(H)$ denote the set of all such colorings of H. A vertex is called *monochromatic* if the four edges incident to it all have the same color. Let $\alpha(C)$ be the number of monochromatic vertices in a coloring $C \in \mathcal{C}_k(H)$.

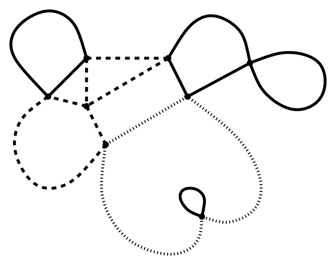

Figure 3: A 3-coloring of H with 2 monochromatic vertices.

Theorem 1 *Let G be a connected plane graph and let H be the medial graph of G. Let k be a positive integer. Then*

$$\sum_{C \in \mathcal{C}_k(H)} 2^{\alpha(C)} = k \cdot T(G; k+1, k+1).$$

As before, let G be a plane graph and H the medial graph of G. An orientation of the edges of H is called *Eulerian* if each vertex has indegree 2 and outdegree 2. Let $\mathcal{O}(H)$ denote the set of all Eulerian orientations of H. For $O \in \mathcal{O}(H)$, let $\beta(O)$ denote the number of *saddle vertices*, i.e. vertices in which the edges are oriented "in, out, in, out" in cyclic order. The following result is due to Las Vergnas who proved it by a recursive argument [11]. We deduce it as a corollary of Theorem 1.

Corollary 2 (Las Vergnas) *Let G be a connected plane graph and let $\mathcal{O}(H)$ be the set of all Eulerian orientations of the medial graph $H = H(G)$. Then*

$$\sum_{O \in \mathcal{O}(H)} 2^{\beta(O)} = 2 \cdot T(G; 3, 3).$$

Proof: We need to show that the LHS of the formula in Theorem 1 for $k = 2$ equals the LHS of the formula in the Corollary. Therefore, it suffices to construct a bijection $\varphi : \mathcal{C}_2(H) \to \mathcal{O}(H)$, such that $\beta(\varphi(C)) = \alpha(C)$ for all $C \in \mathcal{C}_2(H)$.

Let $C \in \mathcal{C}_2(H)$ be a coloring of the edges of H with two colors, say red and blue. Color a face of H grey if it contains a vertex of G, and color it white otherwise. Each edge of H will be incident to one grey face and one white face. Now orient the red edges so the grey face is on the left, and orient the blue edges so the grey face is on the right (see Figure 4).

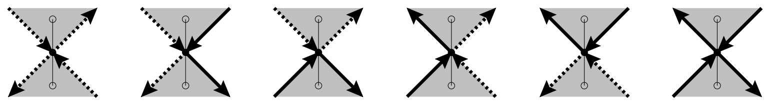

Figure 4: The six possibilities for a vertex of H (solid blue edges, dotted red edges).

It is easy to see that this defines a bijection between $\mathcal{C}_2(H)$ and $\mathcal{O}(H)$ such that monochromatic vertices correspond to saddle vertices. The result follows. ∎

Let us now generalize Theorem 1 to give an interpretation for $T(G; p+1, q+1)$ for any positive integers p and q. Let $p, q \in \mathbb{N}$, and let $n = \mathrm{lcm}(p, q)$ be the least common multiple of p and q. As before, we let H be the medial graph of G, and we color grey those faces of H which contain a vertex of G. Consider labelling the edges of H with integers mod n so that at each vertex, the incident edges satisfy one of the following allowable configurations:

- all four incident edges have the same label,
- two edges which lie on the same grey face are labelled a, the other two edges are labelled b, and $a \equiv b \pmod{\frac{n}{p}}$,
- two edges which lie on the same white face are labelled a, the other two edges are labelled b, and $a \equiv b \pmod{\frac{n}{q}}$.

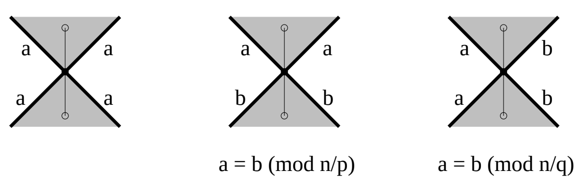

Figure 5: The allowable configurations for a vertex of H.

Let $\mathcal{L}_{p,q}(H)$ denote the set of all such labellings of H. A vertex will be called *monochromatic* if the four edges incident to it all have the same label. Let $\alpha(L)$ be the number of monochromatic vertices in a labelling $L \in \mathcal{L}_{p,q}(H)$.

Theorem 3 *Let G be a connected plane graph and let H be the medial graph of G. Let p and q be positive integers, and let $n = \mathrm{lcm}(p, q)$. Then*

$$\sum_{L \in \mathcal{L}_{p,q}(H)} 2^{\alpha(L)} = n \cdot T(G; p+1, q+1).$$

Note that for $p = q = n = k$, labellings in $\mathcal{L}_{p,q}(H)$ coincide with those in $\mathcal{C}_k(H)$. Therefore, Theorem 3 is an extension of Theorem 1.

3 Wang tilings

The previous result has a nice reformulation when G is a grid graph. Suppose that G is the $x \times y$ *grid graph* defined to be a product of chains of length x and y. Consider the $x \times y$ *Aztec rectangle* defined in Figure 6.

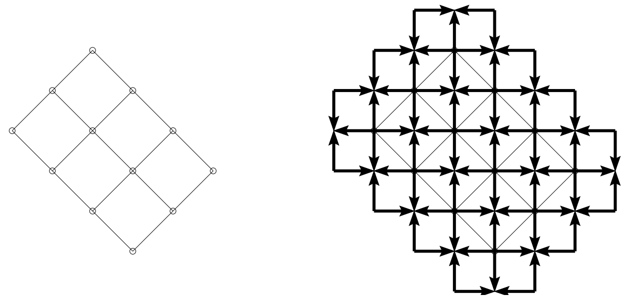

Figure 6: The 3×4 grid, and the 3×4 Aztec rectangle (bold edges).

A set of Wang tiles is a set of squares whose edges are colored, and in this case, directed. Two such tiles can be placed next to one another if and only if the color and orientation match on the edge where they touch. Tiles are not allowed to be rotated. Consider the following set of Wang tiles[1], where the allowable colors are the integers mod n (Figure 7).

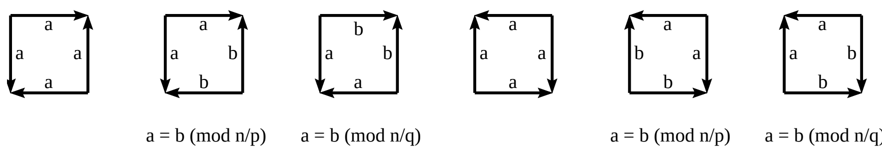

Figure 7: The allowable types of Wang tiles.

