TILE INVARIANTS: NEW HORIZONS

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December 20, 2000

Abstract. Let $\mathbf{T}$ be a finite set of tiles. The group of invariants $\mathbb{G}(\mathbf{T})$, introduced by the author [P], is a group of linear relations between the number of copies of tiles in tilings of the same region. We survey known results about $\mathbb{G}$, the height function approach, the local move property, various applications and special cases.

Introduction

The problem of tileability of a region is very old, and in many instances computationally hard, even for small sets of tiles (see e.g. [MR,Ro]). The subject of this paper is different, although not unrelated. We study a *group of invariants* $\mathbb{G} = \mathbb{G}(\mathbf{T})$, associated with a set of tiles $\mathbf{T}$. This notion was introduced in [P], and further studied in [MuP,MoP]. The elements of $\mathbb{G}$ correspond to linear relations for the number of copies of tiles used in different tiling of every fixed region Γ. Turns out, this group has various nice properties, and in certain special cases can be fully computed.

In this paper we survey much of what is known about $\mathbb{G}$, the basic algebraic properties, some complexity results, as well as some applications and special cases. We describe some examples when coloring arguments do not suffice, while a different technique can be applied. A number of results never appeared before; their proofs will be sketched. We also include conjectures and open problems for further study.

Rather than define the group of invariants here, let us discuss a small but very interesting example of domino tilings, which was one of our motivations. Denote by τ_1, τ_2 the vertical and horisontal domino tiles, and let $\mathbf{T} = \{\tau_1, \tau_2\}$. Let Γ be a connected region on a square grid. The problem of tileability of Γ by $\mathbf{T}$ corresponds to finding a perfect matching in a dual graph, so it can be solved in polynomial time [LP].

Now, let A be a tiling of Γ by dominoes. Denote by $\alpha_1(A)$, $\alpha_2(A)$ the number of times tiles τ_1, τ_2 appear in A. Clearly, $\alpha_1(A) + \alpha_2(A) = |\Gamma|/2$, which follows

Key words and phrases. Polyomino tilings, tile invariants, Conway group, undecidability, height function.

from the area consideration. Also, one can show that $\alpha_1(A) = \text{const}(\Gamma) \mod 2$, where the const depends only on the region Γ, and not on the tiling. This follows from a simple coloring argument [P]. We call the linear relations as above the *tile invariants*. In general, tile invariants are the linear relations of the type

$$(*) \quad c_1\, \alpha_1(A) + c_2\, \alpha_2(A) + \ \dots \ \equiv \text{const}(\Gamma) \mod m,$$

where the const(Γ) depends only on the region Γ, and not on the tiling A of Γ; $c_i \in \mathbb{Z}$, and $m = \infty$ is allowed. The group $\mathbb{G}(\mathbf{T})$ can be defined as the group of such invariants, with addition as a group operation (the precise definition will be given in section 1). In the case of dominoes, the group of invariants is $\mathbb{G}(\mathbf{T}) = \mathbb{Z} \times \mathbb{Z}_2$, generated by the two invariants described above.

Our goal is to determine the group of invariants, and compute it in some special cases. For example, as in the case of dominoes, tile invariants can often be derived from certain colorings of the squares. In section 1 we follow [P] and introduce the *group of valuations* $\mathbb{E} \subset \mathbb{G}$, closely related to the extended coloring arguments. As we mentioned above, in general not all tile invariants can be obtained by the extended coloring arguments. This difference can be underscored by the complexity results. We show that in general case computing $\mathbb{G}$ is NP-hard, and even undecidable when considered on the whole plane. At the same time, $\mathbb{E}$ can be determined in polynomial time (see section 3.)

Now, if the group $\mathbb{G}(\mathbf{T})$ is computed, one can use it to obtain criteria for tileability of regions Γ tileable by $\mathbf{T}$ with a proper subset $\mathbf{T}'$ of tiles. Indeed, in this case the number of times α_i the tiles $\tau_i \in \mathbf{T}'$ can occur in the tiling of Γ must satisfy a number of linear relations. Existence of integral solution of these relations gives a tileability criteria. This approach was pioneered in [CL] and later successfully used in [P] to obtain tileability results which cannot be proved by coloring arguments (see section 9.)

The difficulty with the group of invariants is proving that a suspected relation is indeed a tile invariant. At the moment we see only two ways of proving such a result. The first has to do with the *local move property*. Recall that one can obtain any domino tiling A_1 of a simply connected region Γ to any other domino tiling A_2 of Γ by a sequence of 2×2 moves (see e.g. [LP,T].) Now, in general, it suffices to check that a given relation is preserved by such moves. In fact, one can easily compute the whole group of invariants in this case (see section 4.)

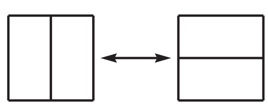

FIGURE 0.1. Local 2×2 move.

Unfortunately, very few sets of tiles have a finite number of local moves. For example, even for dominoes in three dimensions there exist infinitely many principally different simply connected regions which have exactly two domino tilings. In the other direction, even when we believe that there exist a finite number of local moves, even when we conjecture we know them all, the problem of proving this claim may be very hard.

The second and the most successful at the moment approach is based on the notion of height function, and was inspired by the Conway group [CL] and Thurston's article [T]. Roughly, Thurston defined a function from edges in the grid into a line, which maps tileable regions into loops. This approach is useful for proving local move property and finding new tile invariants [T,CL]. In the case of domino tilings, Thurston's height functions proves the connectivity of tilings by the 2×2 moves. It also gives a remarkable linear time algorithm for testing tileability of simply connected regions [Ch,F]. In sections 4, 5 we present general conditions for the technique to succeed.

While our exposition is somewhat brief due to the space limitations, we include a large number of examples and references when the techniques in the survey were successfully applied to various tiling problems. Among others, we present a final result of computation of the ribbon tile invariants [MoP], started earlier in [CL,MuP,P1] (see section 6). We also go at length to describe the Generalized Sperner's Lemma which can also be defined as a tile invariant for a special set of tiles (section 8.1). We conclude with the heuristic method for study of general set of tiles.

Many results are only stated in the main body of the paper. We sketch the proofs of new results in section 10.

Acknowledgements.

We would like to thank David Ingerman, Ezra Miller, Cris Moore, Roman Muchnik, Jim Propp and Richard Stanley for stimulating conversations and encouragement. Without them this work would never be written. We are also grateful to Mike Sipser, Dan Spielman and Santosh Vempala for help with complexity questions.

1. Basic definitions

The most general tiling problem can be formulated as follows. Let Λ be a finite or infinite set, and let $\mathcal{B}$ be a collection of finite subsets, which we call *regions*. Let '$\sim$' be an equivalence relation on $\mathcal{B}$. We will assume that '$\sim$' preserves size (the number of elements in the region). Finally, let $\mathbf{T}$ be a finite subset of $\mathcal{B}$ (the *set of tiles*). Denote by $\widetilde{\mathbf{T}}$ the set of regions $\tau \in \mathcal{B}$ such that $\tau \sim \tau' \in \mathbf{T}$. We assume that $\tau \nsim \tau'$, for all $\tau, \tau' \in \mathbf{T}$.

A typical example is a square grid $\Lambda = \mathbb{Z}^2$ with a set of simply connected regions $\mathcal{B}$ and translation equivalence '$\sim$'. Note that we view tiles here as subsets of squares, for example dominoes correspond to pairs of adjacent squares in the grid.

The problem of tileability by the set of tiles $\mathbf{T}$ is a decision whether a given set $\Gamma \in \mathcal{B}$ can be presented as a disjoint union of regions in $\widetilde{\mathbf{T}}$: $\Gamma = \sqcup \tau_i$, where $\tau_i \in \widetilde{\mathbf{T}}$ for all i. We denote such tilings by A and write $A \vdash \Gamma$. This problem is hard even in some very simple special cases, and will not be studied in this paper. Instead, we will study an abelian group $\mathbb{G}(\mathbf{T}, \mathcal{B})$ which can be defined as follows.

Let $\mathbf{T} = \{\tau_1, \dots, \tau_k\}$ be the set of tiles, where $k = |\mathbf{T}|$. For every tiling A of a region $\Gamma \in \mathcal{B}$ denote by $\alpha_i(A)$ the number of tiles $\tau \in A$ such that $\tau \sim \tau_i$. Now let

$$\mathbb{G}(\mathbf{T}, \mathcal{B}) = \mathbb{Z}^k / \mathbb{Z}\big\langle\big(\alpha_1(A) - \alpha_1(A'), \dots, \alpha_k(A) - \alpha_k(A')\big), \forall \Gamma \in \mathcal{B}, \forall A, A' \vdash \Gamma\big\rangle,$$

where on the right hand side we have a subgroup of k-vectors with A, A' any two tilings by $\mathbf{T}$ of the same region $\Gamma \in \mathcal{B}$. This is a *group of invariants*, the main subject of this paper. The elements of $\mathbb{G}(\mathbf{T},\mathcal{B})$ are called *tile invariants*.

In general, $\mathbb{G}(\mathbf{T},\mathcal{B})$ may depend heavily on the set of regions (all regions vs. simply connected regions) as well as a set of tiles (adding one tile may destroy most of the tile invariants). Note also that if $\mathcal{B}_1 \subset \mathcal{B}_2$, then $\mathbb{G}(\mathbf{T},\mathcal{B}_1) \supset \mathbb{G}(\mathbf{T},\mathcal{B}_2)$. Similarly, if $\mathbf{T}_1 \subset \mathbf{T}_2$, then

$$\mathbb{G}(\mathbf{T}_2,\mathcal{B}) \subset \mathbb{G}(\mathbf{T}_1,\mathcal{B}) \times \mathbb{Z}^{|\mathbf{T}_2|-|\mathbf{T}_1|}.$$

Define a *coloring group*

$$\mathbb{O}(\mathbf{T}) = \mathbb{Z}^{\Lambda}/\mathbb{Z}\langle x_1 + \cdots + x_r = 0, \ \forall \tau = \{x_1,\ldots,x_r\} \in \widetilde{\mathbf{T}}\rangle.$$

One can think of elements of $\mathbb{O}$ as of functions $f : \Lambda \to \mathbb{Z}$, such that $f(\Gamma) = \sum_{x\in\Gamma} f(x)$, and $f(\tau) = 0$ for all $\tau \in \widetilde{\mathbf{T}}$. The function f is called a *coloring map*. Before recently, coloring maps were the main tool to prove untileability [G]. Indeed, if $f(\Gamma) \neq 0$, this immediately implies that Γ is not tileable by $\mathbf{T}$. In this case we say that a *coloring argument* f rejects tileability of Γ. Let us add that any map $f : \Lambda \to G$, where G is abelian, can obtain from the above functions. In other words, if any coloring arguments $f : \Lambda \to G$ rejects tileability of Γ, for some abelian group G, it also rejects tileability for some $f : \Lambda \to \mathbb{Z}_m$.

Now, define an *extended coloring group*

$$\overline{\mathbb{O}}(\mathbf{T}) = \mathbb{Z}^{\Lambda}/\mathbb{Z}\langle x_1 + \cdots + x_r = y_1 + \cdots + y_r\rangle,$$

where $\tau = \{x_1,\ldots,x_r\}$, $\tau' = \{y_1,\ldots,r_r\}$, and $\tau \sim \tau' \in \widetilde{\mathbf{T}}$. Clearly, $\mathbb{O}(\mathbf{T}) \subset \overline{\mathbb{O}}(\mathbf{T})$. One can think of the elements of $\overline{\mathbb{O}}(\mathbf{T})$ as of functions $f : \Lambda \to \mathbb{Z}$, which are constant on equivalent tiles in $\widetilde{\mathbf{T}}$. We call such functions an *extended coloring maps*.

