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On the Number of Faces of Certain Transportation Polytopes

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Define the transportation polytope $T_{n,m}$ to be a polytope of non-negative $n \times m$ matrices with row sums equal to m and column sums equal to n. We present a new recurrence relation for the numbers f_k of the k-dimensional faces for the transportation polytope $T_{n,n+1}$. This gives an efficient algorithm for computing the numbers f_k, which solves the problem known to be computationally hard in a general case.

1. INTRODUCTION

Let $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_n)$ be two non-negative integer vectors, such that $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. Define a *transportation polytope* $T(a, b)$ to be a set of matrices $(x_{i,j}) \in \mathbb{R}^{m \times n}$, $1 \le i \le m$, $1 \le j \le n$, which satisfy the following linear inequalities and equations:

$$\begin{cases} x_{i,j} \ge 0, \\ \sum_{i=1}^{m} x_{i,j} = b_j, \\ \sum_{j=1}^{n} x_{i,j} = a_i, \end{cases} \quad \text{where } 1 \le i \le m,\ 1 \le j \le n. \tag{1}$$

These polytopes arise in many problems of enumerative combinatorics, statistics, combinatorial optimization, and linear programming (see [3, 5, 9, 11–13, 15]). Problems such as computing the volume or sampling the integer points are known to be hard and have been considered earlier (see [3, 4]). In this paper we study the problem of computing the number of k-dimensional faces in a special case. As in [8], the problem is reduced to an enumeration of certain labeled trees, which have simpler combinatorial structure.

In a celebrated paper [8], Klee and Witzgall described the facets and vertices of the transportation polytopes. Nevertheless, little is known about the structure of their faces. Even when there exists a full description (see [1]), it is often general combinatorial, and useless for fast numerical computations. On the other hand, it is well known that if the polytope is simple, then theoretically one can obtain the number of k-dimensional faces from its graph (see [7, 15]). Again, known algorithms run in exponential time (cf. [10]).

In this paper we present an efficient algorithm for computing the number of faces of transportation polytopes in a special case when $n = m + 1$, $a = (m + 1, \ldots, m + 1)$ and $b = (m, \ldots, m)$. Denote this polytope P_m. The dimension of P_m is $d = m(m - 1)$. Let $f_{m,k}$ be the number of k-dimensional faces of P_m, $0 \le k \le d$.

THEOREM 1. *The set of numbers $f_{m,k}$ for $1 \le m \le n$, $0 \le k \le n(n-1)$, can be computed in a time polynomial in n.*

This family of polytopes P_n is of special interest for several reasons. First, they are simple polytopes (see [5]). Second, their vertices and edges have a nice combinatorial description (see [5, 8]). Third, they are known to have the maximal number of vertices amongst all the transportation polytopes with $m = n - 1$. Finally, these polytopes can also be defined as Newton polytopes of products of the leading minors in matrices $n \times (n + 1)$ and related to the study of the hypergeometric functions on grassmannian $G_{2n+1,n}$ (see [6, 14]).

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Proof of the Theorem follows from a new combinatorial identity for the numbers $f_{n,k}$. While somewhat cumbersome, it suffices for the proof of Theorem 1. Formally, we prove the following key result.

THEOREM 2. *Let $F_n(q) = \sum_k f_{n,k} q^k$ be the generating polynomial for the number of k-dimensional faces. Then*

$$F_n(q) = \sum_{k=0}^{n-1} \binom{n-1}{k} (q+1)^{k(n-k)} C_{n,k}(q+1) F_k(q) F_{n-k-1}(q), \tag{2}$$

where $F_0 = 1$ and

$$C_{n,k}(q) = \sum_{1 \le i < j \le n+1} q^{i-1} \sum_{l=0}^{j-i-1} \binom{j-i-1}{l} \binom{n-j+i}{n-k-l-1} q^l. \tag{3}$$

Note that Theorem 1 follows immediately from Theorem 2. Indeed, computation of all the polynomials $C_{m,k}(q)$, $1 \le m \le n$, according to (3) clearly takes a time polynomial in n. Now use recurrence relation (2) to compute all the polynomials F_m, $1 \le m \le n$. This proves the result.