Consider a tiling of the $x \times y$ Aztec rectangle with these Wang tiles, subject to the condition that every edge on the boundary is labelled 0, and is oriented as shown in Figure 6. (It is easy to see that the orientation of the boundary edges as in Figure 6 determines the orientation of the interior edges). Let $\mathcal{W}_{p,q}(x,y)$ denote the set of all such tilings. A tile is called *monochromatic* if its four edges all have the same label. If $W \in \mathcal{W}_{p,q}(x,y)$, let $\alpha(W)$ denote the number of *interior* monochromatic tiles (tiles which do not share an edge with the edge of the Aztec rectangle).

Theorem 4 *Let G be the $x \times y$ grid graph, let p and q be positive integers and let $n = \mathrm{lcm}(p,q)$. Then*

$$\sum_{W \in \mathcal{W}_{p,q}(x,y)} 2^{\alpha(W)} = p \cdot T(G; p+1, q+1).$$

[1]The traditional definition of Wang tiles usually does not involve oriented edges [2]. Notice that we can transform this setup into a format which does not involve orientations by doing the following. Let the set of allowable colors be the set of ordered pairs (x,d), where x is an integer mod n, and d is a direction (north, south, east, or west), corresponding to the direction the edge is pointing. This increases the number of colors needed, but eliminates the need for orientations. However, for simplicity we use the oriented version here.

4 Ribbon Graphs

In this section we extend results in section 2 to graphs embedded in surfaces of higher genus. We start by defining *ribbon graphs*. A ribbon graph is a surface with boundary which is the union of two sets of closed disks (the vertices and the edges) which satisfy certain restrictions on how they may intersect. Intuitively, one should think of the vertices as being small circular disks, and the edges as thin strips which connect the vertices, all living in three dimensions, as in Figure 8 (see [5]).

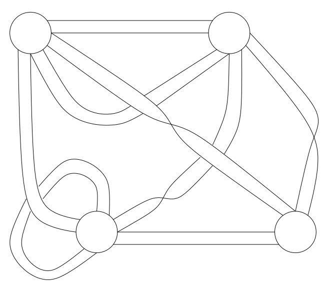

Figure 8: A ribbon graph.

In [4, 5], Bollobás and Riordan define a generalization of the Tutte polynomial to ribbon graphs, which we review here. Let G be a ribbon graph. Let $v(G)$ be the number of vertices of G, and let $e(G)$ be the number of edges of G. Let $c(G)$ be the number of connected components of G. Let $r(G) = v(G) - c(G)$ be the *rank* of G, and let $n(G) = e(G) - r(G)$ be the *nullity* of G. Let $bc(G)$ be the number of components of the boundary of G, where the ribbon graph G is considered as a surface with boundary. For example, if G is the ribbon graph in Figure 8, then $bc(G) = 2$. A spanning subgraph of a ribbon graph is defined in the natural way, as a subset of the ribbon graph which contains all the vertices, and a subset of the edges.

The *Bollobás-Riordan polynomial* $R(G; x, y, z)$ is defined as follows.[2] Let G be a ribbon graph. Then

$$R(G; x, y, z) = \sum_{J \subset G} x^{r(G)-r(J)} y^{n(J)} z^{c(J)-bc(J)+n(J)},$$

where the sum is over all spanning subgraphs $J \subset G$.

It is clear to see that this is a generalization of the Tutte polynomial; indeed, it is immediate from the definitions that

$$R(G; x-1, y-1, 1) \;=\; T(G; x, y).$$

However, there is another observation to be made in the case when G is a *planar ribbon graph*, defined to be a ribbon graph which can be drawn on the plane without the edges crossing or twisting. It follows from Euler's formula that for any planar ribbon graph J, we

[2]This version of the polynomial differs from the one given in [4] (and its generalization in [5]) by a simple change of variables.

have $c(J) - bc(J) + n(J) = 0$. As a result, the variable z never appears. Hence when G is a planar ribbon graph we have a stronger result:

$$(\blacklozenge) \qquad R(G; x-1, y-1, z) \;=\; T(G; x, y).$$

There is a strong connection between ribbon graphs and graphs drawn without intersection on closed surfaces. Given a graph drawn on a surface, a ribbon graph can be formed by taking a neighborhood of the graph in the surface. In doing this, the information we lose is the topology of each face of the embedding. Given a ribbon graph, one can complete it to a closed surface by taking each component of the boundary and gluing a topological disk there. This defines a *canonical embedding* of the ribbon graph into a surface.

With this in mind, one can define the *medial graph* $H = H(G)$ of a ribbon graph G as follows. Take the canonical embedding, include a vertex for every edge of G, and include an edge for every pair of edges of G which are consecutive along the boundary of some face. Note that H is drawn on a closed surface, hence we may think of H as a ribbon graph.

Let G be a ribbon graph, and let H be its medial graph. Consider a coloring of the edges of H with k colors subject to the following rules:

- each vertex is incident to an even number of edges of each color,
- there does not exist a vertex v and distinct colors c_1, c_2 such that the edges incident to v are colored c_1, c_2, c_1, c_2 in cyclic order.

Here the cyclic order is well defined since the medial graph is embedded in a surface.

Let $\mathcal{C}_k(H)$ denote the set of all such colorings of H. A vertex will be called *monochromatic* if the four edges incident to it all have the same color. Let $\alpha(C)$ be the number of monochromatic vertices in a coloring $C \in \mathcal{C}_k(H)$.

Theorem 5 *Let G be a connected ribbon graph and let H be its medial graph. Let k be a positive integer. Then*

$$\sum_{C \in \mathcal{C}_k(H)} 2^{\alpha(C)} \;=\; k \cdot R\big(G; k, k, \frac{1}{k}\big).$$

Let us show that Theorem 5 generalizes Theorem 1. Indeed, let G be a plane graph. By ($\blacklozenge$) we have $R\big(G; k, k, \frac{1}{k}\big) \;=\; T(G; k+1, k+1)$, which implies the claim. Note that the more general Theorem 3 does not extend to ribbon graphs.

5 Labellings of incidences

Note that in previous sections we gave "weighted sum evaluations" of the Tutte polynomial rather than "direct evaluations". In this section we present our first such "direct evaluation".

Let G be a connected plane graph. We now give a combinatorial evaluation of the Tutte polynomial $T(G; x, y)$, as the number of colorings of another plane graph $U(G)$, with vertices corresponding to vertices, edges and faces of G, and edges corresponding to *incidences* of pairs vertex-edge and edge-face.

Formally, define the graph $U = U(G)$ as follows. Include a "circular" vertex x_v for each vertex v of G, include a "square" vertex y_e for each edge e of G, and include a "triangular" vertex z_f for each face f of G. Connect the vertices x_v and y_e if the vertex v is incident to the edge e in G. Connect the vertices y_e and z_f if the edge e lies on the face f in G (see Figure 9). Notice that each square vertex is adjacent to exactly two circular vertices and two triangular vertices, and that every face of U has degree four.

