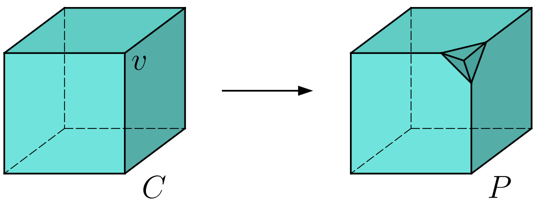
INFLATING THE CUBE WITHOUT STRETCHING

IGOR PAK

Consider the surface S of a unit cube C in $\mathbb{R}^3$. Think of S being a container made of cardboard and full of water. Now, can one add *more* water into container? In other words, we are asking whether one can bend S without tearing and stretching in such a way that the volume increases? While one is tempted to say no, the following result gives a positive answer:

Theorem. *There exists a non-convex polyhedron whose surface is isometric to the surface of a cube of smaller volume.*

Here by *isometric* we mean that the geodesic distance between pairs of points on the non-convex polyhedron is always equal to the geodesic distance between of the corresponding pairs of points on a cube. Alternatively, it means that two surfaces can be triangulated in such a way that they now consist of congruent triangles which are glued according to the same combinatorial rules. For example, if we push a vertex v of a cube C inside as shown in the Figure below, we obtain a polyhedron P whose surface ∂P is isometric to $S = \partial C$. Of course, $\mathrm{vol}(P) < \mathrm{vol}(C)$ in this case.



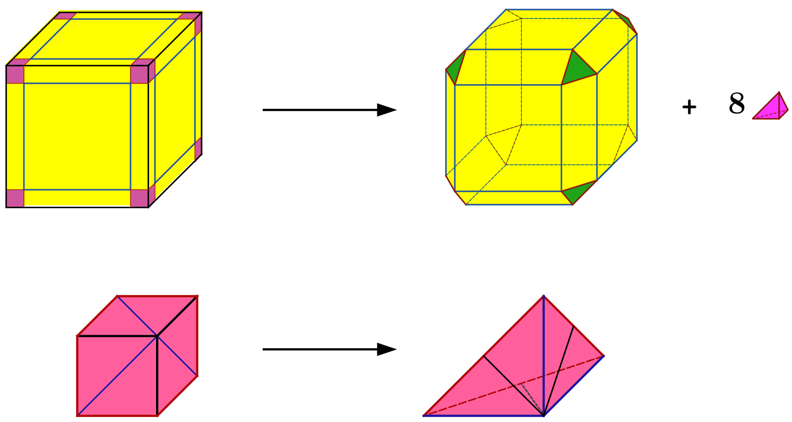
The theorem is somewhat unusual in a sense that one tends to assume that convex objects always maximize the volume. On the other hand, the Alexandrov uniqueness theorem [1] states that convex polyhedra are uniquely determined (up to a rigid motion) by the intrinsic geometry of their surfaces. This means that *all* polyhedra with surfaces isometric to S are either congruent to C or necessarily non-convex.

Before we present the proof of the theorem, let us say a few words about the history of the problem. Define a *bending* to be a continuous isometric deformation of a surface. About ten years ago, two different constructions of volume-increasing bending have appeared [3, 4] (see also [2, 6]). In fact, Bleecker [3] showed that one can start with any convex polytope P and bend it into a (non-convex) polyhedron of larger volume. Unfortunately, both constructions are technically involved. Below we present a simple volume-increasing bending of a unit cube. Our construction is based on the work of Milka [5], where all symmetric bendings of regular polyhedra were classified.

Date: 25 July 2006.

Proof of the Theorem. Consider a cube C with side length 1 and the surface $S = \partial C$. Fix a parameter $\varepsilon \in (0, \frac{1}{2})$. Think of ε as being very small. On each face of a cube, from every corner remove a square of size-length ε. Denote by $R \subset S$ the resulting surface with boundary. On every face of the cube there are four boundary points which form a square. Call these points corners. Move each square away from the center of the cube (without extending it or breaking the symmetry) until all of the distances between the nearest corners on adjacent faces reach 2ε. Take the convex hull of the corners to obtain a polytope Q_ε. To each triangular face of Q_ε attach a triangular pyramid whose base is equilateral with side-length 2ε and whose other faces are right triangles. Denote by P_ε the resulting (non-convex) polyhedron.