There is a natural map $\nu : \overline{\mathbb{O}}(\mathbf{T}) \to \mathbb{Z}^{\mathbf{T}}$ which maps the functions to their values on tiles in $\mathbf{T}$. We have $\mathbb{O}(\mathbf{T}) = \nu^{-1}(0)$. By definition, the value $f(\Gamma)$ of a function in $\overline{\mathbb{O}}(\mathbf{T})$ is independent on the tiling by $\mathbf{T}$, so ν extends to the quotient group $\mathbb{G}(\mathbf{T})$. Denote by $\mathbb{E}(\mathbf{T})$ the image of ν in $\mathbb{G}(\mathbf{T})$. We call $\mathbb{E}(\mathbf{T})$ the *group of valuations* of the set of tiles $\mathbf{T}$. From above,

$$\mathbb{E}(\mathbf{T}) \simeq \overline{\mathbb{O}}(\mathbf{T})/\mathbb{O}(\mathbf{T}).$$

By definition, the subgroup $\mathbb{E}(\mathbf{T}) \subset \mathbb{G}(\mathbf{T})$ consists of all tile invariants which follow from the extended coloring maps.

Computing the coloring group and the group of valuations is of interest, so as to see which tileability criteria and which group invariants are "easy to obtain".

Unless stated otherwise, for the rest of the paper we will assume that $\Lambda \subseteq \mathrm{Z}^2$, where Z^2 denotes the square grid with elements - 1×1 squares. Denote by $\mathcal{B}$, $\mathcal{B}_{\mathrm{sc}}$, $\mathcal{B}_N$ the set of all regions, of all simply connected regions, and the set of regions in $N \times N$ square. The equivalence relation consists of parallel translations of the regions (no rotation or reflection is allowed). Let the set of tiles $\mathbf{T}$ consist of some k tiles, each of size $\leq R$. By abuse of notation, we use $\tau \in \mathbf{T}$ to denote $\tau \in \widetilde{\mathbf{T}}$.

The main questions of this paper can be stated as follows:

Group of Invariants Problem (GI) :
Given $\mathbf{T} \subset \mathrm{Z}^2$, compute $\mathbb{G}(\mathbf{T}, \mathcal{B})$ (or $\mathbb{G}(\mathbf{T}, \mathcal{B}_{\mathrm{sc}})$, $\mathbb{G}(\mathbf{T}, \mathcal{B}_N)$).

Tileability Problem (T) :
Given $\mathbf{T} \subset \mathrm{Z}^2$, $\Gamma \in \mathcal{B}$ (or $\mathcal{B}_{\mathrm{sc}}$, $\mathcal{B}_N$), decide whether Γ is tileable by $\mathbf{T}$.

Group of Valuations Problem (GV) :
Given $\mathbf{T} \subset \mathrm{Z}^2$, compute $\mathbb{E}(\mathbf{T})$.

Coloring Group Problem (CG) :
Given $\mathbf{T} \subset \mathrm{Z}^2$, compute $\mathbb{O}(\mathbf{T})$.

The last two problems are very much related, but we decided to separate them for convenience.

We say that a tile invariant is *finite* (*infinite*) if the order of the element in $\mathbb{G}$ is finite (infinite). Using definition $(*)$ in the introduction, the invariant is infinite if $m = \infty$. We will come back to tile invariants in the next section.

Remark 1.1 Much of this survey can be understood with conventional definitions of the tilings on a square grid. The point of this somewhat overgeneralized section was to introduce the general concepts and notation we use throughout the paper, as well as to prepare the reader to possible extensions and generalizations. While much of the results in the paper can be generalized by verbatim, we decided to keep the presentation simple for the sake of clarity. At the same time we hope that after reading this section the reader is fully equipped to generalize the results to any appropriate level.

Remark 1.2 One should keep in mind that the tile invariants were implicitly introduced in [CL] in order to obtain new tileability criteria. Although we downplay the connection in this paper, the results that are obtained in this direction can be judges as the most unexpected. See section 9 for for details.

2. Algebraic aspects

Fix a set of tiles $\mathbf{T} = \{\tau_1, \ldots, \tau_k\} \subset \mathrm{Z}^2$. Consider $\mathbb{G} = \mathbb{G}(\mathbf{T}, \mathcal{B})$. Since $\mathbb{G}$ is abelian, it can be presented as

$$\mathbb{G} \simeq \mathbb{Z}^r \times (\mathbb{Z}_2)^{m_2} \times (\mathbb{Z}_3)^{m_3} \times \cdots \times (\mathbb{Z}_{p^c})^{m_{p^c}} \times \ldots,$$

where $r \leq k$ is called the *free rank* of $\mathbb{G}$, denotes $\mathrm{rk}(G)$, and $\mathbb{Z}^{\mathrm{rk}} \subset \mathbb{G}$ is called the *free subgroup* of $\mathbb{G}$. Similarly, denote by $M = \sum_{q=p^c} m_q$ the *torsion rank* of $\mathbb{G}$, and $\mathbb{T} = (\mathbb{Z}_2)^{m_2} \times (\mathbb{Z}_3)^{m_3} \times \ldots \subset \mathbb{G}$ is called the *torsion subgroup* of $\mathbb{G}$. By construction, the torsion subgroup is always finite.

Proposition 2.1 *For N sufficiently large, we have $\mathbb{G}(\mathbf{T}, \mathcal{B}_N) = \mathbb{G}(\mathbf{T}, \mathcal{B})$.*

Sketch of proof. Consider a sequence of subgroups $\mathbb{G}_N = \mathbb{G}(\mathbf{T}, \mathcal{B}_N)$. Recall that $\mathbb{G}_N \supset \mathbb{G}_{N+1}$. By Hilbert Basis Theorem, this sequence stabilizes. $\square$

Now let us turn to signed tilings and the coloring group. Denote by $\chi(\Gamma) \in \mathbb{R}^\Lambda$ the characteristic function of a region Γ. One can think of a tiling of Γ by $\mathbf{T}$ as of decomposition $\chi(\Gamma) = \chi(\tau) + \chi(\tau') + \dots$, where $\tau, \tau', \dots \in \mathbf{T}$. The signed tiling is similar decomposition, where each tile is used with a positive or negative sign. Note that the notion of the coloring argument extends to signed tilings as well.

Theorem 2.2 [P] *A region* Γ *has a signed tiling by* $\mathbf{T}$ *if and only if there is no coloring argument which would reject tileability.*

Sketch of proof. Note that signed tilings by $\mathbf{T}$ form a group $\mathbb{S}(\mathbf{T})$, with addition as an operation. By definition, we have $\mathbb{O}(\mathbf{T}) = \mathbb{Z}^{\mathbf{T}}/\mathbb{S}(\mathbf{T})$, which is a reformulation of the result. $\square$

Similarly to the coloring arguments, consider the extended coloring arguments for signed tilings. Define $\mathbb{E}_\circ(\mathbf{T}) = \mathbb{E}(\mathbf{T} \cup -T)$, where $-\mathbf{T}$ contains the negative tiles $-\tau$, with $\chi_{-\tau} = -\chi_\tau$. We claim that

$$\mathbb{E}_\circ(\mathbf{T}) \simeq \mathbb{E}(\mathbf{T}).$$

Indeed, let $f : \Lambda \to Z$ be any extended coloring map. Since $\chi_{-\tau} + \chi_\tau = 0$, we have $f(-\tau) = -f(\tau)$ and thus $\mathbb{E}_\circ(\mathbf{T}) \subset \mathbb{E}(\mathbf{T})$. On the other hand, $\mathbb{E}(\mathbf{T}) \subset \mathbb{E}_\circ(\mathbf{T})$ since every extended coloring map by definition corresponds to an extended coloring map for signed tiles $\mathbf{T} \cup -\mathbf{T}$, and therefore defines a proper valuation on $\mathbf{T} \cup -\mathbf{T}$.

An interesting class of tile invariants are the *abelian invariants*, which are defined as tile invariants which remain invariants for signed tilings. Define *group of abelian invariants* $\mathbb{A}(\mathbf{T}) = \mathbb{G}(\mathbf{T} \cup -\mathbf{T})$. From above, we conclude that $\mathbb{E}(\mathbf{T}) \subset \mathbb{A}(\mathbf{T})$. In fact, this is an identity:

Theorem 2.3 $\mathbb{A}(\mathbf{T}) = \mathbb{E}(\mathbf{T})$. $\square$

The real meaning of Theorem 2.3 can be seen in the following observation. If for some reason we have an abelian invariant, we can conclude that there exists a coloring map which defines it. In practice, finding such coloring map can be complicated. We leave the proof to the reader.

3. Complexity aspects

It is well known that the tileability problem is NP-complete when Γ is finite [GJ]. It is also undecidable when Γ is the whole plane [Be,Ri]. We shall prove that the similar situation holds for GI Problem. But first we need to state it as a decision problem.

GI-rank Problem: Given $\mathbf{T}$, r, decide whether $\operatorname{rk} \mathbb{G}(\mathbf{T}, \mathcal{B}) \geq r$.

Bounded GI-rank Problem: Given $\mathbf{T}$, r, N, decide whether $\operatorname{rk} \mathbb{G}(\mathbf{T}, \mathcal{B}_N) \geq r$.

Theorem 3.1 *The GI-rank Problem is undecidable. Similarly, the Bounded GI-rank Problem is* NP*-hard.*

The proof is given below in section 10. Roughly, Theorem 3.1 implies that computationally GI is intractible. A simple check shows that Theorem 3.1 extends to simply connected regions as well (i.e. computing the rank of $\mathbb{G}(\mathbf{T}, \mathcal{B}_{\rm sc})$ is also undecidable). It seems likely that the proof can be modified to show that computing any of the exponents m_p in the torsion group is also undecidable.

Now, let us fix the set of tiles $\mathbf{T}$. Recall that $\mathrm{rk}(\mathbb{G}) \le |\mathbf{T}|$. Proposition 2.1 implies that the negative answer to the Bounded GI-rank Problem can be obtained by an exhaustive search for some finite $N = N(\mathbf{T})$. In other words, a sequence of Bounded GI-rank Problems is in co-NP (as N grows). The certificate for $\mathrm{rk}(\mathbb{G}) < r$ is a collection of $l > n - r$ bounded regions Γ_i, $1 \le i \le l$, and two collections of tilings $A_i, A_i' \vdash \Gamma_i$, such that

$$\mathrm{rk}\,\mathbb{Z}\big\langle\big(\alpha_1(A_i) - \alpha_1(A_i'), \ldots, \alpha_k(A_i) - \alpha_k(A_i')\big),\, i = 1 \ldots l\big\rangle \,>\, n - r.$$

In a way this makes it unlikely that there is a good generic way to establish the tile invariants for general sets of tiles. For example, if height functions exist for a given set of tiles, this puts the Bounded GI-rank Problem into NP. However, it is believed that an NP-hard problem cannot be in NP $\cap$ co-NP [GJ]. We will not attempt to formalize and extend this observation.

