HOOK FORMULAS FOR SKEW SHAPES II. COMBINATORIAL PROOFS AND ENUMERATIVE APPLICATIONS

ALEJANDRO H. MORALES*, IGOR PAK*, AND GRETA PANOVA†

Abstract. The *Naruse hook-length formula* is a recent general formula for the number of standard Young tableaux of skew shapes, given as a positive sum over *excited diagrams* of products of hook-lengths. In [MPP1] we gave two different q-analogues of Naruse's formula: for the skew Schur functions, and for counting reverse plane partitions of skew shapes. In this paper we give an elementary proof of Naruse's formula based on the case of border strips. For special border strips, we obtain curious new formulas for the *Euler* and *q-Euler numbers* in terms of certain Dyck path summations.

1. Introduction

In Enumerative Combinatorics, when it comes to fundamental results, one proof is rarely enough, and one is often on the prowl for a better, more elegant or more direct proof. In fact, there is a wide belief in multitude of "proofs from the Book", rather than a singular best approach. The reasons are both cultural and mathematical: different proofs elucidate different aspects of the underlying combinatorial objects and lead to different extensions and generalizations.

The story of this series of papers is on our effort to understand and generalize the *Naruse hook-length formula* (NHLF) for the number of standard Yound tableaux of a skew shape in terms of *excited diagrams*. In our previous paper [MPP1], we gave two q-analogues of the NHLF, the first with an algebraic proof and the second with a bijective proof. We also gave a (difficult) "mixed" proof of the first q-analogue, which combined the bijection with an algebraic argument. Naturally, these provided new proofs of the NHLF, but none which one would call "elementary".

This paper is the second in the series. Here we consider a special case of border strips which turn out to be extremely fruitful both as a technical tool and as an important object of study. We give two elementary proofs of the NHLF in this case, both inductive: one using weighted paths argument and another using determinant calculation. We then deduce the general case of NHLF for all skew diagrams by using the *Lascoux–Pragacz identity* for Schur functions. Since the latter has its own elementary proof [HaG] (see also [CYY]), we obtain an elementary proof of the HLF.

But surprises do not stop here. For the special cases of the *zigzag strips*, our approach gives a number of curious new formulas for the *Euler* and and two types of *q-Euler numbers*, the second of which seems to be new. Because the excited diagrams correspond to Dyck paths in this case, the resulting summations have *Catalan number* terms. We also give type B analogues, which have similar feel but with $\binom{2n}{n}$ terms. Despite their strong "classical feel" all these formulas are new and quite mysterious.

1.1. **Hook formulas for straight and skew shapes.**
Let us recall the main result from the first paper [MPP1] in this series. We assume here the reader is familiar with the basic definitions, which are postponed until the next two sections.

Key words and phrases. Hook-length formula, excited tableau, standard Young tableau, flagged tableau, reverse plane partition, alternating permutation, Dyck path, Euler numbers, Catalan numbers, factorial Schur function.

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*Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: {ahmorales,pak}@math.ucla.edu.

†Department of Mathematics, UPenn, Philadelphia, PA 19104. Email: panova@math.upenn.edu.

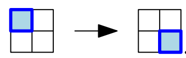
The *standard Young tableaux* (SYT) of straight and skew shapes are central objects in enumerative and algebraic combinatorics. The number $f^\lambda = |\operatorname{SYT}(\lambda)|$ of standard Young tableaux of shape λ has the celebrated *hook-length formula* (HLF):

Theorem 1.1 (HLF; Frame–Robinson–Thrall [FRT])**.** *Let λ be a partition of n. We have:*

$$\text{(HLF)} \qquad f^\lambda \;=\; \frac{n!}{\prod_{u\in[\lambda]} h(u)}\,,$$

where $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the hook-length *of the square $u = (i,j)$.*

Most recently, Naruse generalized (HLF) as follows. For a skew shape λ/μ, an *excited diagrams* is a subset of the Young diagram $[\lambda]$ of size $|\mu|$, obtained from the Young diagram $[\mu]$ by a sequence of *excited moves*:



Such move $(i,j) \to (i+1,j+1)$ is allowed only if cells $(i,j+1)$, $(i+1,j)$ and $(i+1,j+1)$ in $[\lambda]$ are unoccupied (see the precise definition and an example in §3.1). We use $\mathcal{E}(\lambda/\mu)$ to denote the set of excited diagrams of λ/μ.

Theorem 1.2 (NHLF; Naruse [Naru])**.** *Let λ,μ be partitions, such that $\mu\subset\lambda$. We have:*

$$\text{(NHLF)} \qquad f^{\lambda/\mu} \;=\; |\lambda/\mu|!\sum_{D\in\mathcal{E}(\lambda/\mu)}\;\prod_{u\in[\lambda]\setminus D}\frac{1}{h(u)}\;.$$

When $\mu=\varnothing$, there is a unique excited diagram $D=\varnothing$, and we obtain the usual HLF.