The rest of the article is organized as follows. In Section 2 we recall some definitions and known results on the structure of polytopes P_n. In Section 3 we prove the recurrence relation for $F_n(q)$ with some coefficients. In Section 4 we prove an explicit formula (3) for polynomials $C_{n,k}(q)$.

2. COMBINATORICS OF P_n

Recall several definitions. Let P be a convex polytope in a vector space V of dimension $d = \dim(P)$. The polytope P is called *simple* if each vertex is adjacent to exactly d edges. It is known that P_n is a simple polytope of dimension $n(n-1)$ (see [5, 8]).

Define an *f-vector* of a polytope P to be a sequence $f = (f_0, f_1, \ldots, f_d)$, where $d = \dim(P)$ and f_k is the number of k-dimensional faces in P. Characterization of f-vectors of polytopes is an important yet largely unsolved problem (see [13, 15]).

Let $\varphi \in V^*$ be a linear function on V which is *generic*, i.e., not constant on the edges of P. Define an orientation of edges of P to be in the direction of increase of the linear function φ. Let v be a vertex of P. Define an *index* of v (denoted $\mathrm{ind}_\varphi(v)$) to be the number of edges which are leaving v. Denote by g_i the number of vertices of P such that $\mathrm{ind}_\varphi(v) = i$. Define a *$g$-vector* of a polytope P to be a sequence $g = (g_0, g_1, \ldots, g_d)$. It is known that when P is simple, the g-vector does not depend on φ, and

$$F(q) = G(q+1), \tag{4}$$

where $G(q) = \sum_k g_k q^k$, $F(q) = \sum_k f_k q^k$ (see [2, 15]).

Of all the different linear functions φ, one will be particularly useful for our purposes. Let $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ be a linear basis in the vector space V. Define a *lexicographic orientation* of edges of the polytope P by the following rule. Let edge (v_1, v_2) be oriented from v_1 to v_2 if the first non-zero coordinate of a point $(v_2 - v_1)$ is positive. It is easy to see that this orientation can also be obtained by using a linear function $\varphi_E(x) = \sum_{i=1}^m a_i x_i$, where $a_1 = 1$ and $a_1 \gg a_2 \gg \cdots \gg a_m > 0$.

Define *colored trees* to be trees with n edges labeled (colored) by the numbers from 1 to n, and with $n+1$ vertices labeled by the numbers from 1 to $n+1$. Let $\mathcal{R}_n$ be the set of all *colored trees* with n edges. By the Cayley formula, we have

$$|\mathcal{R}_n| = n!\,(n+1)^{n-1}.$$

Now let V be a vector space of $n \times (n+1)$ real matrices. For any $t \in \mathcal{R}_n$, let $a_{i,j}(t)$ be the number of vertices $j' \neq j$ in t, such that the shortest path from j to j' starts with an edge i, $1 \leq i \leq n$, $1 \leq j \leq n+1$. Define a map $\gamma : \mathcal{R}_n \to V$ by the values $a_{i,j}(t)$. It is known that γ gives a bijection between the set $\mathcal{R}_n$ and the vertices of P_n (see [8]). By an abuse of speech we will refer to vertices when we talk about them corresponding to trees.

Now let us describe the edges of P_n in terms of trees. A pair of trees (t_1, t_2) corresponds to an edge of P_n if and only if t_2 can be obtained from t_1 by deleting an edge $i = (j, j_1)$ and adding an edge $i = (j, j_2)$, such that the shortest path from j to j_2 in t_1 starts with an edge i (see [14]). We refer to the edges containing the vertex $v \in P_n$, $v = \gamma(t)$ by the ordered pair of vertices $(j, j_2) \in t$, which are non-adjacent by an edge in the corresponding colored tree t. Note that their number is always $n(n-1)$, which agrees with the dimension $\dim(P_n) = n(n-1)$.