For the signed tilings, one can define the Signed Tileability Problem (ST) by analogy. Observe that Theorem 2.2 can be used now to establish the certificates for $\mathrm{rk}(\mathbb{O}) \ge r$, $m_p(\mathbb{O}) \ge m$. Using the logic as above one would conclude that ST and CG must have efficient solutions. This is true indeed.

Bounded CG-rank Problem: Given $\mathbf{T}$, r, N, decide whether $\mathrm{rk}\,\mathbb{O}(\mathbf{T}, \mathcal{B}_N) \ge r$.

Bounded GV-rank Problem: Given $\mathbf{T}$, r, N, decide whether $\mathrm{rk}\,\mathbb{E}(\mathbf{T}, \mathcal{B}_N) \ge r$.

Theorem 3.2 *Bounded CG-rank Problem and Bounded GV-rank Problem are in* P.

The proof is based on a simple reduction to a linear algebra problem, and is given in section 10. We believe that currently known algorithms for solving linear equations over the integers (see [BK,LLL,Sc]) can be used to determine the full groups $\mathbb{O}(\mathbf{T}, \mathcal{B}_N)$, $\mathbb{E}(\mathbf{T}, \mathcal{B}_N)$. Further, we conjecture that there exist an efficient algorithm for computing $\mathbb{O}(\mathbf{T}, \mathcal{B})$, $\mathbb{E}(\mathbf{T}, \mathcal{B})$. We hope to return to this problem in the future.

4. Height functions

There seem to be no general agreement as to what exactly is the method of height functions, especially when dimension increases. Here we present our personal approach with no attempt to justify it.

Suppose $\mathbf{T}$ is a fine set of tiles of the plane Z^2, or any other plane graph L with straight edges for that matter (for example L can be triangular of hexagonal lattice). Let V be a different plane, which will also be fixed. Suppose the edges of L are oriented, and there is a function $\varphi : L \to V$ which maps oriented edges into vectors in V. Also, let $\varphi(x, y) = -\varphi(y, x)$ for all edges $(x, y) \in L$ oriented from y

to x. Now, every path $x_1 \to x_2 \to x_3 \to \ldots$ can be mapped to a path in V (up to translation): $v_1 \to v_2 \to v_3 \to \ldots$, where $v_{i+1} - v_i = \varphi(x_i, x_{i+1})$. We think about the image of the path on a graph as a polygon in V with straight edges.

The function φ is called a *height function* if the following condition is satisfied:

($\star$) *For every simply connected region* Γ *tileable by a set of tiles* $\mathbf{T}$*, the image* $\varphi(\partial\Gamma)$ *is a closed loop.*

Here the boundary $\partial\Gamma$ is a closed path with any fixed starting point and oriented counterclockwise. We will always assume that there is a finite number of equivalence classes of values $\varphi(x, y)$ for all $(x, y) \in L$. The condition ($\star$) may seem difficult to check, so the following result helps to simplify it.

Theorem 4.1 *It suffices to check* ($\star$) *only for the tiles* $\tau \in \mathbf{T}$.

The theorem follows easily by induction from the following lemma of independent interest.

Lemma 4.2 *Let* $\Gamma \subset \mathbb{R}^2$ *be a simply connected region and is tiled by simply connected regions* $\tau_1, \ldots, \tau_k$. *Then there exist* i *such that* $\Gamma - \tau_i$ *is also simply connected.*

Lemma 4.2 seems to be well known in geometric group theory, although we were unable to obtain any reference to that. In this context it was sketched in the pioneer paper [CL]. A simple proof can be found in [MP] (see also [Pr]).

Let us remark that in 3 and more dimensions Lemma 4.2 as stated is incorrect[1]. On the other hand, proof of Theorem 4.1 requires a result somewhat weaker that that in the lemma. For example, one can change the statement to *"there exist* $i_1, \ldots, i_l$ *such that regions* $\tau_{i_1} \cup \ldots \cup \tau_{i_l}$ *and* $\Gamma - \left(\tau_{i_1} \cup \ldots \cup \tau_{i_l}\right)$ *are simply connected*[2]*"*. We do not believe that even this weaker condition holds. It would be interesting to find an explicit counterexample to that.

Now, once the height function is given, it can be used to prove certain tile invariants for the set of tiles $\mathbf{T}$, not unlike the extended coloring arguments. Indeed, consider any extended coloring argument $f : V \to G$ (G is abelian), where now we require the value $f(\varphi(\tau))$ to be invariant of the location of the τ on the plane. By construction, $f(\varphi(\Gamma))$ is always the sum of the $f(\varphi(\tau_i))$ and is independent of the tiling. Therefore the values $c_i = f(\varphi(\tau_i))$, $\tau \in \mathbf{T}$ define a tile invariant for $\mathbf{T}$.

Formally, denote by $\mathbb{E}_\varphi(\mathbf{T})$ the group of valuations of extended coloring arguments on V for the set of tiles $\varphi(\tau_i)$. Then

$$(**) \quad \mathbb{E}_\varphi(\mathbf{T}) \subseteq \mathbb{G}(\mathbf{T}, \mathcal{B}_{\text{sc}}).$$

This means that in certain cases when there exists a height function, one can obtain proofs of certain tile invariants by finding an appropriate extended coloring

[1] A counterexample is a family of six blocks which form a three dimensional cross shape figure, and is hard to disassemble. In this case no block can be removed without the remaining union of five blocks having a hole inside. Versions of this puzzle can be often found in toy stores.

[2] Actually, we need a slightly stronger condition on the intersection of the two simply connected parts.

argument in V. In other words, one can sometimes compute the whole group of invariants $\mathbb{G}(\mathbf{T}, \mathcal{B}_{\rm sc})$.

We should note here that condition $(\star)$ does not necessarily imply that $\varphi(A)$, $A \vdash \Gamma$ is a tiling of Γ with tiles $\varphi(\tau_i)^3$. Rather, we obtain a signed tiling of $\varphi(\Gamma)$. Still, the conclusion $(**)$ remains valid in view of results in section 3.

Let us emphasize once again, that the relationship

$$\textit{height functions} \;\longleftrightarrow\; \textit{tile invariants}$$

seem to go smoothly only on a plane. In principle, of course, neither Λ nor V have to be planar. There are several interesting example of the height functions when V is a line and dimension of Λ varies. We will come back to such examples in the next section. Let us note also that we don't seem to have any nontrivial example of two-dimensional height functions when Λ is not planar, and nothing at all when V is three and more - dimensional.

5. Local moves

5.1 One-dimensional height functions.

Let $\mathbf{T}$ be a finite set of tiles, $\mathcal{B}$ be any set of finite regions. We say that $\mathbf{T}$ satisfies *local move property with respect to* $\mathcal{B}$ if there exists a finite set of regions $\Gamma_1, \dots, \Gamma_\ell \in \mathcal{B}$, and two collections of tilings $A_i, A'_i \vdash \Gamma_i$, for all $1 \le i \le \ell$ (cf. section 3), such that

$(\diamond)$ *For every* $\Gamma \in \mathcal{B}$ *and two tilings* $A, A' \vdash \Gamma$, *there exists a sequence of tilings* $A = B_0 \to B_1 \to \ \dots \ \to B_t = A'$, *where the arrow* $X \to Y$ *is between two tilings which differ in a region* $\Gamma' \sim \Gamma_i$, *with the tilings* X, Y *restricted to* $\Gamma' \subset \Gamma$, *being the tilings* A_i *and* A'_i.

Theorem 5.1 *If* $\mathbf{T}$ *satisfies local move property with respect to* $\mathcal{B}$, *then the GI-rank Problem is in* P.

The main problem with the local move property is scarcity of the sets of tiles which have it and difficulty of proving it in this case. Most known approaches are more or less ad hoc, with a small exception of the height function approach. Again, there seem to be no consensus of how this should work in general. We describe here a version of it, following [T,Ch,ST].

Let $\Lambda \subset \mathbb{R}^d$ be a d-dimensional structure (set of lattice cubes, simplices, etc.) For every $\Gamma \subset \Lambda$ denote by $\widehat{\Gamma}$ the set of points $x \in \mathbb{R}^d$ inside Γ. Suppose $\varphi : \Lambda \to \mathbb{R}$ is a one dimensional height function, such that $\varphi : \tau \to \mathbb{R}$ can be defined at all points $x \in \widehat{\tau}$ (by using piecewise linearity, or otherwise). This defines a function $\varphi_A : \widehat{\Gamma} \to \mathbb{R}$ for every tiling $A \vdash \Gamma$. We say that $\varphi(A) \le \varphi(A')$, where $A, A' \vdash \Gamma$, if for all points $x \in \Gamma$ we have $\varphi_A(x) \le \varphi_{A'}(x)$. Finally, denote by '$\prec$' a partial linear order on tilings $A, A' \vdash \Gamma$:

$$A \prec A' \ \text{ if and only if } \ \varphi(A) \le \varphi(A').$$

[3]The tiles $\varphi(\tau_i) \subset V$ may also not be uniquely defined. The extended coloring argument f defined above must be constant on all such tiles though.

Note that a priori there could be incomparable tilings.

Now, suppose the "suspected" set of local moves

$$(\bigcirc) \quad \{(A_i \to A_i'), A_i, A_i' \vdash \Gamma_i, 1 \le i \le \ell\}$$

satisfied the following properties:

($\bullet$) *Either $A_i \prec A_i'$ or $A_i \succ A_i'$ for all $1 \le i \le \ell$.*

($\bullet\bullet$) *If $x \in \widehat{\Gamma} - \partial\Gamma$, is a local maximum of φ_A, $A \vdash \Gamma$, then there exists a local move $A \to A'$ such that $A' \prec A$.*

($\bullet\bullet\bullet$) *For all $x \in \partial\Gamma$ there exists a unique tile τ_x, $\widehat{\tau} \ni x$, such that if x is a local maximum of φ_A, $A \vdash \Gamma$, then $A \ni \tau$.*

Theorem 5.2 *Let $\mathcal{B} = \mathcal{B}_{\rm sc}$ and $d = 2$. If ($\bigcirc$) and a one-dimensional height function φ satisfies ($\bullet$) – ($\bullet\bullet\bullet$) for all $\Gamma \in \mathcal{B}$, then $\mathbf{T}$ satisfies the local move property with respect to $\mathcal{B}$, with ($\bigcirc$) as a set of local moves. Further, the maximum number M of local moves to be made satisfies $M \le c\,|\Gamma|^2$, where $c = c(\mathbf{T})$ does not depend on Γ. Finally, the Tileability Problem is in* P *in this case.*

To avoid problems related to generalizations of Lemma 4.2, the above result covers only the case $d = 2$. For $d \ge 3$ we need an additional geometric condition to compensate for absence of the Lemma. Formally, consider the following property:

($\clubsuit$) For every local maxima $x \in \partial\Gamma$, $\Gamma \in \mathcal{B}$ we always have $\Gamma - \tau_x = \Gamma' \sqcup \Gamma'' \sqcup \ldots$, where $\Gamma', \Gamma'', \ldots \in \mathcal{B}$.

It is easy to see that $\mathcal{B}_{\rm sc}$ satisfies ($\clubsuit$) for $d = 2$, so the following result is a generalization of Theorem 5.2.

Theorem 5.2$'$ *If in condition of Theorem 5.2 the property ($\clubsuit$) is also satisfied, then conclusion of Theorem 5.2 holds for all $d \ge 2$.*

Note that the conclusion of Theorem 5.2 implies, by Theorem 5.1, that the GI Problem is also in P in this case. As we shall see, the examples include domino tilings, zonotopal tilings, etc. It would be interesting to find analogs of ($\bullet$) for two-dimensional height function. This could positively resolve the connectivity conjecture for ribbon tilings.