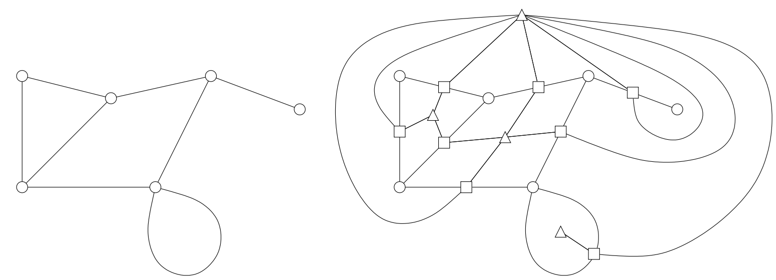

Figure 9: A graph G, and the graph $U = U(G)$.

Let p and q be positive integers, and let $n = \mathrm{lcm}(p, q)$. Consider a labelling of the edges of U which satisfies the following properties.

- Each edge is either labelled with an integer mod n, or it is given the label $\star$.
- For each face of U, exactly two of its boundary edges are labelled $\star$, and the other two contain the same integer mod n.
- For each square vertex, the incident edges are labelled a, $\star$, b, $\star$, in cyclic order; if the edges labelled a and b connect to circular vertices, then it must hold that $a \equiv b \pmod{\frac{n}{p}}$; if they connect to triangular vertices, then $a \equiv b \pmod{\frac{n}{q}}$.

Let $\mathcal{Q}_{p,q}(U)$ denote the set of all such labellings.

Theorem 6 *Let G be a connected plane graph, and let $U = U(G)$. Then*

$$|\mathcal{Q}_{p,q}(U)| = n \cdot T(G; p + 1, q + 1).$$

The theorem is proved in section 8.4 via reduction to Theorem 3. While the latter does not generalize to ribbon graphs (see section 9), the special case $p = q = n$ of Theorem 1 does generalize as follows.

Suppose G is a ribbon graph. Use the canonical embedding to define the graph $U = U(G)$ and the set of edge colorings $\mathcal{Q}_{k,k}(U)$ of U.

Theorem 7 *Let G be a connected ribbon graph, and let $U = U(G)$. Then*

$$|\mathcal{Q}_{k,k}(U)| = k \cdot R\big(G; k, k, \frac{1}{k}\big).$$

Note that this theorem neither generalizes nor follows from Theorem 6.

6 Claw coverings

For a connected plane graph G, define the graph $F = F(G)$ as follows. Include a red vertex $r_{v,e,f}$ for every triple (v, e, f) where v is a vertex of G, e is an edge of G incident to v, and f is a face of G incident to e. Also include two blue vertices $s_{v,f}$ and $t_{v,f}$ for every pair (v, f) where v is a vertex of G and f is a face of G incident to v. Connect $r_{v,e,f}$ to $r_{w,e,f}$ (where v and w are the endpoints of e), and connect $r_{v,e,f}$ to $r_{v,e,g}$ (where f and g are the faces incident to e). Also connect $r_{v,e,f}$ to $s_{v,f}$ and $t_{v,f}$. Finally, let F be the resulting graph (see Figure 10).

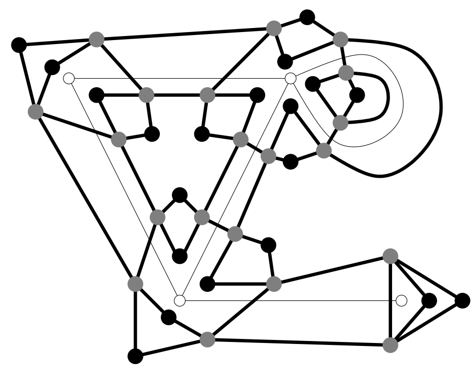

Figure 10: A graph G (thin edges), and the graph $F = F(G)$ (bold edges). (Red vertices are light while blue vertices are dark.)

Define a *claw* to be the bipartite graph $K_{1,3}$ (Figure 11). For a graph J, define a *claw covering* of J to be a spanning subgraph of J such that every component is isomorphic to a claw. Figure 12 shows a claw covering of the graph F as in Figure 10. Let $\mathcal{Y}(J)$ denote the set of all claw coverings of J.

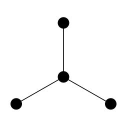

Figure 11: A claw.

Let us note here that claw coverings of J have a nice reformulation when J is a grid graph; this will be explored in the next section.

Theorem 8 *Let G be a connected plane graph, and let $F = F(G)$. Then*

$$|\mathcal{Y}(F)| = 2 \cdot T(G; 3, 3).$$

This theorem has a straightforward generalization to ribbon graphs. For every ribbon graph G use the canonical embedding to define a graph $F = F(G)$, and let $\mathcal{Y}(F)$ be the set of claw coverings of F.

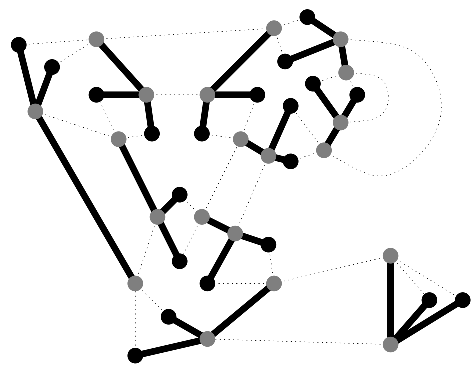

Figure 12: The bold edges form a claw covering of $F(G)$.

Theorem 9 *Let G be a connected ribbon graph, and let $F = F(G)$. Then*

$$|\mathcal{Y}(F)| = 2 \cdot R\big(G; 2, 2, \frac{1}{2}\big).$$

7 T-tetromino tilings

For a simply connected region on a square grid $\Gamma \subset \mathbb{Z}^2$, denote by $\mathcal{T}(\Gamma)$ the set of tilings of Γ by T-tetrominoes as shown in Figure 13.

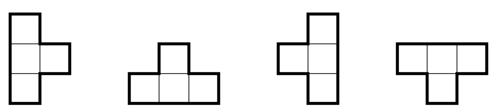

Figure 13: The four orientations of a T-tetromino.

Let $\mathcal{T}_{x,y}$ denote the set of tilings of a $4x \times 4y$ rectangle with T-tetrominoes. The following result, which appears in [10], is a corollary of Theorem 8.