Note that the surface ∂Q_ε without triangular faces between the corners is isometric to R. Similarly, three ε squares meeting at a vertex of the cube can be bent into three faces of a pyramid (see the Figure below). This easily implies that the surface ∂P_ε is isometric to the surface S.



Let us calculate the volume of the polyhedron P_ε. Cut Q_ε with six planes, each parallel to a square face and containing the nearest edges of its four neighborhood square faces. This subdivides Q_ε into one (interior) cube, six slabs (along the faces), twelve right triangular prisms (along the edges), and eight pyramids (one per cube vertex). Observe that the cutting planes are at distance $d = 2\varepsilon/\sqrt{2} = \sqrt{2}\,\varepsilon$ from the sides of the interior cube. We have:

$$\mathrm{vol}(Q_\varepsilon) \;=\; (1-2\varepsilon)^3 + 6\cdot(1-2\varepsilon)^2 d + 12\cdot(1-2\varepsilon)\,\frac{d^2}{2} + 8\cdot\frac{d^3}{6}\,.$$

Since $\mathrm{vol}(P_\varepsilon) = \mathrm{vol}(Q_\varepsilon) + 8\cdot d^3/6$, we conclude that

$$\mathrm{vol}(P_\varepsilon) \;=\; 1 \;+\; 6(\sqrt{2}-1)\,\varepsilon \;+\; k_2\,\varepsilon^2 \;+\; k_3\,\varepsilon^3\,,$$

where k_2 and k_3 are constants. For small $\varepsilon > 0$, the terms $1 + 6(\sqrt{2}-1)\varepsilon$ dominate, so that $\mathrm{vol}(P_\varepsilon) > 1 = \mathrm{vol}(C)$, as desired. $\square$

Remark. That the polyhedra P_ε are indeed non-convex may not be immediately obvious. As we mentioned earlier, this follows from the Alexandrov theorem. We leave the checking to the reader.

When $\varepsilon \to \frac{1}{2}$, the polyhedra Q_ε converge to a regular octahedron, and P_ε converges to a nice stellated polyhedron consisting of 18 right triangular faces. The volume of the latter is < 0.95, as the reader can deduce from the calculations above. Thus at some point the volume of P_ε stops increasing.

Acknowledgements. The author is grateful to David Jerison and Ezra Miller for interesting discussions. The author was partially supported by the NSF.

References

[1] A. D. Alexandrov, *Convex polyhedra* (in Russian), M.: Gostekhizdat, 1950; English translation in Springer Monographs in Mathematics, Springer, Berlin, 2005.

[2] V. A. Alexandrov, How one can crush a milk carton in such a way as to enlarge its volume (in Russian), *Soros Educational J.* **51** (2000), 121–127.

[3] D. D. Bleecker, Volume increasing isometric deformations of convex polyhedra, *J. Diff. Geom.* **43** (1996), 505–526.

[4] Yu. D. Burago and V. A. Zalgaller, Isometric piecewise-linear embeddings of two-dimensional manifolds with a polyhedral metric into $\mathbf{R}^3$, *St. Petersburg Math. J.* **7** (1996), 369–385.

[5] A. D. Milka, Linear bendings of regular convex polyhedra (in Russian), *Mat. Fiz. Anal. Geom.* **1** (1994), 116–130.

[6] I. Kh. Sabitov, On some recent results in the metric theory of polyhedra, *Rend. Circ. Mat. Palermo (2)*, Suppl. No. 65, part II, (2000), 167–177.

Igor Pak
Department of Mathematics
M.I.T.
Cambridge, MA 02139
pak@math.mit.edu