NHLF has two natural q-analogues which were proved in the previous paper in the series.

Theorem 1.3 ([MPP1])**.** *We have:*

$$\text{(first } q\text{-NHLF)} \qquad s_{\lambda/\mu}(1,q,q^2,\ldots) \;=\; \sum_{S\in\mathcal{E}(\lambda/\mu)}\;\prod_{(i,j)\in[\lambda]\setminus S}\frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}}\,.$$

Theorem 1.4 ([MPP1])**.** *We have:*

$$\text{(second } q\text{-NHLF)} \qquad \sum_{\pi\in\operatorname{RPP}(\lambda/\mu)} q^{|\pi|} \;=\; \sum_{S\in\mathcal{P}(\lambda/\mu)}\;\prod_{u\in S}\frac{q^{h(u)}}{1-q^{h(u)}}\,,$$

where $\mathcal{P}(\lambda/\mu)$ is the set of pleasant diagrams (*see* Definition 4.1)*.*

The second theorem employs a new family of combinatorial objects called *pleasant diagrams*. These diagrams can be defined as subsets of complements of excited diagrams (see [MPP1, §6]), and are technically useful. This allows us to write the RHS of (second q-NHLF) completely in terms of excited diagrams.

1.2. **Combinatorial proofs.** Our approach of the combinatorial proof of the NHLF in Section 5-7 is as follows. We start by proving the case of border strips (connected skew shapes with no 2×2 square). In this case the NHLF is more elegant,

$$\frac{f^{\lambda/\mu}}{|\lambda/\mu|!} = \sum_\gamma \prod_{(i,j)\in\gamma}\frac{1}{h(i,j)},$$

where the sum is over lattice paths γ from $(\lambda'_1,1)$ to $(1,\lambda_1)$ that stay inside $[\lambda]$. We give two self contained inductive proofs of this case. The first proof in Section 6 is based on showing a multivariate identity of paths. The second proof in Section 7 uses determinants to show that a multivariate identity of paths equals a ratio of *factorial Schur functions*.

We then use a corollary of the Lascoux–Pragacz identity for skew Schur functions: if $(\theta_1, \ldots, \theta_k)$ is a decomposition of the shape λ/μ into outer border strips θ_i (see Section 7.5) then

$$\frac{f^{\lambda/\mu}}{|\lambda/\mu|!} = \det\left[\frac{f^{\theta_i\#\theta_j}}{|\theta_i\#\theta_j|!}\right]_{i,j=1}^{k},$$

where $\theta_i\#\theta_j$ is a certain substrip of the outer border strip of λ.

Combining the case for border strips and this determinantal identity we get

$$\frac{f^{\lambda/\mu}}{|\lambda/\mu|!} = \det\left[\sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}}\ \prod_{(r,s)\in\gamma}\frac{1}{h(r,s)}\right]_{i,j=1}^{k},$$

where (a_j, b_j) and (c_i, d_i) are the endpoints of the border strip $\theta_i\#\theta_j$. Lastly, using the Lindström–Gessel–Viennot lemma this determinant is written as a weighted sum over non-intersecting lattice paths in λ. By an explicit characterization of excited diagrams in Section 3, the supports of such paths are exactly the complements of excited diagrams. The NHLF then follows.

A similar approach is used in Section 5 to give a combinatorial proof of the first q-NHLF for all skew shapes given in [MPP1]. The Hillman–Grassl inspired bijection in [MPP1] remains the only combinatorial proof of the second q-NHLF.

Remark 1.5. We should mention that our inductive proof is involutive, but basic enough to allow "bijectification", i.e. an involution principle proof of the NHLF. We refer to [K1, Rem, Zei] for the involution principle proofs of the (usual) HLF.

1.3. **Enumerative applications.** In sections 8 and 9, we give enumerative formulas which follow from NHLF. They involve q-analogues of Catalan, Euler and Schröder numbers. We highlight several of these formulas.

Let $\mathsf{Alt}(n) = \{\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots\} \subset S_n$ be the set of *alternating permutations*. The number $E_n = |\mathsf{Alt}(n)|$ is the n-th *Euler number* (see [S3] and [OEIS, A000111]), with the g.f.

$$1 + \sum_{n=1}^{\infty} E_n \frac{x^n}{n!} = \tan(x) + \sec(x)\,. \tag{1.1}$$

Let $\delta_n = (n-1, n-2, \ldots, 2, 1)$ denotes the staircase shape and observe that $E_{2n+1} = f^{\delta_{n+2}/\delta_n}$. Thus, the NHLF relates Euler numbers with excited diagrams of δ_{n+2}/δ_n. It turns out that these excited diagrams are in correspondence with the set $\mathsf{Dyck}(n)$ of *Dyck paths* of length $2n$ (see Corollary 8.1). More precisely,

$$|\mathcal{E}(\delta_{n+2}/\delta_n)| = |\mathsf{Dyck}(n)| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

where C_n is the n-th Catalan number, and $\mathsf{Dyck}(n)$ is the set of lattice paths from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that stay on or above the x-axis (see e.g. [S5]). Now the NHLF implies the following identity.