Corollary 10 (Korn-Pak) *Let G be the $x \times y$ grid graph. The number $|\mathcal{T}_{x,y}|$ of tilings of a $4x \times 4y$ rectangle by T-tetrominoes satisfies*

$$|\mathcal{T}_{x,y}| \;=\; 2 \cdot T(G; 3, 3).$$

Before we prove the Corollary, let us first recall a result of Walkup [17], who showed that only $4x \times 4y$ rectangles are tileable by T-tetrominoes. It follows from the proof that in any tiling of a rectangle by T-tetrominoes, there are certain "walls" that no tile may cross (see Figure 14). These walls form a periodic pattern and give an additional structure to T-tetromino tilings of these rectangles.

Proof: A T-tetromino tiling of a $4x \times 4y$ rectangle corresponds naturally to a claw covering of the $4x \times 4y$ grid graph $G_{4x,4y}$ (see Figure 14). Thus it suffices to give a bijection between claw coverings of $G_{4x,4y}$, and those in $F_{4x,4y} := F(G_{x,y})$.

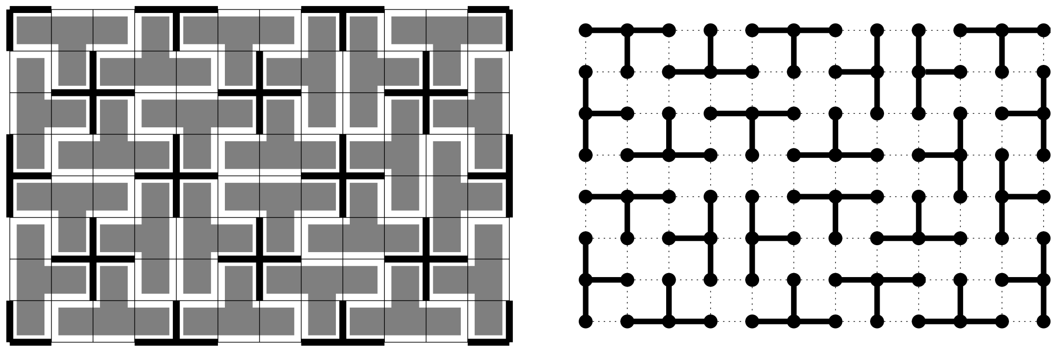

Figure 14: A T-tetromino tiling of a rectangle (bold grid lines indicate "walls"), and its representation as a claw covering of $G_{4x,4y}$.

Recall that T-tetromino tilings never intersect Walkup's walls (as in Figure 14). These walls correspond to edges of $G_{4x,4y}$ which can never occur in a claw covering. Let us delete these edges and call the resulting graph $Z_{4x,4y}$.

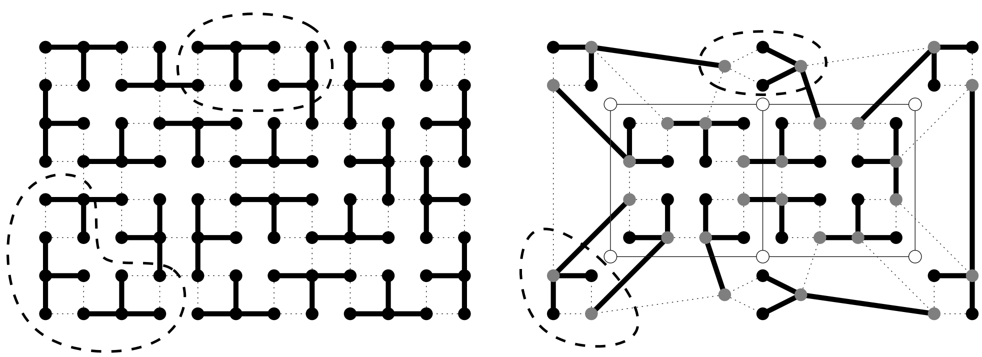

Figure 15: A claw covering of $Z_{4x,4y}$, and the corresponding claw covering of $F_{4x,4y}$.

Now let us compare the graphs $Z_{4x,4y}$ and $F_{4x,4y}$. The graph $Z_{4x,4y}$ differs from $F_{4x,4y}$ only near the boundary, and the difference is that a square in $F_{4x,4y}$ is replaced by a chain of squares in $Z_{4x,4y}$ (as inside the dotted regions in Figure 15). However, a chain of squares can be covered with claws in only two ways—the same as a single square (see Figure 16).

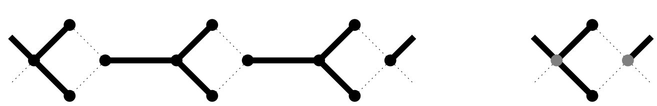

Figure 16: A chain of squares, and a single square.

To summarize, we have shown that T-tetromino tilings of the $4x \times 4y$ rectangle are in one-to-one correspondence with claw coverings of $G_{4x,4y}$. The latter coincide with claw coverings of $Z_{4x,4y}$, which are in one-to-one correspondence with claw coverings of $F_{4x,4y} = F(G_{x,y})$.

By Theorem 8, the number of claw coverings of $F(G_{x,y})$ is $2 \cdot T(G_{x,y}; 3, 3)$, which completes the proof of the corollary. ■

Denote by $\mathbb{T}_{x,y}$ a quotient of $\mathbb{Z}^2$ by a lattice $x\,\mathbb{Z} \times y\,\mathbb{Z} \subset \mathbb{Z}^2$. Clearly, $\mathbb{T}_{x,y}$ is topologically a torus. We will call the corresponding ribbon graph the $x \times y$ *torus ribbon graph.*

Now, think of a torus $\mathbb{T}_{4x,4y}$ as a region which can be tiled by T-tetrominoes. Clearly, T-tetromino tilings no longer respect Walkup's walls, simply because any translation of a tiling is still a tiling.

Define *Walkup tilings* to be T-tetromino tilings of $\mathbb{T}_{4x,4y}$ which do not cross Walkup's walls defined as above. Denote by $\mathcal{T}^{\diamond}_{x,y}$ the set of all such tilings.

Corollary 11 *Let G be the $x \times y$ torus ribbon graph. The number $|\mathcal{T}^{\diamond}_{x,y}|$ of Walkup tilings by T-tetrominoes of the torus region $\mathbb{T}_{4x,4y}$ equals*

$$|\mathcal{T}^{\diamond}_{x,y}| \;=\; 2 \cdot R\big(G; 2, 2, \frac{1}{2}\big).$$

The proof follows verbatim the proof of Corollary 10 via reduction to Theorem 9. In fact, the proof actually simplifies as there is no boundary effect in this case.

8 Proofs

8.1 Proof of Theorem 3

Proof: Denote by $A(G; p, q)$ the LHS of the formula in Theorem 3:

$$A(G; p, q) = \sum_{L \in \mathcal{L}_{p,q}(H)} 2^{\alpha(L)}.$$

It suffices to show that

- $A(G; p, q) = n(p + 1)$ if G is a single non-loop edge,
- $A(G; p, q) = n(q + 1)$ if G is a single loop,
- $A(G; p, q) = (p + 1) \cdot A(G/e; p, q)$ if e is a bridge,
- $A(G; p, q) = (q + 1) \cdot A(G\backslash e; p, q)$ if e is a loop,
- $A(G; p, q) = A(G\backslash e; p, q) + A(G/e; p, q)$ if e is neither a loop nor a bridge.