Conjecture 5.3 *If $\mathbf{T}$ satisfies the local move property with respect to $\mathcal{B}_{\rm sc}$, then Tileability Problem for regions $\Gamma \in \mathcal{B}_{\rm sc}$ is in* P.

While we have only few known examples of the local moves property, the conjecture seem to hold. Theorem 5.2 seem to support the conjecture. Note that if $\Gamma \in \mathcal{B}$ is untileable, then ($\diamond$) holds by default. Heuristicly, the conjecture suggests that for any set of local moves one should be able to define a "generalized one-dimensional height function", and apply the analog of the last part of Theorem 5.2.

5.2 Tiling Polytope.

Let us conclude this section with a polytopal interpretation of the local moves. Define *rational tilings* (cf. [SU]) to be decompositions $\chi(\Gamma) = \kappa\,\chi(\tau) + \kappa'\,\chi(\tau') + \ldots$, where $\tau, \tau', \dots \in \mathbf{T}$, $\kappa, \kappa', \dots \in \mathbb{Q}_+$.

Theorem 5.4 *Rational Tileability Problem is in* P.

Proof. Let '$\prec$' be a lexicographic order on Λ. For any $\tau \in \mathbf{T}$, denote by τ_x the unique tile $\sim \tau$, such that $x \prec y$ for all $y \in \tau_x$. In other words, let τ_x be the tile obtained by translation of τ such that x is the smallest element in τ_x.

Let $k = |\mathbf{T}|$. For any region $\Gamma \in \mathcal{B}$, consider a polytope $\mathbf{P}_\Gamma \subset \mathbb{R}^{k|\Gamma|} = \mathbb{R}\langle a_{x,\tau},\ x \in \Gamma, \tau \in \mathbf{T}\rangle$, defined by the following linear equations and inequalities:

$$\begin{cases} a_{x,\tau} \geq 0, \quad \forall\, x \in \Gamma, \tau \in \mathbf{T}, \\ \displaystyle\sum_{x,\tau:\ \tau_x \ni y} a_{x,\tau} = 1,\ \forall\, y \in \Gamma. \end{cases}$$

Now, every rational point (a) in the polytope $\mathbf{P}_\Gamma$ corresponds to a rational tiling with $\kappa_{\tau_x} = a_{x,\tau}$. Since the system is rational, the rational tileability is equivalent to $\mathbf{P}_\gamma$ being empty or not. The latter can be determined in polynomial time (see e.g. [Sc]). $\square$

Proposition 5.5 *Let* $\mathbf{P}_\Gamma$ *be the polytope defined in the proof of Theorem 5.4. Then the integer points in* $\mathbf{P}_\Gamma$ *correspond to the (usual) tilings of* Γ *with the set of tiles* $\mathbf{T}$. $\square$

One can think of the points in $\mathbf{P}_\Gamma$ as of nonnegative real tilings of Γ. All the vertices are the rational tilings. Unfortunately, not all of them are integer (the usual) tilings. Denote by $\widehat{\mathbf{P}}_\Gamma \subset \mathbf{P}_\Gamma$ a convex hull of the integer points. We call $\widehat{\mathbf{P}}_\Gamma$ the *tiling polytope*. By definition, $\widehat{\mathbf{P}}_\Gamma$ is a $0-1$ polytope.

Let $A, A' \vdash \Gamma$. We say that a local move $A \to A'$ is *primitive* if for no $B \vdash \Gamma$ we can have two nonintersecting local moves $A \to B$ and $B \to A'$.

Theorem 5.6 *The primitive moves* $A \to A'$*, where* $A, A' \vdash \Gamma$*, are in one-to-one correspondence with edges in the tiling polytope* $\widehat{\mathbf{P}}_\Gamma$.

We should mention here that for large Γ the set of edges of the tiling polytope is much larger than the set of local moves described in the beginning. Indeed, while the local moves can be (and usually are) primitive moves, the minimal set of local moves is a very small subset of primitive moves which can be compositions of a number of (intersecting) local moves.

It is tempting to study the simplex method or other optimization problems on tiling polytopes. The difficulty is that the minimum number of linear relations and inequalities which define $\widehat{\mathbf{P}}_\Gamma$ is probably exponential in $|\Gamma|$ (it's superpolynomial unless P=NP).

5.3 Zonotopal tilings.

It was noted on many occasions that one can think of tilings by "lozenges" (analogues of dominoes in the triangular lattice) as of projection of the cubic surface, at least for certain nice simply connected regions. In fact, Thurston's height function coincides with the height of the surface in these cases (see [T,ST]). Let us briefly mention here that one can consider zonotopal tilings which extend this observation.

Let M be a finite set of vectors in $V = \mathbb{R}^d$ and suppose $\langle M \rangle = V$. Consider a polytope P_M defined as a Minkowski sum of elements in M (considered as intervals). Such polytopes are called *zonotopes*. Call *basis blocks* zonotopes P_B such that $B \subset M$, $\langle B \rangle = B = d$. Polyhedral subdivision of P_m into basis blocks are called *zonotopal tilings*. They have a number of interesting properties, in particular the basis blocks in every zonotopal tiling are in one to one correspondence with bases of a matroid M [BLSWZ,St,Z]. In fact, much of the information about P_M and zonotopal tilings can be obtained from from the (oriented) matroid structure of M (see references above).

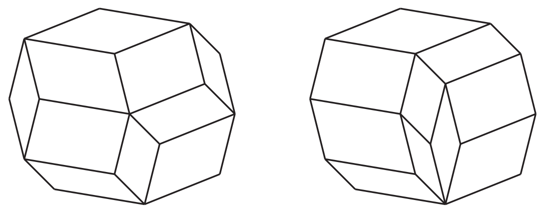

FIGURE 5.1. Two zonotopal tiling of a centrally symmetric 10-gon.

Among the most interesting properties of zonotopal tilings is (non)existence of a one-dimensional height functions. The latter correspond to the so-called 1-extensions of M (into $\mathbb{R}^{d+1}$). One can show that all zonotopal tilings that arise from every such extension are connected by "local moves" (in zonotopes generated by $d+1$ vectors). While 1-extensions of M may generate all tilings, all 1-extensions can make a graph of zonotopal tilings connected (there is a related notion of a coherent subdivision [GKZ,Z]). Still, there exist zonotopal tilings disconnected from the others. We refer to the above mentioned [BLSWZ,GKZ,St,Z] and the references therein.

6. Ribbon tiles

6.1 Basic definitions.

Let $\Lambda = \mathrm{Z}^2$ be the square grid. Let $x = (i, j) \in \Lambda$ be the square in Z^2 with i increasing downward and j increasing to the right. As before, let '$\sim$' be defined by translations.

Fix an integer $n \geq 2$. A region $\tau \in \mathcal{B}_{\mathrm{sc}}$ is called a *ribbon tile* if every diagonal $i - j = \mathrm{const}$ contains at most one square of τ. Denote by $\mathbf{T}_n$ the set of ribbon tiles with n squares. It is easy to see that $|\mathbf{T}_n| = 2^{n-1}$, with tiles τ encoded by $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1})$, $\epsilon_i \in \{0, 1\}$ as follows. Start in the lower left corner of τ and move northeast; each upward move encode with 1, each right move with 0. Denote by τ_ϵ the tile as above, and by $\alpha_\epsilon(A)$ the number of times tile τ_ϵ occurs in a tiling A.

Define 2-moves to be the local moves which involve exactly two ribbon tiles. For description of all such moves see [P]. As observed by Adin [Ad], the total number of such moves is $\binom{|\mathbf{T}_n|}{2}$. This formula is somewhat misleading since not all pairs of ribbon tiles can form a 2-move, while some pairs can form it in several ways.

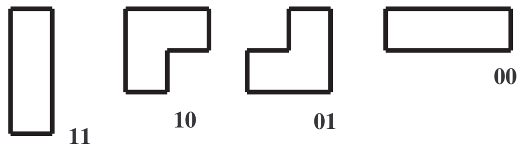

FIGURE 6.1. Ribbon trominoes.

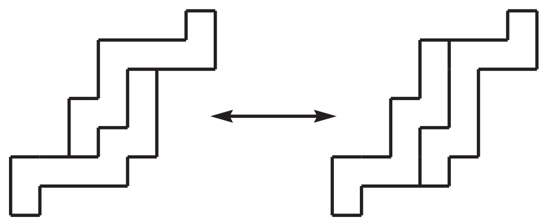

FIGURE 6.2. Example of 2-move for ribbon tiles.

The main object of this section is the successful computation of $\mathbb{G}(\mathbf{T}_n)$, and the local move property with respect to 2-moves. Note that there is an obvious *area invariant* which states that the total number of tiles τ is $|\Gamma|/n$.

6.2 Dominoes.

This is a classical example studied for decades (see e.g. [G,Ka,LP,TF]). Thurston [T]. defined an important one-dimensional height function φ which became a model for our generalization in section 5. Color the squares with two colors (black and white) in a checkerboard fashion. Orient all edges upward and to the right. The map φ is defined on edges in Z^2, and is $+1$ (-1) if the edge is moving counterclockwise (clockwise) around a black square.

One can show that the above height function with the set of 2-moves satisfies $(\bullet) - (\bullet\bullet\bullet)$. From here we obtain the local move property for 2-moves with respect to $\mathcal{B}_{\mathrm{sc}}$ as an immediate conclusion of Theorem 5.2. An elementary example shows that this does not hold for non simply connected regions. We should mention here that the result can be generalized to any planar regular graph with a bipartite dual graph [Ch]. Also, a careful look at the tileability algorithm reveals that it has cost $O(|\Gamma|)$, faster than other (general) matching algorithms [LP,Sc]. This result can be extended to non simply connected regions as well [F].

As mentioned in the introduction, the group of invariants $\mathbb{G}(\mathbf{T}_2) \simeq \mathbb{E}(\mathbf{T}_2) \simeq \mathbb{Z} \times \mathbb{Z}_2$ in this case.

6.3 Ribbon Trominoes.

The set of ribbon trominoes is the celebrated example, studied Conway and

Lagarias [CL][4]. They defined a two-dimensional height function φ which maps edges of the square lattice into a Cayley graph of a specially chosen group embedded in $\mathbb{R}^2$. The latter consists of hexagons and triangles. The sum of the winding numbers around centers of hexagons gives a nonabelian tile invariant:

$$\alpha_{01} - \alpha_{10} = \mathrm{const}(\Gamma).$$

One can conclude from here that the group of invariants $\mathbb{G}(\mathbf{T}_3) \simeq \mathbb{Z}^2$. On the other hand, direct computation shows that $\mathbb{E}(\mathbf{T}_3) \simeq \mathbb{Z} \times \mathbb{Z}_3$ [CL,P], so the infinite tile invariant above cannot be proved by means of coloring arguments.

The local move property for 2-moves with respect to $\mathcal{B}_{\mathrm{sc}}$ remains open (see below). A special case was considered in [We] for the starecase shaped regions introduced in [CL] (see also [P]).

Before we conclude, let us mention here that the approach was later modified by Muchnik and the author [MuP] to prove that $\mathbb{G}(\mathbf{T}_4) \simeq \mathbb{Z}^2 \times \mathbb{Z}_2$. At the same time, $\mathbb{E}(\mathbf{T}_4) \simeq \mathbb{Z} \times \mathbb{Z}_4$ [P].