Corollary 1.6. *We have:*

$$\sum_{\mathsf{p}\in\mathsf{Dyck}(n)}\ \prod_{(a,b)\in\mathsf{p}}\frac{1}{2b+1} = \frac{E_{2n+1}}{(2n+1)!}\,, \tag{EC}$$

where $(a, b) \in \mathsf{p}$ denotes a point (a, b) of the Dyck path p.

Consider the following two q-analogues of E_n :

$$E_n(q) := \sum_{\sigma\in\mathsf{Alt}(n)} q^{\mathrm{maj}(\sigma^{-1})} \qquad \text{and} \qquad E_n^*(q) := \sum_{\sigma\in\mathsf{Alt}(n)} q^{\mathrm{maj}(\sigma^{-1}\kappa)}\,,$$

where $\operatorname{maj}(\sigma)$ is the *major index* of permutation σ in S_n and κ is the permutation $\kappa = (13254\ldots)$. See examples 8.6 and 9.5 for the initial values.

Now, for the skew shape δ_{n+2}/δ_n, Theorem 1.3 gives the following q-analogue of Corollary 1.6.

Corollary 1.7. *We have:*

$$\sum_{\mathsf{p} \in \mathsf{Dyck}(n)} \prod_{(a,b)\in \mathsf{p}} \frac{q^b}{1-q^{2b+1}} \;=\; \frac{E_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,.$$

Similarly, Theorem 1.4 in this case gives a different q-analogue.

Corollary 1.8. *We have:*

$$\sum_{\mathsf{p} \in \mathsf{Dyck}(n)} q^{H(\mathsf{p})} \prod_{(a,b)\in \mathsf{p}} \frac{1}{1-q^{2b+1}} \;=\; \frac{E^*_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,,$$

where

$$H(\mathsf{p}) \;=\; \sum_{(c,d)\in \mathcal{HP}(\mathsf{p})} (2d+1)\,,$$

and $\mathcal{HP}(\mathsf{p})$ denotes the set of peaks (c,d) in p with height $d>1$.

All three corollaries are derived in sections 8 and 9.

Section 8 considers the special case when λ/μ is a thick strip shape δ_{n+2k}/δ_n, which give the connection with Euler and Catalan numbers. In Section 9, we consider the pleasant diagrams of the thick strip shapes, establishing connection with Schröder numbers. We also state conjectures on certain determinantal formulas. We conclude with final remarks and open problems in Section 10.

2. Notation and Background

2.1. **Young diagrams.** Let $\lambda = (\lambda_1,\ldots,\lambda_r), \mu = (\mu_1,\ldots,\mu_s)$ denote integer partitions of length $\ell(\lambda) = r$ and $\ell(\mu) = s$. The *size* of the partition is denoted by $|\lambda|$ and λ' denotes the *conjugate partition* of λ. We use $[\lambda]$ to denote the Young diagram of the partition λ. The *hook length* $h_{ij} = \lambda_i - i + \lambda'_j - j + 1$ of a square $u = (i,j) \in [\lambda]$ is the number of squares directly to the right and directly below u in $[\lambda]$. The *Durfee square* $\square^\lambda$ is the largest square inside $[\lambda]$; it is always of the form $\{(i,j), 1 \le i,j \le k\}$.

A *skew shape* is denoted by λ/μ. A skew shape can have multiple edge connected components. For an integer k, $1-\ell(\lambda) \le k \le \lambda_1 - 1$, let d_k be the diagonal $\{(i,j) \in \lambda/\mu \mid i-j = k\}$, where $\mu_k = 0$ if $k > \ell(\mu)$. For an integer t, $1 \le t \le \ell(\lambda)-1$ let $\mathsf{d}_t(\mu)$ denote the diagonal $\mathsf{d}_{\mu_t - t}$ where $\mu_t = 0$ if $\ell(\mu) < t \le \ell(\lambda)$.

Given the skew shape λ/μ, let $P_{\lambda/\mu}$ be the poset of cells (i,j) of $[\lambda/\mu]$ partially ordered by component. This poset is *naturally labelled*, unless otherwise stated.