Let G^* be the plane dual of G. Recall that the medial graph of G^* (call it H^*) coincides with the medial graph of G, except with white and grey faces switched. Hence an element of $\mathcal{L}_{p,q}(H)$ is also an element of $\mathcal{L}_{q,p}(H^*)$, and $A(G; p, q) = A(G^*; q, p)$. Furthermore, a loop of G is a bridge of G^*, and deletion of an edge in G corresponds to contraction of an edge in G^*. Therefore, it suffices to show the first, third, and fifth items in our list; the remaining two follow by duality (⋇) (see section 1).

If G is a single non-loop edge, the medial graph H will consist of a vertex with two loop edges (see Figure 17). There are n ways to label the first loop; say we label it a. If the second loop is labelled b, we must have $b \equiv a \pmod{\frac{n}{p}}$. Hence there are p ways to label the second loop. Exactly n of these np labellings have a monochromatic vertex. Hence in this case

$$A(G;p,q) \;=\; (np-n)+n\cdot 2^1 \;=\; n(p+1),$$

as desired.

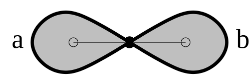

Figure 17: A single edge, and its medial graph.

Now assume G has at least 2 edges. Suppose e is a bridge of G. Let v be the vertex of H which corresponds to the edge e, and let w_1, w_2, w_3, and w_4 be the neighbors of v in H as shown in Figure 18. Let H' be the medial graph of G/e.

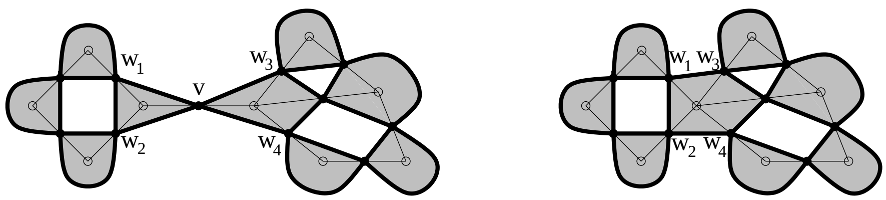

Figure 18: The graph G with its medial graph H, and the graph G/e with its medial graph H'.

Observe that in a valid labelling of H, the edges (v, w_1) and (v, w_2) must have the same label, call it l_1. (The set of all edges with label l has even degree at every vertex, thus it must cross each cut an even number of times; hence (v, w_1) and (v, w_2) cannot have different labels.) Similarly, the edges (v, w_3) and (v, w_4) must also have the same label, call it l_2. Meanwhile, in the graph H', the edges (w_1, w_3) and (w_2, w_4) must also receive the same label.

We wish to establish a mapping ϱ between labellings in $\mathcal{L}_{p,q}(H)$ and labellings in $\mathcal{L}_{p,q}(H')$. Let $L \in \mathcal{L}_{p,q}(H)$ be a labelling of H. If $l_1 = l_2$, then let $\varrho(L)$ be the labelling of H' which matches L everywhere. If $l_1 \neq l_2$, then add $l_1 - l_2$ to the label of every edge to the right of v, and let $\varrho(L)$ be the labelling of H' which matches that labelling everywhere. It is clear that ϱ maps surjectively onto $\mathcal{L}_{p,q}(H')$.

Upon applying ϱ, the only information lost is the value of $l_1 - l_2$. Since $l_1 \equiv l_2 \pmod{\frac{n}{p}}$, there are p possible values of $l_1 - l_2$. Note that $\varrho(L)$ contains the same number of monochromatic vertices as L, unless $l_1 = l_2$, in which case L has one more. Thus $A(G;p,q) = (p+1)\cdot A(G/e;p,q)$, as desired.

Now assume e is an edge which is neither a loop nor a bridge, and again let v be the vertex of H corresponding to e. Let H' be the medial graph of $G\backslash e$, and let H'' be the medial graph of G/e (see Figure 19).

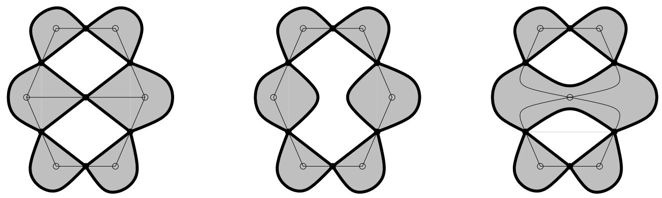

Figure 19: The graph G with its medial graph H, the graph $G\backslash e$ with its medial graph H', and the graph G/e with its medial graph H''.

Observe that for two edges lying on the same white face, adjacent or not, if they are labelled a and b, it must hold that $a \equiv b \pmod{\frac{n}{p}}$. (This is obviously true for adjacent edges on the same white face, hence it holds for all edges on that face, by transitivity.) Similarly, for two edges lying on the same grey face, we have $a \equiv b \pmod{\frac{n}{q}}$.

We wish to establish a mapping ϑ_1 from $\mathcal{L}_{p,q}(H')$ to $\mathcal{L}_{p,q}(H)$ and another mapping ϑ_2 from $\mathcal{L}_{p,q}(H'')$ to $\mathcal{L}_{p,q}(H)$. Suppose L' is a labelling of H'. Define $\vartheta_1(L')$ to be the labelling of H which matches L' everywhere. (Notice that the observation in the preceding paragraph guarantees us that $\vartheta_1(L')$ will be a valid labelling of H.) The function ϑ_1 is a bijection between $\mathcal{L}_{p,q}(H')$ and the set of labellings of H where the edges sharing a grey face at v have the same label. We define ϑ_2 similarly. The function ϑ_2 is a bijection between $\mathcal{L}_{p,q}(H'')$ and the set of labellings of H where the edges sharing a white face at v have the same label.