6.4 The general case.

It was recently shown in [MoP] that for all $n \geq 2$:

$$\mathbb{G}(\mathbf{T}_n, \mathcal{B}_{\mathrm{sc}}) \simeq \begin{cases} \mathbb{Z}^m, & \text{if } n = 2m+1, \\ \mathbb{Z}^{m-1} \times \mathbb{Z}_2, & \text{if } n = 2m. \end{cases}$$

This proved the conjecture of the author [P], previously known only for $n \leq 4$. The main result of [P] is a similar result for a smaller set of regions $\mathbb{G}(\mathbf{T}_n, \mathcal{B}_{\mathrm{rc}})$, where $\mathcal{B}_{\mathrm{rc}})$ is the set of row convex regions. The author in [P] also found an explicit basis for the group:

$$\sum_{\epsilon:\ \epsilon_i=0,\,\epsilon_{n-i}=1} \alpha_\epsilon \ - \sum_{\epsilon:\ \epsilon_i=1,\,\epsilon_{n-i}=0} \alpha_\epsilon \ = \ \mathrm{const}(\Gamma),\ \ 1 \leq i < n/2,$$

and

$$\sum_{\epsilon:\ \epsilon_{n/2}=0} \alpha_\epsilon \ = \ \mathrm{const}(\Gamma) \mod 2, \quad n = 2m.$$

On the other hand, it was shown in [P] that $\mathbb{E}(\mathbf{T}_n) \simeq \mathbb{Z} \times \mathbb{Z}_n$, and all tile invariants in the basis do not follow from the extended coloring arguments.

The technique used in [MoP] is notable since it used a new construction of the two-dimensional height function φ, which mapped the edges of the square lattice into $\{\omega^k, 0 \leq k \leq n-1\} \subset \mathbb{C}$, where $\omega = \exp(2\pi i/n)$. Then the authors take a signed area in $\mathbb{C}$ as a the generalized coloring argument. Remarkably, this single real-valued invariant contains all tile invariants presented above.

Denote by $\mathcal{B}_{\mathrm{y}}$ and $\mathcal{B}_{\mathrm{sy}}$ the set of regions with Young diagram and skew Young diagram shape (see e.g. [M,JK]). It was shown in [P] that $\mathbf{T}_n$ has local move property (for 2-moves) with respect to $\mathcal{B}_{\mathrm{y}}$. The result, already more general than

[4]They actually considered one additional disconnected tile which we ignore. This set of tiles appeared after translation of the trominoes in hexagonal lattice into the square lattice [CL].

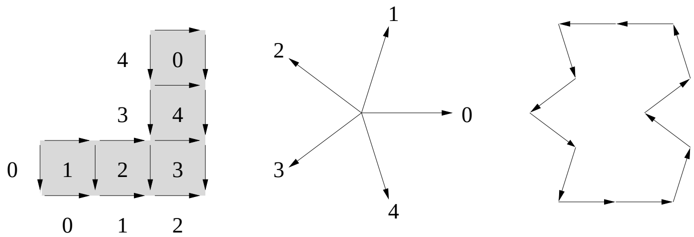

FIGURE 6.4. Ribbon tile $\tau = \tau_{0011}$, vectors ω^k, height function $\varphi(\tau)$.

[We], was later extended by the author to include $\mathcal{B}_{\rm sy}$ (unpublished). Following [P], we conjecture the local move property with respect to all simply connected regions. The computation of $\mathbb{G}(\mathbf{T}_n, \mathcal{B}_{\rm sc})$ and the height function arguments [MoP] seem to support the conjecture.

7. Small sets of tiles

7.1 T-tetrominoes.

It was shown in [Wa] that four rotations of T-tetromino can tile a $m \times n$ rectangle if and only if 4 divides both m and n. It is easy to see that the result cannot be proved by the coloring arguments. Nevertheless, no height function argument is known.

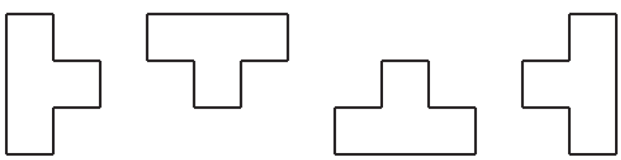

FIGURE 7.1. Four T-tetrominoes.

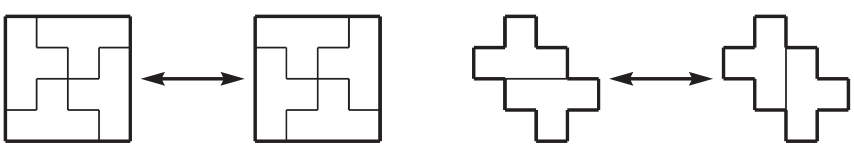

FIGURE 7.2. Local moves: 2-move and 4-move.

The set of tiles is of interest since it also seem to have a local move property. Observe that besides the 2-moves there is also a 4-move involving a reflection in a 4×4 square. We conjecture that these local move suffice. It seems that the combinatorial technique in [Wa] can be extended to prove the local move property with respect to rectangular regions.

7.2 Bars and Rectangular shapes.

Let $\mathbf{T}$ be a set of two "bars", i.e. of $m \times 1$ and $1 \times n$ rectangles. Claire and Rick Kenyon found a remarkable application of the height functions in this case [KK]. They introduced a tree-valued height function, and proved properties $(\bullet)-(\bullet\bullet\bullet)$ in this case. From here they deduced the local move connectivity (the only local move required is $A_1 \to A_2$, where $A_1, A_2 \vdash m \times n$ rectangle), obtain the general bound on the distance (it's $O(|\Gamma|^{3/2})$ in that case) and present a linear algorithm for testing tileability. The authors show that their analysis can be modified to rectangular regions $m \times n$ and $n \times m$. In particular, the authors present a quadratic algorithm for tileability and prove the local move property for 2×3 and 3×2 rectangles.

While the authors do not compute the group of invariants, it can be easily determined from either local move property or coloring arguments. Let us note that the polynomial algorithms for tileability exist only for simply connected regions, as in general case the problem is NP-complete [Ro] (see also [BJLS]).

7.3 L-trominoes.

Let $\mathbf{T}$ be the set of four rotations of L-trominoes. We showed in [P] that $\mathbb{G}(\mathbf{T}, \mathcal{B}) = \mathbb{E}(\mathbf{T}) = \mathbb{Z} \times \mathbb{Z}_3^2$. The proof involves some explicit coloring arguments.

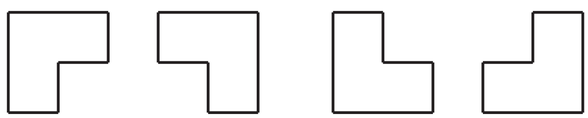

FIGURE 7.3. Four L-trominoes.

The set $\mathbf{T}$ has no local move property, as shown in [P]. There, we constructed large regions with exactly two tilings. Also, for general regions the tileability is NP-complete [MR]. It would be interesting to see if the same is true for simply connected regions. Let us mention here an old result that a $n \times n$ square with one square deleted can be tiled with $\mathbf{T}$ unless n is divisible by three [CJ].

7.4 Skew and square tetromino.

This example wa introduced by Propp, who found a very nice application of the height function approach [Pr]. The group of invariants $\mathbb{G}$ can be computed completely in this case, by using the coloring arguments and a nonabelian tile invariant presented in [Pr], which implies that $\mathrm{rk}(\mathbb{G}) = 2$. There are two interesting features in this case. First, the authors makes a distinction between "odd" and "even" 2×2 squares. In principle, this can be done in other special cases, by taking a smaller group of translations. Still, this is by far the most interesting such example, as the infinite tile invariant becomes a finite tile invariant when odd and even squares are identified.

For the second feature, Propp in [Pr] defines a tile invariant as a signed area, refraining from the "winding number" approach in [CL]. This was the approach

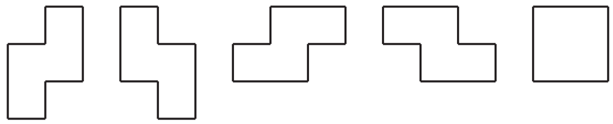

FIGURE 7.4. Skew and square tetromino.

continued in [MoP]. We hope the reader will enjoy this well written article and completes the computation of the full group of invariants as an exercise.

7.5 Dominoes again.

Let Γ be a simply connected region, and let k be a fixed integer. Consider all domino tilings of Γ with exactly k vertical domino. Recall that k can vary for different domino tilings, although its parity remains fixed. It was noted by Gupta [Gu] that sometimes one can make a connected graph $G(\Gamma, k)$ on these domino tilings by introducing 2×3 moves (see Figure 7.5). He showed that $G(\Gamma, k)$ is connected when Γ is a rectangle, the Aztec diamond, etc., but not in general case. We refer to [Gu] for the details.

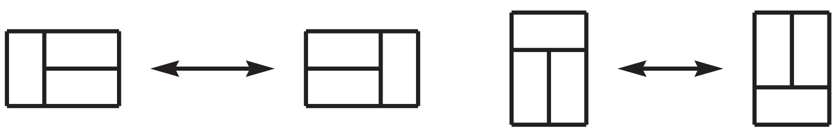

FIGURE 7.5. 2×3 moves.

In general, suppose $\mathbf{T}$ is a finite set of tiles and Γ is a tileable region. One can ask whether local connectivity exists for tilings $A \vdash \Gamma$ with given set of numbers $\alpha_i(A)$, defined as in the introduction. The work of Gupta suggests that certain nice sets of tiles and certain regions might satisfy this remarkable property.

7.6 More examples.

Consider the following two sets of tiles $\mathbf{T}_1$, $\mathbf{T}_2$. The first contains two rotations of T-tetromino and skew tetromino which fit into 2-row strip (see Figure 7.6). The second contains two rotations of T-pentamino, S-pentamino and skew tetromino, which fit into 3-row strip (see Figure 7.7). As before, we allow only translations of the tiles.

We are interested whether either or both sets have nonabelian tile invariants, local move property, height functions, etc. It is an exercise to establish these properties for regions which fit in 2-row and 3-row strip tiled by $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively. Also, replacing skew tetrominoes with a square tetromino gives an interesting modification of $\mathbf{T}_2$. We challenge the reader to resolve these problems.

7.7 Other lattices.

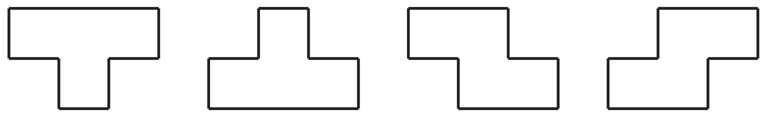

FIGURE 7.6. 2-row skew and T-tetrominoes.

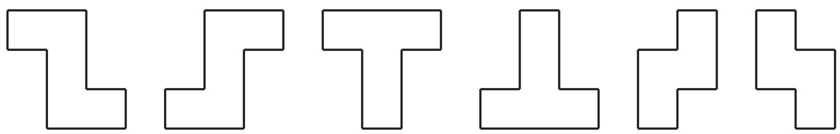

FIGURE 7.7. S and T - pentaminoes and skew tetrominoes.