2.2. **Young tableaux.** A *reverse plane partition* of skew shape λ/μ is an array $\pi = (\pi_{ij})$ of nonnegative integers of shape λ/μ that is weakly increasing in rows and columns. We denote the set of such plane partitions by $\mathrm{RPP}(\lambda/\mu)$. A *semistandard Young tableau* of shape λ/μ is a RPP of shape λ/μ that is strictly increasing in columns. We denote the set of such tableaux by $\mathrm{SSYT}(\lambda/\mu)$. A *standard Young tableau* (SYT) of shape λ/μ is an array T of shape λ/μ with the numbers $1,\ldots,n$, where $n = |\lambda/\mu|$, each i appearing once, strictly increasing in rows and columns. For example, there are five SYT of shape $(32/1)$:

$$\begin{ytableau} \none & 1 & 2 \\ 3 & 4 \end{ytableau} \qquad \begin{ytableau} \none & 1 & 3 \\ 2 & 4 \end{ytableau} \qquad \begin{ytableau} \none & 1 & 4 \\ 2 & 3 \end{ytableau} \qquad \begin{ytableau} \none & 2 & 3 \\ 1 & 4 \end{ytableau} \qquad \begin{ytableau} \none & 2 & 4 \\ 1 & 3 \end{ytableau}$$

The *size* of a RPP or tableau T is the sum of its entries. A *descent* of a SYT T is an index i such that $i+1$ appears in a row below i. The *major index* $\operatorname{tmaj}(T)$ is the sum $\sum i$ over all the descents of T.

2.3. **Skew Schur functions.** Let $s_{\lambda/\mu}(\mathbf{x})$ denote the *skew Schur function* of shape λ/μ in variables $\mathbf{x} = (x_0, x_1, x_2, \ldots)$. In particular,

$$s_{\lambda/\mu}(\mathbf{x}) \;=\; \sum_{T\in \mathrm{SSYT}(\lambda/\mu)} \mathbf{x}^T\,, \qquad s_{\lambda/\mu}(1,q,q^2,\ldots) \;=\; \sum_{T\in \mathrm{SSYT}(\lambda/\mu)} q^{|T|}\,,$$

where $\mathbf{x}^T = x_0^{\#0s\text{ in }(T)} x_1^{\#1s\text{ in }(T)} \ldots$ Recall, the skew shape λ/μ can have multiple edgewise connected components $\theta_1, \ldots, \theta_m$. Since then $s_{\lambda/\mu} = s_{\theta_1} \cdots s_{\theta_k}$ we assume without loss of generality that λ/μ is edgewise connected.

2.4. **Determinantal identities for $s_{\lambda/\mu}$.** The Jacobi-Trudi identity (see e.g. [S4, §7.16]) states that

$$s_{\lambda/\mu}(\mathbf{x}) \;=\; \det\big[h_{\lambda_i-\mu_j-i+j}(\mathbf{x})\big]_{i,j=1}^n, \tag{2.1}$$

where $h_k(\mathbf{x}) = \sum_{i_1\le i_2\le\cdots\le i_k} x_{i_1}x_{i_2}\cdots x_{i_k}$ is the k-th *complete symmetric function.*

There are other determinantal identities of (skew) Schur functions like the Giambelli formula (e.g. see [S4, Ex. 7.39]) and the *Lascoux–Pragacz* identity [LasP]. Hamel and Goulden [HaG] found a vast common generalization to these three identities by giving an exponential number of determinantal identities for $s_{\lambda/\mu}$ depending on *outer decompositions* of the shape λ/μ. We focus on the Lascoux–Pragacz identity that we describe next through the Hamel–Goulden theory (e.g. see [CYY]).

A *border strip* is a connected skew shape without any 2×2 squares. The starting point and ending point of a strip are its southeast and northeast endpoints. Given λ, the *outer border strip* is the strip containing all the boxes sharing a vertex with the boundary of λ., i.e. $\lambda/(\lambda_2-1,\lambda_3-1,\ldots)$. A *Lascoux–Pragacz* decomposition of λ/μ is a decomposition of the skew shape into k maximal outer border strips $(\theta_1,\ldots,\theta_k)$, where θ_1 is the outer border strip of λ, θ_2 is the outer border strip of the remaining diagram $\lambda\setminus\theta_1$, and so on until we start intersecting μ. In this case, we continue the decomposition with each remaining connected component. The strips are ordered $\succeq$ by the contents of their northeast endpoints. See Figure 2.4, left, for an example.

We call the border strip θ_1 the *cutting strip* of the decomposition and denote it by τ [CYY]. For integers p and q, let $\phi[p,q]$ be the substrip of τ consisting of the cells with contents between p and q. By convention, $\theta[p,p]=(1)$, $\phi[p+1,p]=\varnothing$ and $\phi[p,q]$ with $p>q+1$ is undefined. The strip $\theta_i\#\theta_j$ is the substrip $\phi[p(\theta_j),q(\theta_i)]$ of τ, where $p(\theta_i)$ and $q(\theta_i)$ are the contents of the starting point and ending point of θ_i.

Theorem 2.1 (Lascoux–Pragacz [LasP], Hamel–Goulden [HaG])**.** *If $(\theta_1,\ldots,\theta_k)$ is a Lascoux–Pragacz decomposition of λ/μ, then*

$$s_{\lambda/\mu} = \det\left[\,s_{\theta_i\#\theta_j}\,\right]_{i,j=1}^k. \tag{2.2}$$

where $s_\varnothing = 1$ and $s_{\phi[p,q]}=0$ if $\phi[p,q]$ is undefined.