Suppose L is a labelling of H. If v is not monochromatic, then L is either in the image of ϑ_1 or ϑ_2, but not both. If v is monochromatic, then L is in the image of both ϑ_1 and ϑ_2. However, in this case, L has one more monochromatic vertex than its counterparts in $\mathcal{L}_{p,q}(H')$ and $\mathcal{L}_{p,q}(H'')$. It follows that $A(G;p,q) = A(G\backslash e;p,q) + A(G/e;p,q)$, as desired. ■

8.2 Proof of Theorem 4

Proof: We wish to reduce this problem to an instance of Theorem 3. Let G be the $x \times y$ grid graph, and let H be the medial graph of G. Now Theorem 3 gives:

$$\sum_{L \in \mathcal{L}_{p,q}(H)} 2^{\alpha(L)} \;=\; n \cdot T(G; p+1, q+1).$$

Suppose $L \in \mathcal{L}_{p,q}(H)$ is a labelling of the edges of H. As in the proof of Theorem 3, observe that for two edges lying on the same white face, adjacent or not, if they are labelled a and b, it must hold that $a \equiv b \pmod{\frac{n}{p}}$. (This is obviously true for adjacent edges on the same white face, hence it holds for all edges on that face, by transitivity.) Let $\mathcal{L}^{0}_{p,q}(H)$ denote the set of all labellings of H where the edges lying on the unbounded (white) face of H have labels congruent to 0 $\pmod{\frac{n}{p}}$. By symmetry, $|\mathcal{L}^{0}_{p,q}(H)| = \frac{p}{n} \cdot |\mathcal{L}_{p,q}(H)|$. Moreover, the symmetry also gives:

$$\sum_{L \in \mathcal{L}^{0}_{p,q}(H)} 2^{\alpha(L)} \;=\; \frac{p}{n} \;\cdot \sum_{L \in \mathcal{L}_{p,q}(H)} 2^{\alpha(L)} \;=\; p \cdot T(G; p+1, q+1).$$

Below we construct a bijection $\varphi : \mathcal{L}^{0}_{p,q}(H) \to \mathcal{W}_{p,q}(x,y)$, such that $\alpha(\varphi(L)) = \alpha(L)$ for every $L \in \mathcal{L}^{0}_{p,q}(H)$. Together with the formula above this implies the result.

First, notice that a Wang tiling can be viewed as a labelling of the edges of the Aztec rectangle such that each square satisfies certain conditions. Observe that every edge of the Aztec rectangle (except those on the boundary) crosses exactly one edge of the medial graph H (see Figure 20). So any labelling in $\mathcal{L}^{0}_{p,q}(H)$ induces a labelling of the edges of the Aztec rectangle (recall that the boundary edges are necessarily labelled 0). It is easy to check that any labelling in $\mathcal{L}^{0}_{p,q}(H)$ induces a labelling of the Aztec rectangle which is a valid Wang tiling.

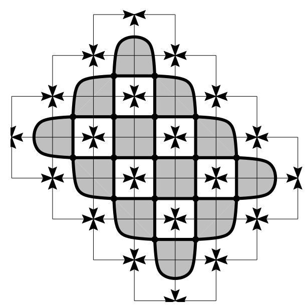

Figure 20: The 3×4 Aztec rectangle (thin edges), and H, the medial graph of the 3×4 grid.

In the other direction, we need to construct a labelling in $\mathcal{L}^{0}_{p,q}(H)$ from a Wang tiling. To do this, assign to each edge of H the same label as the edge of Aztec rectangle that it crosses. This is straightforward, except for the edges of H lying on the unbounded face, which cross more than one edge of the Aztec rectangle. However, it is easy to see that these edges of the Aztec rectangle must all have the same label, so there is no conflict. Again, it is easy to check that a labelling of H constructed in this way in fact belongs to $\mathcal{L}^{0}_{p,q}(H)$.

To summarize, we showed that labellings in $\mathcal{L}^{0}_{p,q}(H)$ are in one-to-one correspondence with Wang tilings, and monochromatic vertices in H correspond to interior monochromatic tiles. This completes the proof. ■

8.3 Proof of Theorem 5

Proof: Let us begin by simplifying the Bollobás-Riordan polynomial in this case. We have:

$$\begin{aligned} k \cdot R\big(G; k, k, \frac{1}{k}\big) &= \sum_{J \subset G} k \cdot k^{r(G)-r(J)} k^{n(J)} k^{-c(J)+bc(J)-n(J)} \\ &= \sum_{J \subset G} k^{1+r(G)-r(J)-c(J)+bc(J)} \\ &= \sum_{J \subset G} k^{1+(v(G)-c(G))-(v(J)-c(J))-c(J)+bc(J)} \\ &= \sum_{J \subset G} k^{1+v(G)-1-v(G)+c(J)-c(J)+bc(J)} \\ &= \sum_{J \subset G} k^{bc(J)}. \end{aligned}$$

The rest of the proof is bijective: we give a combinatorial interpretation of the RHS of the formula above and show that it is in bijection with that in the LHS of the formula in Theorem 5.

First, define a *bit labelling* of G to be a function which assigns a bit (0 or 1) to each edge of G. Bit labellings are in one-to-one correspondence with spanning subgraphs—a bit labelling corresponds to the spanning subgraph which consists of just those edges labelled 1.

For a ribbon graph G, define a set of *control points* as follows. Consider G as a surface with boundary, and place a control point on the boundary of each vertex between every pair of consecutive edges (see Figure 21.) Observe that there is a natural one-to-one correspondence between control points and edges of the medial graph H (both sets correspond to incidences of a vertex and a face of G). As such, we may think of a k-coloring as either a coloring of the edges of H or a coloring of the control points.

We will say two control points are *V-neighbors* if they lie consecutively on the same vertex. Similarly, we will say two control points are *F-neighbors* if they lie consecutively on the same boundary component (see Figure 21). Notice that each edge of G is bordered by four control points. Call an edge *monochromatic* if these four control points all have the same color. This is essentially the same notion as before; for a given k-coloring, an edge of G is monochromatic if and only if the corresponding vertex in H is monochromatic.

We will say that a k-coloring of the control points *agrees with* a bit labelling if the following conditions hold (see Figure 22):

- for every edge of G which is labelled 0, the two pairs of V-neighbors bordering that edge have the same color,
- for every edge of G which is labelled 1, the two pairs of F-neighbors bordering that edge have the same color.

(Having all four control points the same color is always allowed.)

Let us count the number of pairs (C, J) where C is a k-coloring which agrees with a bit labelling J. Suppose C is a k-coloring. For a non-monochromatic edge of G, there is only

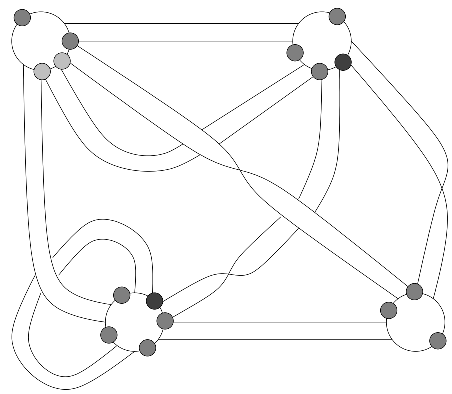

Figure 21: The shaded dots are control points. The lightest dots, for instance, are V-neighbors, while the darkest dots are F-neighbors.