It was realized rather early that tiling problems are of interest on other lattices as well [G]. The original question in [CL] comes from a hexagonal lattice, and the running example in [T] is the set of "lozenges", analogues of dominoes on a triangular lattice. A number of results for small sets of tiles on a triangular lattice was discovered recently by Rémila [Ré]. The author's approach is somewhat different from this article's main theme, and we strongly suggest it as a complimentary reading. Finally, a nice local connectivity result for squares-and-octagons was obtained by Gupta in [Gu].

8. Tilings in many dimensions

There is little known about tilings in many dimensions, although there seem to be no clear reason for that. As mentioned before, we do not know of any nonabelian tile invariant even for three-dimensional tiles. Without attempt to review the subject, let us present few examples that seem relevant.

8.1 Generalized Sperner's Lemma.

The Sperner's Lemma is the following classical result. Let Λ be a triangular lattice, Γ be a n-triangle with deleted three corner triangles. Color the vertices of the triangle with colors $\{0,1,2\}$, so that the sides are colored with $0, 1, 2$ (clockwise). Then there exists a $(0,1,2)$ colored triangle. In fact, the number of $(0,1,2)$ triangles minus the number of $(0,2,1)$ triangles (reading colors clockwise) is always 1.

While the Sperner's Lemma is often associated with Brouwer's fixed point theorem (see e.g. [Sh]), its generalizations are easier to obtained in the context of the Stokes Theorem. We present here the Generalized Sperner's Lemma, which implies an abelian tile invariant for a certain set of tiles. While the generalization below is probably well known (and follows easily from Stokes Theorem) the interpretation of it in the language of tile invariants seems new and will be presented here along with a short proof of the lemma.

Let us state the Generalized Sperner's Lemma first in two, and then in all dimensions. Let Γ be any region on a triangular lattice colored with $\{0,1,2\}$. Denote

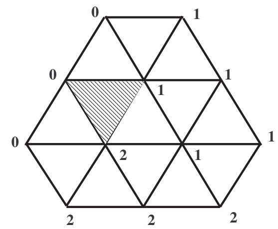

FIGURE 8.1. Sperner's Lemma.

by $\alpha_+(\Gamma)$ and $\alpha_-(\Gamma)$ the number of triangles with all three colors $(0, 1, 2)$, going clockwise and counterclockwise respectively. Then $\alpha_+ - \alpha_- = \mathrm{const}(\partial\Gamma)$, where $c = \mathrm{const}(\partial\Gamma)$ depends only on the coloring of the boundary. Note that we do not require Γ to be simply connected. The boundary $\partial\Gamma$ may be disconnected, but the coloring must be fixed on vertices of each connected component.

In general case, let Γ be any region in $V = \mathbb{R}^d$ with a fixed simplicial subdivision. Fix an orientation in $\mathbb{R}^d$ by taking a basis $(e_1, \dots, e_d)$ in V. Consider any coloring of vertices of Γ with $d+1$ colors $\{0, 1, \dots, d\}$. We say that Γ is $(d+1)$-*colored* in this case. We say that a simplex is *positive* (*negative*) if it is $(d+1)$-colored with basis $(\overrightarrow{01}, \overrightarrow{02}, \dots, \overrightarrow{0d})$ having a positive (negative) volume, defined as a determinant of the corresponding linear transformation. Denote by $\alpha_+(\Gamma)$ and $\alpha_-(\Gamma)$ the number of positive and negative simplices in Γ, respectively. Then $\alpha_+ - \alpha_- = \mathrm{const}(\partial\Gamma)$, where the constant depends only on the coloring of $\partial\Gamma$, and not on the interior of Γ. Let us state this result as follows.

Theorem 8.1 (Generalized Sperner's Lemma) *Let Γ be a triangulated region in $\mathbb{R}^d$ with a fixed $(d+1)$-coloring of the boundary $\partial\Gamma$. Let A be a $(d+1)$-coloring of the interior vertices. Then*

$$\alpha_+(A) - \alpha_-(A) = \mathrm{const}(\partial\Gamma),$$

where const *depends only on the coloring of the boundary, and not on coloring A.*

Now, the lemma can be reduced to an infinite tile invariant for a special set of tiles. First, take the tiles to correspond to $(d+1)$-colorings by somewhat changing the boundaries around the vertices in a consistent way which depends on the color (cf. proof of Theorem 3.1). For example, a small simplex can be added to, or subtracted from the sides of a large simplex so that only simplices with the same "color" can fit together (see Figure 8.2). Denote by $\mathbf{T}$ this new set of tiles, corresponding to all possible $(d+1)$-colorings of vertices of d-dimensional simplices. In Figure 8.2 we exhibit one such two-dimensional tile corresponding to $(1, 2, 3)$-coloring.

Now notice that the "coloring" of the boundary uniquely defines the shape of the boundary. Thus the "colorings" of the interior vertices of Γ are in one-to-one correspondence with tilings of Γ with $\mathbf{T}$. Consider the tiles which correspond to $(d+1)$-colorings with distinct colors, with positive and negative orientation. Theorem 8.1 implies that the difference between the number of certain "positive"

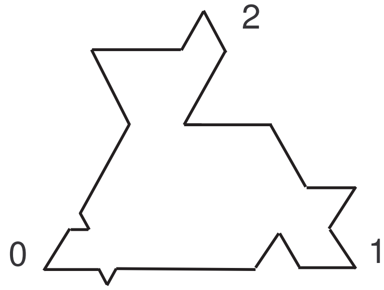

FIGURE 8.2. Modification of a 3-colored triangle.

and "negative" tiles is an fixed integer which depends on the boundary $\partial\Gamma$. We suggest the reader think through this simple, almost classical construction.

Let us note that from the proof (see section 10) it follows through verbatim that the infinite invariant defined in the lemma holds for signed tilings by $\mathbf{T}$ as well. Thus the tile invariant is abelian, and by Theorem 2.3 can be obtained by an extended coloring argument. Interestingly, this coloring argument is not obvious, and depends heavily on the way the set $\mathbf{T}$ is constructed.

Remark 8.2 The Sperner's Lemma has a number of variations, generalizations and applications. Let us first mention a similar in the spirit work [SS] where Sperner's Lemma is used to obtain relations for the volume(s) of simplices in tilings. The first d-dimensional version of the lemma can be found in [BC]. The cubical version, perhaps more acceptable for traditional tiling concepts, can be found in [Wo]. We refer to [Sh] for various application to fixed point results.

8.2 Parity check.

We will adopt the same notion of as in the previous subsection. Consider any triangular lattice $\Lambda \subset \mathbb{R}^d$, such that the dual graph is bipartite. In other words, we assume that the simplices are colored with black and white. An example is a regular partition of the cubic lattice with each cube partitioned into $d!$ simplices corresponding to permutations of basis vectors. The sign of the permutation then determines the color of the simplex.

Now consider colorings of vertices with m colors, $m \geq d$. We say that a simplex is *r-deficient* if it has exactly $(d+1-r)$ distinct colors of the vertices. Let Γ be any region in Λ with a fixed coloring of the boundary, and let A be any coloring of the interior vertices. Denote by $\rho_+(A)$ $(\rho_-(A))$ the number of black (white) 1-deficient simplices. Similarly, denote by $\alpha_+(A)$ $(\alpha_+(A))$ the number of black (white) 0-deficient simplices. Finally, let $\rho = \rho_+ - \rho_-$, $\alpha = \alpha_+ - \alpha_-$.

Theorem 8.3 *We have* $2\rho(A) + (d+1)\alpha(A) = \text{const}$, *where* $\text{const} = \text{const}(\Gamma)$ *depends only on the coloring of the boundary* $\partial\Gamma$ *and not on* A.

The proposition can be restated as an infinite abelian invariant of a certain set of tiles. We leave the details to the reader. As a bonus, the theorem implies that for odd d the total number of 1-deficient tiles has a fixed parity even when black and white tiles are indistinguishable. Even this is a nontrivial finite abelian invariant.

Let us conclude this part by presenting a special case when two independent tile invariants appear from such construction. This result is due to Moore and Newman, and it appeared in [MN]. We follow [Mo] in our presentation.

Consider any triangular lattice $\Lambda \subset \mathbb{R}^2$ with a bipartite dual graph. Fix a black/white coloring of triangles. Let Γ be a region in Λ with a fixed coloring of the boundary with colors $\{1, 2, 3, 4\} = I$. Denote by $\rho_+(i, j, k)$ and $\rho_-(i, j, k)$ the number of black and white triangles colored with $i, j, k \in I$. Let

$$\begin{aligned}
\alpha_\pm &= \rho_\pm(1,1,2) + \rho_\pm(1,2,2) + \rho_\pm(3,4,4) + \rho_\pm(3,3,4),\\
\beta_\pm &= \rho_\pm(1,1,3) + \rho_\pm(1,3,3) + \rho_\pm(2,4,4) + \rho_\pm(2,2,4),\\
\gamma_\pm &= \rho_\pm(1,1,4) + \rho_\pm(1,4,4) + \rho_\pm(2,3,3) + \rho_\pm(2,2,3),\\
\alpha &= \alpha_+ - \alpha_-\,, \quad \beta = \beta_+ - \beta_-\,, \quad \gamma = \gamma_+ - \gamma_-\,.
\end{aligned}$$

Theorem 8.4 ([MN]) *We have* $\alpha(A) - \beta(A) = \mathrm{const}_1$, $\beta(A) - \gamma(A) = \mathrm{const}_2$, *where* $\mathrm{const}_1, \mathrm{const}_2$ *depend only on the coloring of the boundary* $\partial\Gamma$ *and not on* A.

We challenge the reader to obtain a proper generalization of the theorem to higher dimensions [Mo].

8.3 3-dimensional dominoes.

While dominoes on a square grid satisfy the local move property with respect to simply connected regions, this is no longer true for 3-dimensional dominoes. Heuristicly, in three dimensions there is enough space to make large simply connected "local moves". Formally, for any n there exist a simply connected region Γ with exactly two domino tilings $A_1, A_2 \vdash \Gamma$, so that the move $A_1 \to A_2$ involves at least n dominoes.

Indeed, consider a cycle of size $4n$ with a $(n-1) \times (n-1)$ square shaped hole inside. Think of the cycle lying in a (x, y) plane. Color this square with black and white colors in the usual checkerboard fashion. Fill this hole with dominoes pointing up or down (in the direction z), depending on whether the square is black or white. Now notice that there are exactly two domino tilings of this region Γ, as the positions of the vertical dominoes are fixed by the construction, and the only freedom we have is given by two possible tilings of the cycle. The move will involve $2n$ dominoes then, which proves the claim.

The construction naturally extends to tilings in any $d \geq 3$ dimensions. This makes it rather unlikely that there exists a one-dimensional height function as described in section 5.1. On the other hand, the tileability by dominoes is in P for any d (see [LP]).

Let us note that there are other generalizations of the 2-dimensional dominoes. For example, in three dimensions, one can consider $2 \times 2 \times 1$ blocks. The similar construction to the one above shows that there is no local move property with respect to the simply connected regions. It would be interesting to see if the tileability is also in P in this case (cf. [MR]).