Example 2.2. Figure 2.4, right, shows the Lascoux–Pragacz decomposition for the shape $\lambda/\mu = (5441/21)$ into two strips (θ_1,θ_2). Then $\tau=\theta_1=(5441/33)$ and

$$\theta_1\#\theta_1=\theta_1,\quad \theta_1\#\theta_2=\phi[0,4]=(322/11),\quad \theta_2\#\theta_1=\phi[-3,2]=(441/3),\quad \theta_2\#\theta_2=\theta_2.$$

Then by the Lascoux–Pragacz identity $s_{(5441/21)}$ can be written as the following 2×2 determinant

$$s_{\lambda/\mu} = \det\begin{bmatrix} s_{\theta_1\#\theta_1} & s_{\theta_1\#\theta_2}\\ s_{\theta_2\#\theta_1} & s_{\theta_2\#\theta_2}\end{bmatrix} = \det\begin{bmatrix} s_{5441/33} & s_{322/11}\\ s_{441/3} & s_{22/1}\end{bmatrix}.$$

2.5. **Factorial Schur functions.** The *factorial Schur function* (e.g. see [MS]) is defined as

$$s_\mu^{(d)}(\mathbf{x}\mid\mathbf{a}) := \frac{\det\big[(x_i-a_1)\cdots(x_i-a_{\mu_j+d-j})\big]_{i,j=1}^d}{\prod_{1\le i<j\le d}(x_i-x_j)},$$

where $\mathbf{x}=x_1,\ldots,x_d$ and $\mathbf{a}=a_1,a_2,\ldots$ is a sequence of parameters.

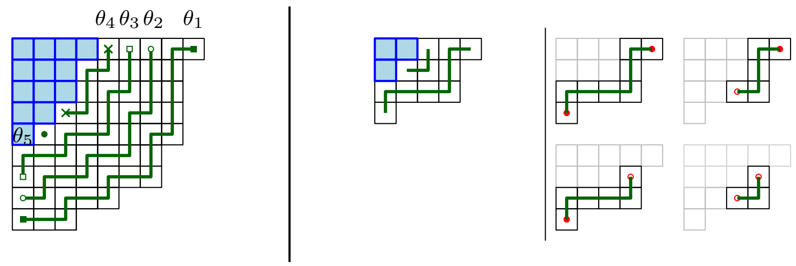


FIGURE 1. Left: Example of a Lascoux–Pragacz outer decomposition. Right: the Lascoux–Pragacz decomposition of the shape $\lambda/\mu = (5441/21)$ and the border strips of the determinantal identity for $s_{\lambda/\mu}$

2.6. **Permutations.** We write permutations of $\{1, 2, \ldots, n\}$ in *one-line notation*: $w = (w_1 w_2 \ldots w_n)$ where w_i is the image of i. A *descent* of w is an index i such that $w_i > w_{i+1}$. The *major index* $\mathrm{maj}(w)$ is the sum $\sum i$ of all the descents i of w.

2.7. **Dyck paths.** A *Dyck path* p of length $2n$ is a lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ that stay on or above the x-axis. We use $\mathsf{Dyck}(n)$ to denote the set of Dyck paths of length $2n$. For a Dyck path p, a *peak* is a point (c,d) such that $(c-1,d-1)$ and $(c+1,d-1) \in \mathsf{p}$. Peak (c,d) is called a *high-peak* if $d > 1$.

3. EXCITED DIAGRAMS

3.1. **Definition.** Let λ/μ be a skew partition and D be a subset of the Young diagram of λ. A cell $u = (i,j) \in D$ is called *active* if $(i+1,j)$, $(i,j+1)$ and $(i+1,j+1)$ are all in $[\lambda] \setminus D$. Let u be an active cell of D, define $\alpha_u(D)$ to be the set obtained by replacing (i,j) in D by $(i+1,j+1)$. We call this replacement an *excited move*. An *excited diagram* of λ/μ is a subdiagram of λ obtained from the Young diagram of μ after a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of λ/μ and $e(\lambda/\mu)$ its cardinality. For example, Figure 2 shows the eight excited diagrams of $(5441/21)$ (for the moment ignore the paths in the complement).

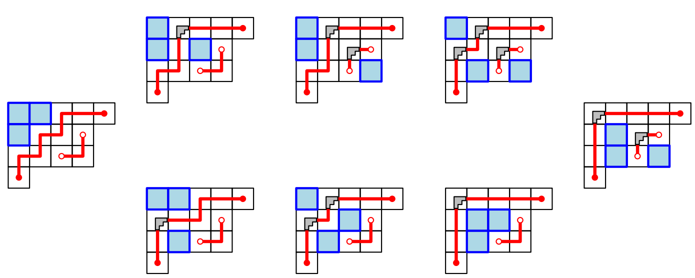


FIGURE 2. The eight excited diagrams (in blue) of shape $\lambda/\mu = (5441/21)$ and the corresponding non-intersecting paths (in red) in their complement. The high peaks of the paths described in Section 4.1 are in gray.