Figure 22: Colorings which agree with a bit labelling.

one bit which can be assigned to it which allows for agreement. For monochromatic edges, either bit value is acceptable. Thus, $2^{\alpha(C)}$ labellings agree with C, which gives a total of $\sum_{C\in\mathcal{C}_k(H)} 2^{\alpha(C)}$ agreeing pairs.

Now, consider a spanning subgraph J of G and the corresponding bit labelling (see above). Let us count the number of k-colorings which agree with J. Consider two control points which lie consecutively on a boundary component of J. They must be either F-neighbors which border an edge in J, or V-neighbors which border an edge not in J (see Figure 23). In either case, they must have the same color in order to agree with the bit labelling J. So control points which lie on the same boundary component of J must have the same color. By similar reasoning, one can see that any k-coloring which satisfies this constraint will agree with J. There are k ways to color each of the $bc(J)$ boundary components of J, so there are $k^{bc(J)}$ different k-colorings which agree with J. Hence there are $\sum_{J\subset G} k^{bc(J)}$ agreeing pairs.

Putting together these two formulas for the number of agreeing pairs, we conclude:

$$\sum_{C\in\mathcal{C}_k(H)} 2^{\alpha(C)} = \sum_{J\subset G} k^{bc(J)},$$

which completes the proof. ■

8.4 Proof of Theorem 6

Proof: We will establish a correspondence between elements of $\mathcal{Q}_{p,q}(U)$ and $\mathcal{L}_{p,q}(H)$, and the result will follow from Theorem 3. Consider any labelling of U. Notice that there is a

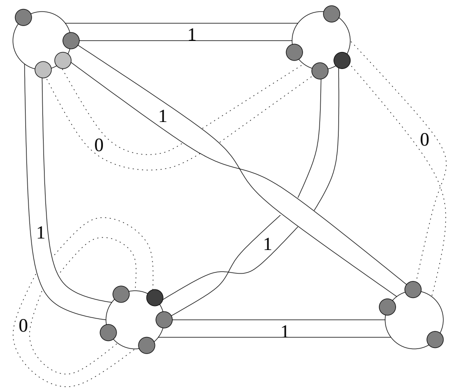

Figure 23: A spanning subgraph $J \subset G$. The lightest points are V-neighbors bordering an edge labelled 0. The darkest points are F-neighbors bordering an edge labelled 1.

one-to-one correspondence between faces of U and edges of H (see Figure 24). Each face of U is bounded by two $\star$'s and two of the same integer a mod n. Label the corresponding edge of H with this integer a mod n. It is straightforward to verify that this is a valid labelling of H. As for the other direction, take a labelling of H. Temporarily label each face of U with the label from the corresponding edge in H. Now, for each edge of U, if the two neighboring faces have the same label, label the edge with this label. If the two neighboring faces have different labels, label the edge with a $\star$. This is not a valid labelling, however. Any square vertex which was a monochromatic vertex in H is now incident to four edges with the same label, and no $\star$'s. For each such vertex, we now must arbitrarily replace two of the incident edges' labels with $\star$'s. There are two ways to do this for each such vertex. It is straightforward to verify that the resulting labellings are valid labellings of U. Each labelling L of H corresponds to $2^{\alpha(L)}$ labellings of U, so the result follows. ■

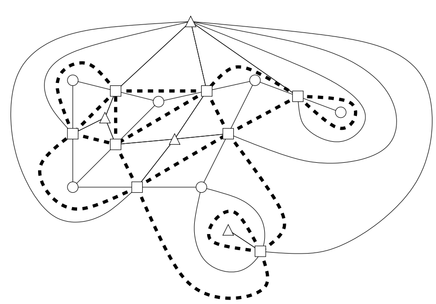

Figure 24: The graphs U (solid edges) and H (dotted edges).

The proof of Theorem 7 follows verbatim the proof above. In this case we establish a correspondence between elements of $\mathcal{Q}_{k,k}(U)$ and colorings in $\mathcal{C}_k(H)$ of the ribbon medial graph $H = H(G)$. The results then follows from Theorem 5. We omit the details.

8.5 Proof of Theorem 8

Proof: We will construct a bijection between $\mathcal{Y}(F)$ and $\mathcal{Q}_{2,2}(U)$ (see section 5). The result will then follow from Theorem 6.

Given a claw covering of F we wish to construct a labelling of U. If U and F are drawn in the standard way, then each edge of U crosses exactly one edge of F. (Specifically, the edge (x_v, y_e) in U crosses the edge $(r_{v,e,f}, r_{v,e,g})$ in F, where f and g are the faces incident to e, and the edge (y_e, z_f) in U crosses the edge $(r_{v,e,f}, r_{w,e,f})$ in F, where v and w are the vertices incident to e.) Orient each edge of U so that every circular vertex is a source and every triangular vertex is a sink. (Each edge is incident to exactly one circular or triangular vertex, so there is exactly one orientation which obeys this criterion.) Let e be an edge of U. If e crosses no claw, give e the label $\star$. If e crosses a claw such that three vertices of the claw lie to the left of e, give e the label 0. If e crosses a claw such that three vertices of the claw lie to the right of e, give e the label 1 (see Figure 25).

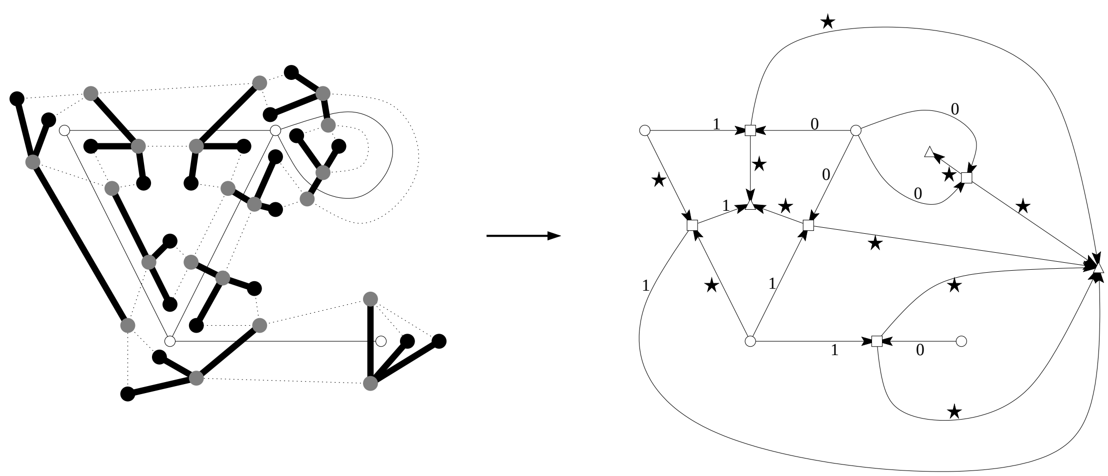

Figure 25: A claw covering of F, and the corresponding labelling of U.