By our definition, $P_n = T_{n,n+1}$ lies in a vector space V with a natural basis $E_n = \{e_{1,1}, \ldots, e_{1,n+1}, e_{2,1}, \ldots, e_{2,n+1}, \ldots, e_{n,1}, \ldots, e_{n,n+1}\}$, where $e_{i,j}$ is a matrix with 1 at (i, j) and 0 elsewhere. Let us fix the lexicographic orientation on P_n corresponding to the basis E and any linear function φ which defines the same orientation. For the rest of the paper φ will stand for this particular linear function.

3. Proof of the Recurrence Relation

In this section we shall obtain the recurrence relation for f and g-vectors by the direct counting of indices of colored trees for lexicographic orientation.

Let $t \in \mathcal{R}_n$ be a colored tree with n edges. Let $\rho(t) = \operatorname{ind}_\varphi(\gamma(t))$ be the number of edges of P_n leaving $v = \gamma(t)$ in lexicographic orientation. For any subtree t_1 of t define $\rho(t_1) = \rho(t_1')$, where t_1' is obtained from t_1 by monotonic relabeling of the vertices.

Let us fix any $t_0 \in \mathcal{R}_n$. Consider an edge $1 = (i_0, j_0)$, $i_0 < j_0$ in t_0. Define the trees t_1 and t_2 to be subtrees of t_0 obtained after removing edge 1, and such that $i_0 \in t_1$ and $j_0 \in t_2$. Below we will express $\rho(t)$ in terms of $\rho(t_1)$ and $\rho(t_2)$. This will enable us to prove the recurrence relation.

Recall that all edges of P_n correspond to non-adjacent pairs of vertices (i, j). The number $\rho(t)$ is the number of those pairs (i, j) such that the linear function φ increases along the corresponding edge (t_0, t). We call these *increasing edges*. We will consider all possible locations of vertices i and j on a tree t_0.

Let S_1 and S_2 be the sets of subtrees $t_1 \setminus \{i_0\}$ and $t_2 \setminus \{j_0\}$. Let $k = |S_1|$. Then $|S_2| = n-k-1$. Consider all possibilities of locations of vertices i and j. Compute the number of such pairs which correspond to increasing edges.

(1) For $i, j \in S_1 \cup \{i_0\}$, the number of increasing edges is equal to $\rho(t_1)$.

(1′) For $i, j \in S_2 \cup \{j_0\}$, the number of increasing edges is equal to $\rho(t_2)$.

(2) For $i \in S_1$, $j \in S_2 \cup \{j_0\}$, the number of increasing edges is equal to $k(n-k)$. Indeed, the lexicographically first changing entry of the matrix $A = (a_{i,j}(t_0)$ is a_{1,i_0}, which is increasing.

(2′) For $i \in S_2$, $j \in S_1 \cup \{i_0\}$, the number of increasing edges is equal to 0. Indeed, the lexicographically first changing entry of $A = (a_{i,j}(t_0))$ is a_{1,i_0}, which is decreasing.

(3) For $i = i_0$, $j \in S_2$, $j < j_0$, all such edges are increasing. Indeed, the lexicographically

first changing entry of $A = (a_{i,j}(t_0))$ is $a_{1,j}$, which increases from 0. Note that a_{1,i_0} remain unchanged.

(4) For $i = i_0$, $j \in S_2$, $j > j_0$, all such edges are decreasing. Indeed, the lexicographically first changing entry of $A = (a_{i,j}(t_0))$ is a_{1,j_0}, which decreases to 0. Note that a_{1,i_0} remain unchanged.

(5) For $i = j_0$, $j \in S_1$, $j < i_0$, all such edges are increasing. Indeed, the lexicographically first changing entry of $A = (a_{i,j}(t_0))$ is $a_{1,j}$, which increases from 0.

(6) For $i = j_0$, $j \in S_1$, $j > j_0$, all such edges are decreasing. Indeed, the lexicographically first changing entry of $A = (a_{i,j}(t_0))$ is a_{1,i_0}, which decreases to 0.