3.2. **Flagged tableaux.** Excited diagrams of λ/μ are equivalent to certain *flagged tableaux* of shape μ (see [MPP1, §3] and [Kre1, §6]). Thus, the number of excited diagrams is given by a determinant, a polynomial in the parts of λ and μ as follows. Consider the diagonal that passes through cell (i, μ_i), i.e. the last cell of row i in μ. Let this diagonal intersect the boundary of λ at a row denoted by $\mathsf{f}_i^{(\lambda/\mu)}$.

Proposition 3.1 ([MPP1]). *In notation above, we have:*

$$e(\lambda/\mu) = \det \left[\binom{\mathsf{f}_i^{(\lambda/\mu)} + \mu_i - i + j - 1}{\mathsf{f}_i^{(\lambda/\mu)} - 1} \right]_{i,j=1}^{\ell(\mu)}.$$

Example 3.2. For the same shape as in Example 2.2 and Figure 2, the number of excited diagrams equals the number of flagged tableaux of shape $(2,1)$ with entries in the first and second row ≤ 2. Thus

$$e(\lambda/\mu) = \det \begin{bmatrix} \binom{4}{2} & \binom{5}{2} \\ \binom{2}{2} & \binom{3}{2} \end{bmatrix} = \det \begin{bmatrix} 6 & 10 \\ 1 & 3 \end{bmatrix} = 8.$$

3.3. **Border strip decomposition formula for** $e(\lambda/\mu)$**.** This first determinantal identity for $e(\lambda/\mu)$ is similar to the Jacobi–Trudi identity for s_μ. In this section we prove a new determinantal identity for $e(\lambda/\mu)$ very similar to the Lascoux–Pragacz identity for $s_{\lambda/\mu}$.

Theorem 3.3. *If* $(\theta_1, \ldots, \theta_k)$ *is the Lascoux–Pragacz decomposition of* λ/μ *into* k *maximal outer border strips then*

$$e(\lambda/\mu) = \det \left[\, e(\theta_i \# \theta_j) \, \right]_{i,j=1}^{k},$$

where $e(\varnothing) = 1$ *and* $e(\phi[p,q]) = 0$ *if* $\phi[p,q]$ *is undefined.*

Example 3.4. For the same shape λ/μ as in Example 2.2 and Figure 2 we have

$$e(\lambda/\mu) = \det \begin{bmatrix} e(5441/33) & e(322/11) \\ e(441/3) & e(22/1) \end{bmatrix} = \det \begin{bmatrix} 10 & 3 \\ 4 & 2 \end{bmatrix} = 8.$$

In order to prove Theorem 3.3 we show a relation between excited diagrams and certain tuples of non-intersecting paths.

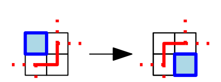
For the connected skew shape λ/μ there is a unique tuple of border-strips (i.e. non-intersecting paths) $\gamma_1^*, \ldots, \gamma_k^*$ in λ with support $[\lambda/\mu]$, where each border strip γ_i^* begins at the southern box (a_i', b_i') of a column and ends at the eastern box (c_i', d_i') of a row [Kre1, Lemma 5.3]. We call this tuple the *Kreiman decomposition* of λ/mu. Let $\mathcal{NI}(\lambda/\mu)$ be the set of k-tuples $\Gamma := (\gamma_1, \ldots, \gamma_k)$ of non-intersecting paths contained in $[\lambda]$ with $\gamma_i : (a_i, b_i) \to (c_i, d_i)$. The supports of the paths in $\mathcal{NI}(\lambda/\mu)$ are exactly the complements of excited diagrams in $\mathcal{E}(\lambda/\mu)$ [Kre1, §5.5]. See Figure 2, for an example.

Proposition 3.5 (Kreiman [Kre1]). *The* k*-tuples of paths in* $\mathcal{NI}(\lambda/\mu)$ *are uniquely determined by their support in* $[\lambda]$ *and moreover these supports are exactly the complements of excited diagrams of* λ/μ*.*

Proof. The fact that paths are uniquely determined by their support in $[\lambda]$ follows by [Kre1, Lemma 5.2]. By abuse of notation we identify the k-tuples of paths in $\mathcal{NI}(\lambda/\mu)$ with their supports. Note that the supports of all k-tuples of paths in $\mathcal{NI}(\lambda/\mu)$ have size $|\lambda/\mu|$.

We now show that the support of k-tuples in $\mathcal{NI}(\lambda/\mu)$ correspond to complements of excited diagrams.