8.4 Generalized ribbon tiles.

During the search of the nonabelian tiling arguments in many dimensions, one may ask as to whether some generalization ribbon tiles have any. Consider the obvious generalization, corresponding to connected d-dimensional tiles with at most one cube in every plane $L_c : x_1 + \ldots + x_d = c$. Denote by $\mathbf{T}_n^d$ the set of such tiles in d dimensions with n cubes. Note that $|\mathbf{T}_n^d| = d^{n-1}$. The problem of finding the tile invariant group $\mathbb{G}(\mathbf{T}_n^d, \mathcal{B}_{\text{sc}})$ remains open in general case. Preliminary computations (for $d = 3$, $n = 3, 4$) suggest that $\operatorname{rk} \mathbb{G}(\mathbf{T}_n^3, \mathcal{B}_{\text{sc}}) = 1$, i.e that there is no infinite nonabelian invariant in this case (area is clearly an infinite abelian invariant). We conjecture that $\operatorname{rk} \mathbb{G}(\mathbf{T}_n^d, \mathcal{B}_{\text{sc}}) = 1$ for all $d \geq 3$. It is conceivable however, that the rank may increase if the set of regions is more restrictive. It would be interesting to find a nontrivial example of that.

9. Final Remarks

Let us begin by saying that in our opinion, papers [T], [CL] had a profound effect on the study of tilings, by introducing new techniques and methods into the field. The notion of tile invariants and the group of invariants [P] were inspired by [CL] and f-vectors in simple polytopes [Z]. Tile invariants have yet to become widely accepted. It is our goal here is to convince the reader that computing $\mathbb{G}(\mathbf{T})$ for various sets of tiles $\mathbf{T}$ is an important problem, which might lead to a better understanding of tilings.

To summarize this paper, me propose a new approach to the study of any fixed set of tiles $\mathbf{T}$. Fist, one can compute the coloring group $\mathbb{O}(\mathbf{T})$, an extended coloring group $\overline{\mathbb{O}}(\mathbf{T})$ and the group of valuations $\mathbb{E}(\mathbf{T})$ (cf. Theorem 3.2). Then one should attempt to determine $\mathbb{G}(\mathbf{T}, \mathcal{B}_{\text{sc}})$ by computing $\mathbb{G}_N = \mathbb{G}(\mathbf{T}, \mathcal{B}_{\text{sc}} \cap \mathcal{B}_N)$ for N large enough. If at some point $\mathbb{G}_N = \mathbb{E}(\mathbf{T})$, this implies that there are no nonabelian invariants (cf. Proposition 2.3), so the set $\mathbf{T}$ is not so interesting.

Suppose, on the other hand, that the calculations suggest existence of some nonabelian invariants in $\mathbb{G}(\mathbf{T})$. Then, one should check whether $\mathbf{T}$ satisfies local move property. If yes, attempt to find a one-dimensional height function which proves that (cf. Theorem 5.2). Then compute $\mathbb{G}(\mathbf{T})$ from local moves. If $\mathbf{T}$ does not satisfy the local move property, one should attempt to find nontrivial height functions φ, and compute groups $\mathbb{E}_\varphi(\mathbf{T}) \neq \mathbb{E}(\mathbf{T})$. Since $\mathbb{E}_\varphi \subset \mathbb{G}(\mathbf{T}, \mathcal{B}_{\text{sc}})$, one might be able to compute the whole group of invariants that way (cf. section 6.3,4).

While Theorem 3.1 seem to suggest that the above prescription works only for special sets of tiles, we consider a success a proof of *any* nonabelian tile invariant or *any* local move property. The theory is still in the early stages of development, so even partial results are of interest.

Few words about the tileability applications. After all, tileability of the starecase shaped regions by the ribbon L-trominoes was the original motivation in [CL]. In general, suppose we are given two sets of tiles $\mathbf{T} \subset \mathbf{T}'$, and a fully computed tiling group $\mathbb{G}(\mathbf{T}', \mathcal{B})$. Now let $\Gamma \in \mathcal{B}$ be a region tileable by $\mathbf{T}'$. This determines all the constants $\text{const}(\Gamma)$ for all tile invariants $(*)$. Now restriction of the tile invariants for $\mathbf{T}'$ to $\mathbf{T}$ gives a number of integer linear equations which may or may not have integer solutions. In the latter case the region is untileable by $\mathbf{T}$ (see [CL,P]).

From the point of view of tileability criteria, this seem like a weak approach. Indeed, in general, we need at least as many invariants as the number of tiles $|\mathbf{T}|$, and these tile invariants are hard to find and to prove. On the other hand, the integrality of solutions helps. In [P] we found several (un)tileability results in this direction. As a bonus, an easily computable coloring group $\mathbb{O}(\mathbf{T})$ can determine whether a certain tileability argument follows from the coloring argument. Or, as it was done in [CL], one can prove untileability of a Γ and then find a signed tilings of Γ by $\mathbf{T} \cup -\mathbf{T}$. By Theorem 2.2 one cannot prove untileability of Γ by the coloring arguments then.

There is a number of open problems that remain unresolved. Beside those mentioned earlier (Conjecture 5.3, questions about various small sets of tiles, etc.), let us stress again that we have yet to find an efficient algorithm for computing $\mathbb{E}(\mathbf{T})$ on the whole plane. It would be interesting to find other approaches to computing the group of invariants, besides the height functions, or find a reasoning why there cannot be any. It would be also very exciting to prove a local move property for some natural large set of tiles.

Let us conclude by saying that the local move property and one-dimensional height functions have important consequences in Statistical Physics and in study of Markov chains. Roughly, random application of local moves gives an easy way to sample random tilings; existence of the height function representation assists one in proving the rapid mixing. We refer to [BH,MN,PW,LRS,RY] for references and details.

10. Proof of Results

Proof of Theorem 3.2 (sketch).

We need to show that given N, $\mathbf{T} = \{\tau_1, \dots, \tau_k\}$, $|\tau_i| \le R$, one can solve Bounded CG-rank and Bounded GV-rank Problems in time polynomial in N, k, and R. Without loss of generality we will assume that $N \ge R$.

Denote by S the $N \times N$ square. Consider first a coloring group $\mathbb{O}(\mathbf{T}, \mathcal{B}_N)$. It is defined as $\mathbb{Z}^S$ quotient by the relations corresponding to translations of the tiles $\tau_i \in \mathbf{T}$ which lie in S. The rank of $\mathbb{O}$ is equal to the dimension of the corresponding real vector space (with the same integer linear equations).

There are at most N^2 translations of each tile, there are k tiles. In total, we need to calculate the rank of the system of at most N^2k equations with N^2 variables. This can clearly be done in polynomial time.

For the extended coloring group $\overline{\mathbb{O}}(\mathbf{T}, \mathcal{B}_N)$, we obtain a somewhat different set of equations. Fix one translation $\tau_i' \subset S$ of each tile $\tau_i \in \mathbf{T}$. Now, each translation τ_i'' gives an equation corresponding having to sum of the function on squares in τ_i'' equal to the sum of the function on squares in τ_i'. Again, we need to calculate the rank of the system of at most N^2k equations with N^2 variables.

Now, for the rank of the group of of valuations we have

$$\operatorname{rk} \mathbb{E}(\mathbf{T}, \mathcal{B}_N) = \operatorname{rk} \overline{\mathbb{O}}(\mathbf{T}, \mathcal{B}_N) - \operatorname{rk} \mathbb{O}(\mathbf{T}, \mathcal{B}_N).$$

This completes the proof. $\square$

Proof of Theorem 4.1 (sketch).

Use induction on the number of tiles in Γ to prove $(\star)$. The base is tautological. For the step of induction, consider τ from Lemma 4.2 such that $\Gamma' = \Gamma - \tau$ is simply connected. Fix a counterclockwise orientation on $\partial\tau$, $\partial\Gamma$, and $\partial\Gamma'$. Let $x \in \partial\Gamma$ be the starting point of the path P along the boundary. The paths P', R along the boundaries of Γ', τ are mapped into loops by inductive assumption. Observe that the intersection $P' \cup R$ will appear twice, once in each direction. On the other hand, $P = (P' - P' \cap R) \sqcup (R - P' \cap R)$. Adding the values of the height function φ along P as above, we obtain that P is also mapped into a loop. This completes the step of induction. $\square$

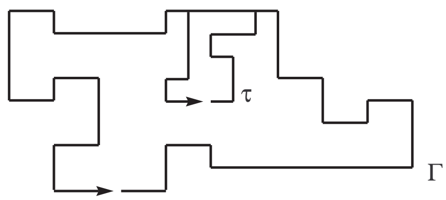

FIGURE 10.5. Simply connected regions Γ, τ and $\Gamma' - \tau$.

Proof of Theorem 5.1 (sketch).

We need to determine the group of invariants $\mathbb{G}(\mathbf{T}, \mathcal{B})$ in time polynomial in $k = |\mathbf{T}|$, ℓ, and $M = \max_i |\Gamma_i|$.

Indeed, tile invariants are precisely the maps $f : \mathbf{T} \to \mathbb{Z}$ which are invariant along the moves. In other words, we have

$$\mathbb{G}(\mathbf{T}) = \mathbb{Z}^{\mathbf{T}} / \mathbb{Z}\big\langle \big(\alpha_1(A_i) - \alpha_1(A_i'), \dots, \alpha_k(A_i) - \alpha_k(A_i')\big),\ 1 \le i \le \ell \big\rangle.$$

Now, calculating all $\alpha_j(A_i)$ is polynomial in k, M. Proceed as in the proof of Theorem 3.2. Indeed, it remains now to determine rank of the system of ℓ linear equations (over $\mathbb{R}$). This can be done in polynomial time [Sc]. $\square$

Proof of Theorem 5.2′ (sketch).

Denote by $\mathcal{A} = \mathcal{A}(\Gamma)$ the poset of all tilings $A \vdash \Gamma$, with '$\prec$' as an order relation. We claim that $\mathcal{A}$ has a minimum element A_0. Indeed, start with any tiling $A \vdash \Gamma$ and calculate φ_A. We claim that there exists a sequence of local moves from A to A_0. First, find any local maximum $x \in \widehat{\Gamma}$. If $x \notin \partial\Gamma$, then apply a local move $A \to A'$, and proceed by induction. If $x \in \partial\Gamma$, then both A, A_0 contain τ_x. Delete τ from Γ. Observe that we obtain either one region with smaller area or several smaller regions. Again proceed by induction. This proves the local connectivity property with respect to $\mathcal{B}$.

The second part follows from the following observation. Denote by A_I the largest element in $\mathcal{A}$. Then $M \le 2\Delta$, where Δ is the number of local moves from A_0 to A_I. Fix a value 0 of any point $z \in \partial\Gamma$. Let $\varphi_0 = \varphi_{A_0}$, $\varphi_I = \varphi_{A_I}$. Let h be the maximum value of φ on edges of Λ. Then for the maximum value H_I of φ_I we have $H_I \leq h|\partial\Gamma| \leq ch|\Gamma|$, where $0 \leq c \leq d2^d$. Similarly, for the smallest value H_0 of φ_0 we have $H \geq -ch|\Gamma|$.

Now, for every $A \vdash \Gamma$ define

$$\psi(A) = \int \varphi_A(x)\, d\mu,$$

where the integration is taken over $\widehat{\Gamma}$ and $d\mu$ is the usual euclidean measure on $\mathbb{R}^d$. We have

$$\psi(A_I) - \psi(A_0) \leq \mu(\widehat{\Gamma})(H_I - H_0) \leq c'|\Gamma|^2,$$

where c' is a constant which depends only on $\mathbf{T}$. Denote by δ the smallest change of ψ under the local move:

$$\delta = \min_{i=1}^{\ell} \left|\psi(A_i) - \psi(A_i')\right| > 0.$$

We conclude that $\Delta \leq (c'/\delta)\,|\Gamma|^2 \leq c''\,|\Gamma|^2$, which proves the claim.