Notice that F contains an equal number of red and blue vertices. Also notice that a claw can contain at most two blue vertices. Therefore, in order to cover all the vertices, each claw must contain exactly two blue vertices. For the blue vertex $s_{v,f}$, the only other blue vertex which can appear in the same claw is $t_{v,f}$. Thus these two vertices must always share a claw.

Now consider the square formed by the vertices $r_{v,e,f}$, $r_{w,e,f}$, $r_{w,e,g}$, and $r_{v,e,g}$ (where v and w are the endpoints of the edge e, and f and g are the faces incident to e). Observe that exactly two edges of this square must belong to claws, and they must be disjoint edges. Thus at the square vertex of U corresponding to the edge e, there will meet four edges which are labelled $\star$, a, $\star$, b, where a and b are either 0 or 1.

Next, consider a face of U. There are four vertices of F which lie in this face, and there are eight ways these vertices can be covered with claws (Figure 26). Notice that in each case, the edges of U are labelled with two $\star$'s and two of the same integer. Therefore, the labelling of U constructed in this way is a valid labelling, and thus belongs to $\mathcal{Q}_{2,2}(U)$.

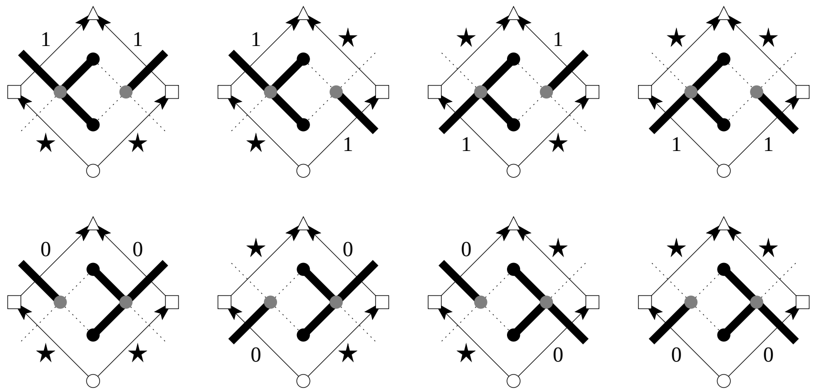

Figure 26: The eight possibilities for a face of $U(G)$.

It is straightforward to check that every labelling in $\mathcal{Q}_{2,2}(U)$ comes from exactly one claw covering in $\mathcal{Y}(F)$. This completes the bijection, proving that $|\mathcal{Y}(F)| = |\mathcal{Q}_{2,2}(U)|$. Therefore, $|\mathcal{Y}(F)| = 2 \cdot T(G; 3, 3)$, which completes the proof of Theorem 8. ■

The proof of Theorem 9 follows verbatim the proof above. The result then follows from Theorem 7. We omit the details.

9 Final Remarks

1. The main result of our previous paper [10] is a local connectivity property of T-tetromino tilings of rectangular and more general regions, which proved a conjecture made by the second author [12]. Basically, we show that one can connect any two such tilings by a sequence of moves involving either two or four T-tetrominoes. The proof involved the construction of a height function specific to this case. It seems these results can be extended to general plane graphs with faces of bounded size. This resembles similar results for perfect matchings. It would be nice to see this approach carried though (see [12] for definitions and further references on height functions).

We should add that the appendix in [10] gives another approach to local connectivity. Our proof of Corollary 2 was in fact suggested by these results.

2. Our motivation for the general Theorem 3 goes to an interesting recent paper by Reiner [13], which extends a paper by Jaeger [8]. The latter in turns extends [11] whose main result is Corollary 2. One can probably derive our Theorem 3 directly from the combinatorial interpretation in [13], but not vice versa, as Jaeger's and Reiner's interpretations hold in a more general context of representable matroids.

To put this in perspective, recall Stanley's classical combinatorial interpretation of the value $|T(G; 2, 0)|$ as the number of acyclic orientations of edges in G [14]. This was later extended to hyperplane arrangements and oriented matroids [19, 7]. We believe we now have a conceptual understanding of Stanley's combinatorial evaluation. It would be interesting to see what can be done in this direction for our evaluations.

3. The discrepancy between pairs of results for plane and ribbon graphs (such as Theorem 3 and Theorem 5) suggests that there might be a general theorem extending for general positive integer values of the Bollobás-Riordan polynomial for ribbon graphs. In fact, we are doubtful that such a generalization exists, but are unable to formalize this disbelief.

We should mention that the importance of the specialization $f(G;k) = R(G;k,k,1/k)$ was already noticed in [5] where the authors showed that it is invariant under duality: $f(G;k) = f(G^*;k)$. Unfortunately, the full polynomial does not satisfy a natural generalization of the duality ($\circledast$) given in section 1.

In a different direction, observe that all our results can be extended to disconnected graphs with only minor revisions, since the Tutte polynomial is multiplicative under disjoint unions of graphs (see section 1). However, for simplicity we chose to use only connected graphs for the statement of our theorems.

4. In section 7 we present a result for a torus graph $\mathbb{T}_{x,y}$. In fact, one can attach sides of a $x \times y$ grid graph in many other ways, which give surfaces of any genus $g \geq 0$. The results extend to these cases without difficulty.

5. There is an indirect way to deduce Theorem 1 from known results in Knot Theory. We refer to [1] for basic definitions on knot and link invariants.

Define *Kauffman's bracket polynomial* as follows:

$$\mathrm{L}(G;A,B,d) \,=\, \sum_K A^{a(K)}\, B^{b(K)}\, d^{c(K)},$$

where the summation goes over splitting as in Figure 19 of all vertices of the medial graph $H = H(G)$; statistics $a(K)$, $b(K)$ are the numbers of splittings of each type, and $c(K)$ is the number of cycles in the resulting diagram. In [9] Kauffman showed that the Jones polynomial and the Tutte polynomial are special cases of $\mathrm{L}(\cdot)$.

Following the proof of Theorem 5, one can show that the LHS of the formula in Theorem 1 is equal to $\mathrm{L}(G;1,1,k)$. Applying Kauffman's Theorem here, this equals to the value of the Tutte polynomial, proving Theorem 1. Our proof, as the proof in [9], uses the recursive formulas, so there seem to be little advantage in using Knot Theory to prove the results. On the other hand, Knot Theory has become a motivation and a driving force of recent investigations of (extensions of) Tutte polynomial, including papers [4, 5]. The interested reader is advised to study these developments; bibliography in [1, 3, 18] can be used as a starting point for references.

6. Note that the existence of a claw covering in a finite graph is a special case of the exact cover problem that is a well known NP complete problem [6]. We believe that the claw cover problem is also NP complete, while the number of claw coverings is #P complete. It would be interesting to see if this holds even if G is planar.

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