First, we show that if $D \in \mathcal{E}(\lambda/\mu)$ then $[\lambda] \setminus D \in \mathcal{NI}(\lambda/\mu)$ by induction on the number of excited moves. Given $[\mu] \in \mathcal{E}(\lambda/\mu)$, its complement $[\lambda/\mu]$ corresponds to Kreiman decomposition $(\gamma_1^*, \ldots, \gamma_k^*) \in \mathcal{NI}(\lambda/\mu)$ as mentioned above. Then excited moves on the diagrams correspond to *ladder moves* on the non-intersecting paths:



The latter do not introduce intersections and preserve the endpoints of the paths. Thus for each $D \in \mathcal{E}(\lambda/\mu)$, its complement $[\lambda] \setminus D$ corresponds to a k-tuple $(\gamma_1, \ldots, \gamma_k)$ of paths in $\mathcal{NI}(\lambda/\mu)$.

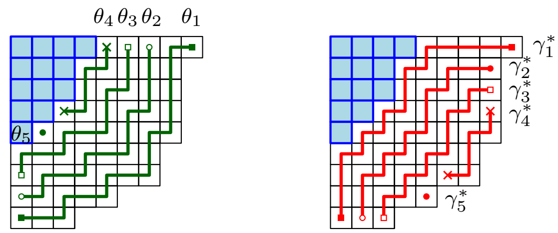


FIGURE 3. Example of the correspondence of the border strips of the Lascoux–Pragacz outer decomposition $(\theta_1,\ldots,\theta_5)$ and Kreiman outer decomposition $(\gamma_1^*,\ldots,\gamma_5^*)$ of λ/μ. The contents of the endpoints and the lengths of θ_i and γ_i^* are the same.

Conversely, consider the support S of a k-tuple of paths in $\mathcal{NI}(\lambda/\mu)$. The set S has the following property: the subset $S_k := S\cap\square_k^\lambda$ has no descending chain bigger than the length of the kth diagonal of λ/μ. Such sets S are called *pleasant diagram* of λ/μ [MPP1, §6] (see Section 4). Since $|S| = |\lambda/\mu|$, by [MPP1, Thm. 6.5] the set S is the complement of an excited diagram in $\mathcal{E}(\lambda/\mu)$, as desired. □

Corollary 3.6. $e(\lambda/\mu) = |\mathcal{NI}(\lambda/\mu)|$.

Next, we show a correspondence between the Kreiman decomposition and the Lascoux–Pragacz decomposition of λ/μ.

Lemma 3.7. *There is a correspondence between the border strips of the Lascoux–Pragacz and the Kreiman decomposition of λ/μ that preserves the lengths and contents of the starting and ending points of the paths/border strips:*

$$p(\theta_i) = b_i - a_i, \quad q(\theta_i) = d_i - c_i.$$

Proof. Let $(\theta_1,\ldots,\theta_k)$ and $(\gamma_1^*,\ldots,\gamma_t^*)$ be the Lascoux–Pragacz and the Kreiman decomposition of the shape λ/μ. We prove the result by induction on k. Note that θ_1 and γ_1^* have the same endpoints $(\lambda_1',1)$ and $(1,\lambda_1)$. Thus the strips have the same length and their endpoints have the same respective contents.

Next, note that the skew shapes λ/μ with θ_1 removed and λ/μ with γ_1^* removed are the same. The Lascoux–Pragacz of this new shape is $(\theta_2,\ldots,\theta_k)$ with the contents of the endpoints unchanged. Similarly, the Kreiman decomposition of this new shape is $(\gamma_2^*,\ldots,\gamma_t^*)$ with the contents of the endpoints unchanged. By induction $k-1=t-1$ and the strip θ_i and γ_i^* for $i=2,\ldots,k$ have the same length and their endpoints have the same content. This completes the proof. □

Lastly, we need a Lindström–Gessel–Viennot type Lemma to count (weighted) non-intersecting paths in $[\lambda]$. To state the Lemma we need some notation. Its proof follows the usual sign-reversing involution on paths that intersect (e.g. see [S4, §2.7]). Let $(a_1,b_1),\ldots,(a_k,b_k)$ and $(c_1,d_1),\ldots,(c_k,d_k)$ be cells in $[\lambda]$ and let

$$h((a_j,b_j)\to(c_i,d_i),\mathbf{y}) := \sum_\gamma \prod_{(r,s)\in\gamma} y_{r,s},$$

where the sum is over paths $\gamma:(a_j,b_j)\to(c_i,d_i)$ in $[\lambda]$ with steps $(1,0)$ and $(0,1)$, and the product is over cells (r,s) of γ. Let also

$$N_\lambda((a_i,b_i)\to(c_i,d_i);\mathbf{y}) := \sum_{(\gamma_1,\ldots,\gamma_k)} \prod_{i=1}^k \prod_{(r,s)\in\gamma_i} y_{r,s},$$

where the sum is over k-tuples $(\gamma_1,\ldots,\gamma_k)$ of non-intersecting paths $\gamma_i:(a_i,b_i)\to(c_i,d_i)$ in $[\lambda]$.