For the last part, consider the following algorithm. Compute φ on $\partial\Gamma$. From above, the local maxima of $\varphi_0 = \varphi_{A_0}$ are on the boundary. Find a maximum value of $x \in \partial\Gamma$. This is clearly a local maximum of φ_0. Now delete τ_x from Γ and proceed accordingly. Eventually we either determine A_0 completely, or at some point we have to delete τ_x from Γ in an impossible situation. Since A_0 is unique, this implies untileability of Γ in that case. Note that the cost of the algorithm is $O(|\Gamma|^2\ell k)$. This completes the proof of the theorem. $\square$

Proof of Theorem 5.6.

First, observe that tilings $A \vdash \Gamma$ correspond to vertices of $\mathbf{P}_\Gamma$. Indeed, suppose otherwise. By abuse of notation we can write this as $A = \beta_1\, B_1 + \beta_2\, B_2 + \dots$, where $\beta_1, \beta_2, \dots \in \mathbb{R}_+$. But that means that zeroes of $(a_{x,\tau})$ on the left hand side correspond to zeroes on the right hand side, i.e. $B_1, B_2, \dots = A$. This proves the claim.

Similarly, consider two tilings $A_1, A_2 \vdash \Gamma$. Let

$$A_\lambda = \lambda\, A_1 + (1-\lambda)\, A_2 = \beta_1\, B_1 + \beta_2\, B_2 + \dots,$$

where $0 < \lambda < 1$. The point A_λ lies on the interval $[A_1, A_2]$. By the observation above, only tiles that are in A_1, A_2 can appear in B_i. Therefore all tiles that lie in $A_1 \cap A_2$ must also appear in each of the B_i. On the other hand, a tile $\tau_x \in A_1$ must appear in B_i with the total weight λ. Having or not having τ_x splits the set of tilings B_i into two subsets. Since every element $y \in \Lambda$ must belong to some tile, the total set of tiles splits between tiles that contain and don't contain τ_x. Denote these sets of indices by I and J. The above implies that either every $B_i = A_1$, $i \in I$, every $B_j = A_2$, $j \in J$, or there exist B_i, B_j, $i \in I$, $j \in J$, such that $A_1 \to C$ and $C \to A_2$ are non-intersecting local moves (and the same is true for $A_1 \to D$ and $D \to A_2$). This completes the proof. $\square$

Proof of Generalized Sperner's Lemma 8.1 (sketch).

Define an orientation of the $(d-1)$-dimensional simplices on the boundary to agree with orientation of $V = \mathbb{R}^d$. Formally, we say that a simplex on the boundary is positive (negative) if it is colored with d colors $\in \{0, 1, \ldots, d\}$ and coloring the remaining vertex of a unique d-dimensional simplex in Γ with the remaining color would make this simplex positive (negative). Denote by $\beta_+(\partial\Gamma)$ and $\beta_-(\partial\Gamma)$ the number of positive and negative simplices on the boundary. A simplex (of any dimension) with repeated colors we call *neutral*.

Let us prove by induction that in conditions of the theorem we have:

$$\mathrm{const}(\partial\Gamma) = (d+1)\bigl(\beta_+(\partial\Gamma) - \beta_-(\partial\Gamma)\bigr).$$

First, let us prove the base of induction. Indeed, for a single positive (negative) d-dimensional simplex all $(d+1)$ simplices on the boundary are positive (negative). If the d-dimensional simplex is neutral, then the symmetry argument implies that $\mathrm{const} = 0$ in this case.

For the step of induction, we can delete *any* d-dimensional simplex from Γ. Now observe that $\mathrm{const}(\partial\Gamma)$ is additive with respect to such division since the intersection of the boundaries is taken with opposite signs, and thus cancel each other (cf. proof of Theorem 4.1). We omit the easy details. $\square$

Proof of Theorem 8.2.

Consider all 0-deficient $(d-1)$-dimensional simplices in Γ, i.e. $(d-1)$-dimensional faces with d distinct colors. Each such face is either on the boundary or is a boundary of one black and one white d-dimensional simplex. Denote by Δ the number of such faces. Denote by δ_+ (δ_-) the number of of such faces on the boundary, so that the adjacent simplex is black (white). By counting Δ separately, as a boundary of black or white squares, we obtain

$$\Delta = 2\rho_+ + (d+1)\alpha_+ + \delta_- = 2\rho_- + (d+1)\alpha_- + \delta_+.$$

Subtracting the sides in the last equality, we conclude

$$2\rho + (d+1)\alpha = \delta_+ - \delta_-.$$

This proves the result. $\square$

References

[Ad] R. Adin, personal communication (1999).

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, *Oriented Matroids*, Cambridge U. Press, Cambridge, UK, 1999.

[Be] R. Berger, *The undecidability of the domino problem*, Mem. Amer. Math. Soc. **66** (1966).

[BJLS] F. Berman, D. Johnson, T. Leighton, P. Shor, *Generalized planar matching*, J. Algorithms **11** (1990), 153–184.

[BC] A. B. Brown, S. S. Cairns, *Strengthening of Sperner's lemma applied to homology theory*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 113–114.

[BH] J. K. Burton, Jr., C. L. Henley, *A constrained Potts antiferromagnet model with an interface representation*, J. Phys. A **30** (1997), 8385–8413.

[Ch] T. Chaboud, *Domino tiling in planar graphs with regular and bipartite dual*, Theoret. Comput. Sci. **159** (1996), 137–142.

[CJ] I.-P. Chu, R. Johnsonbaugh, *Tiling deficient boards with trominoes*, Math. Mag. **59** (1986), 34–40.

[CL] J. H. Conway, J. C. Lagarias, *Tilings with polyominoes and combinatorial group theory*, J. Comb. Theory, Ser. A **53** (1990), 183–208.

[F] J. C. Fournier, *Tiling pictures of the plane with dominoes*, Discrete Math. **165/166** (1997), 313–320.

[GJ] M. Garey, D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-completeness*, Freeman, San Francisco, CA, 1979.

[GKZ] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, MA, 1994.

[G] S. Golomb, *Polyominoes*, Scribners, New York, 1965.

[GO] J. E. Goodman, J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, CRC Press, Boca Raton, FL, 1997.

[Gu] D. Gupta, *Some tilings moves explored* Ph.D. Thesis, MIT, 1998, pp. 135 pp..

[JK] G. James, A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, Reading, MA, 1981.

[Ka] P. W. Kastelyn, *The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice*, Phisica **27** (1961), 1209–1225.

[KK] C. Kenyon, R. Kenyon, *Tiling a polygon with rectangles*, Proc. 33rd Symp. Foundations of Computer Science (1992), 610–619.

[Ke] R. Kenyon, *A note on tiling with integer-sided rectangles*, J. Combin. Theory, Ser. A **74** (1996), 321–332.

[LLL] A. K. Lenstra, H. W. Lenstra, L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. **261** (1982), 515–534.

[LP] L. Lovász, M. D. Plummer, *Matching theory*, North-Holland, Amsterdam, 1986.

[LRS] M. Luby, D. Randall, A. Sinclair, *Markov chain algorithms for planar lattice structures*, Proc. 36th Symp. Foundations of Computer Science (1995), 150–159.

[M] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, London, 1979.

[Mo] C. Moore, personal communication (2000).

[MN] C. Moore, M. E. J. Newman, *Height Representation, Critical Exponents, and Ergodicity in the Four-state Triangular Potts Antiferromagnet*, J. Stat. Phys. **99** (2000), 661–690.

[MoP] C. Moore, I. Pak, *Ribbon tile invariants from signed area*, J. Comb. Theory, Ser. A, to appear (2001).

[MR] C. Moore, J. M. Robson, *Hard tiling problems with simple tiles*, Discrete and Computational Geometry, to appear (2001).

[MuP] R. Muchnik, I. Pak, *On tilings by ribbon tetrominoes*, J. Comb. Theory, Ser. A **88** (1999), 188–193.

[P] I. Pak, *Ribbon tile invariants*, Trans. AMS **352** (2000), 5525–5561.

[Pr] J. Propp, *A pedestrian approach to a method of Conway, or, A tale of two cities*, Math. Mag. **70** (1997), 327–340.

[PW] J. Propp, D. Wilson, *Exact Sampling with Coupled Markov Chains and Applications to Statistical Mechanics*, Random Structures and Algorithms **9** (1996), 223–252.

[RY] D. Randall, G. Yngve, *Random three-dimensional tilings of Aztec octahedra and tetrahedra: an extension of domino tilings*, Proc. SODA 2000, 636–645.

[Ré] E. Rémila, *Tiling groups: new applications in the triangular lattice*, Discrete Comput. Geom. **20** (1998), 189-204.

[Ri] R. M. Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Invent. Math. **12** (1971), 177–209.

[Ro] J. M. Robson, *Sur le pavage de figure du plan par des barres*, in Actes des Journées Polyominos et Pavages (1991), 95–103.

[ST] N. C. Saldanha, C. Tomei, *An overview of domino and lozenge tilings*, Resenhas **2** (1995), 239–252.

[Sc] A. Schrijver, *Theory of Integer and Linear Programming*, John Wiley, New York, 1988.

[SU] E. R. Scheinerman, D. H. Ullman, *Fractional graph theory*, Wiley, New York, 1997.

[Si] M. Sipser, *Introduction to the Theory of Computation*, PWS Publishing Company, New York, 1997.

[Sh] Yu. A. Shashkin, *Fixed points*, AMS, Providence, RI, 1991.

[St] R. P. Stanley, *Enumerative Combinatorics,* Vol. 1, Wadsworth & Brooks/Cole, California, 1986.

[SS] S. K. Stein, S. Szabó, *Algebra and tiling*, Carus Mathematical Monographs, **25**, MAA, Washington, DC, 1994.

[TF] H. N. V. Temperley, M. E. Fisher, *Dimer problem in statistical mechanics – An exact result*, Philos. Mag. **6** (1961), 1061–1063.

[T] W. Thurston, *Conway's tiling group*, Amer. Math. Monthly **97** (1990), 757–773.

[Wa] D. W. Walkup, *Covering a rectangle with T-tetrominoes*, Amer. Math. Monthly **72** (1965), 986–988.

[We] D. C. West, *An elementary proof of two triangle-tiling theorems of Conway and Lagarias*, unpublished manuscript (1990), 6 pp.

[Wo] L. A. Wolsey, *Cubical Sperner lemmas as applications of generalized complementary pivoting*, J. Comb. Theory, Ser. A **23** (1977), 78–87.

[Z] G. Ziegler, *Lectures on Polytopes,* Graduate Texts in Mathematics 152, Springer, 1995.

Added in Print:

In the past year few advances has been made. First, Scott Sheffield resolved most of the open problem on ribbon tilings in "*Ribbon tilings and multidimensional height functions*", arXiv preprint math.CO/0107095. Among other things, he proved local connectivity conjecture (see section 6.4) and found linear time algorithm for testing tileability.

Second, Cris Moore, Ivan Rapaport and Eric Remila defined a height function and proved a local connectivity property for the set of colored square tiles similar to that in section 8.2. Their paper "*Tiling groups for Wang tiles*" will appear in Proc. SODA'2002.

Finally, the author resolved affirmatively the question whether computing (unbounded) group $\mathbb{E}(\mathbf{T})$ is decidable ("*Computational complexity of tile invariants*", preprint, 2001.)