Summarizing all the previous cases we obtain the following result

$$\rho(t_0) = \rho(t_1) + \rho(t_2) + k(n-k) + \alpha(S_2) + \beta(S_1), \tag{5}$$

where

$$\alpha(S_2) = |\{j \in S_2,\ j < j_0\}| \tag{6}$$

$$\beta(S_2) = |\{j \in S_1,\ j < i_0\}|. \tag{7}$$

Now define the polynomials $C_{n,k}(q)$ as follows:

$$C_{n,k}(q) = \sum_{1 \le i_0 < j_0 \le n+1} \ \sum_{\substack{S_1, S_2,\ S_1 \cap S_2 = \emptyset, \\ S_1 \cup S_2 = \{1,\ldots,n+1\} \setminus \{i_0, j_0\}}} q^{\alpha(S_2)+\beta(S_1)}. \tag{8}$$

In the next section we will show that these polynomials are given by formula (3). Assuming that, we prove our recurrence relation (2). Indeed, raise q into power on both sides of (5) and sum this over all possible pairs of subtrees (t_1, t_2). Substitution of (8) gives

$$G_n(q) = \sum_{k=0}^{n-1} \binom{n-1}{k} q^{k(n-k)} C_{n,k}(q) G_k(q) G_{n-k-1}(q). \tag{9}$$

Replacing q by $q+1$ in (9) and applying (4), we obtain (2). This proves the first part of Theorem 2.

4. Computing Coefficients $C_{n,k}(q)$

We shall prove the following lemma, which implies the second part of Theorem 2.

LEMMA 3. *For any* $1 \le i < j \le n+1$ *we have*

$$\sum_{\substack{S_1, S_2, S_1 \cap S_2 = \emptyset, \\ S_1 \cup S_2 = \{1,\ldots,n+1\} \setminus \{i_0, j_0\}}} q^{\alpha(S_2)+\beta(S_1)} = q^{i-1} \sum_{r=0}^{j-i-1} \binom{j-i-1}{r} \binom{n-j+i}{n-k-r-1} q^r. \tag{10}$$

PROOF. For fixed i and j define $\tau(S_2)$ as follows

$$\tau(S_2) = |\{l \in S_2, i < l < j\}|. \tag{11}$$

Observe that

$$\alpha(S_2) + \beta(S_1) = i - 1 + \tau(S_2). \tag{12}$$

Define the coefficients $b_{n,k}^{i,j,r}$ from the following identity:

$$\sum_{\substack{S_2, |S_2|=n-k-1 \\ S_2 \subset \{1,\ldots,n+1\} \setminus \{i_0, j_0\}}} q^{i-1+\tau(S_2)} = \sum_r b_{n,k}^{i,j,r} q^{i-1+r}. \tag{13}$$

Note that the left-hand side of (10) is equal to the left-hand side of (13).

By definition, $b_{n,k}^{i,j,r}$ is the number of subsets $S_2 \subset \{1, \ldots, n+1\} \setminus \{i_0, j_0\}$ such that $\tau(S_2) = r$. Divide S_2 into a union of two non-intersecting subsets $S_2 = S_2' \cup S_2''$, where $S_2' = \{l \in S_2, i < l < j\}$ and $S_2'' = S_2 \setminus S_2'$. It is easy to see that the number of ways to choose S_2' is $\binom{j-i-1}{r}$ and the number of ways to choose S_2'' is $\binom{n-j+i}{n-k-r-1}$. We conclude that

$$b_{n,k}^{i,j,r} = \binom{j-i-1}{r}\binom{n+i-j}{n-k-r-1}. \tag{14}$$

Now (14) combined with (13) gives (10). This completes the proof of the lemma. □

REMARK 4. Note the nice multiplicative formulas for several components of the f-vector. Say, $f_0 = n!\,(n+1)^{n-1}$, $f_1 = n!\,(n+1)^{n-1}\binom{n}{2}$ and $f_{d-1} = n\,(n+1)$. We challenge the reader to obtain similar formulas for other components.

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