Lemma 3.8 (Lindström–Gessel–Viennot)**.**

$$N_\lambda((a_i,b_i)\to(c_i,d_i),\mathbf{y}) = \det\left[h((a_j,b_j)\to(c_i,d_i),\mathbf{y})\right]_{i,j=1}^k.$$

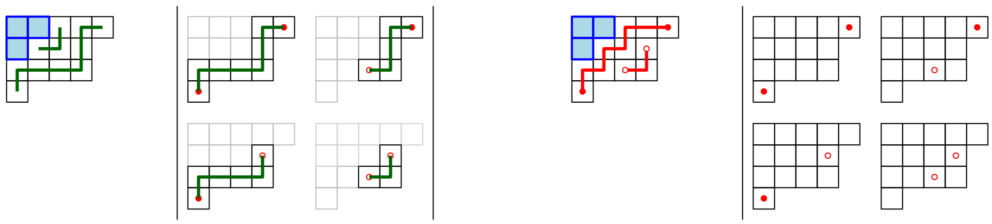


FIGURE 4. For the shape $\lambda/\mu = (5441/21)$: on the left the Lascoux–Pragacz decomposition of λ/μ and the border strips of the determinantal identity for $s_{\lambda/\mu}$; on the right the Kreiman decomposition of λ/μ and the corresponding endpoints of the paths in the determinantal identity (3.1) to calculate $e(\lambda/\mu)$.

Proof of Theorem 3.3. Combining Corollary 3.6 and Lemma 3.8 with weights $y_{r,s} = 1$ it follows that $e(\lambda/\mu)$ can be calculated by a determinant.

$$e(\lambda/\mu) = \det\left[\#\{\text{ paths } \gamma \mid \gamma \subseteq \lambda, \gamma : (a_j, b_j) \to (c_i, d_i)\,\}\right]_{i,j=1}^{k}. \tag{3.1}$$

Now, by Corollary 3.6, the number of paths in each matrix entry in the RHS of (3.1) is also the number of excited diagrams of a border strip θ. Since in Kreiman's decomposition, the endpoints (a_j, b_j) and (c_i, d_i) are in the bottom boundary of λ, then θ is a substrip of the outer border strip θ_1 of λ/μ going from (a_j, b_j) to (c_i, d_i),

$$\#\{\text{ paths } \gamma \mid \gamma \subseteq \lambda, \gamma : (a_j, b_j) \to (c_i, d_i)\,\} = e(\theta).$$

Next, we claim that the substrip θ of θ_1 described above is precisely the substrip $\theta_i\#\theta_j$ of $\tau = \theta_1$ from the Lascoux–Pragacz identity (2.2). This follows from Lemma 3.7 since the starting point (a_j, b_j) of γ_j^* has the same content as the starting point of θ_j and the end point (c_i, d_i) of γ_i^* has the same content as the end point of θ_i, i.e. $p(\theta_j) = b_j - a_j$ and $q(\theta_i) = d_i - c_i$. Thus $\theta = \phi[p(\theta_j), q(\theta_i)] = \theta_i\#\theta_j$ and so the previous equation becomes

$$\#\{\text{ paths } \gamma \mid \gamma \subseteq \lambda, \gamma : (a_j, b_j) \to (c_i, d_i)\,\} = e(\theta_i\#\theta_j). \tag{3.2}$$

Finally, the result follows by combining (3.1) and (3.2). $\square$

4. PLEASANT DIAGRAMS

4.1. Definition and characterization.

Definition 4.1 (Pleasant diagrams [MPP1])**.** A diagram $S \subset [\lambda]$ is a *pleasant diagram* of λ/μ if for all integers k with $1 - \ell(\lambda) \le k \le \lambda_1 - 1$, the subarray $S_k := S \cap \square_k^\lambda$ has no descending chain bigger than the length s_k of the diagonal d_k of λ/μ, i.e. for every k we have $dc_1(S_k) \le s_k$. We denote the set of pleasant diagrams of λ/μ by $\mathcal{P}(\lambda/\mu)$ and its size by $p(\lambda/\mu)$.

Pleasant diagrams can be characterized in terms of complements excited diagrams.

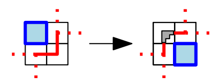
Theorem 4.2 ([MPP1])**.** *A diagram $S \subset [\lambda]$ is a pleasant diagram in $\mathcal{P}(\lambda/\mu)$ if and only if $S \subseteq [\lambda]\backslash D$ for some excited diagram $D \in \mathcal{E}(\lambda/\mu)$.*

Recall that by Proposition 3.5 for an excited diagram D, its complement corresponds to a tuple of non-intersecting paths in $\mathcal{NI}(\lambda/\mu)$ and that such paths are characterized by their support of size $|\lambda/\mu|$. Next, we give a formula for $p(\lambda/\mu)$ from [MPP1]. In order to state it we need to define a peak statistic for the non-intersecting paths associated to the complement of an excited diagram D.

To each tuple Γ of non-intersecting paths we associate recursively, via ladder/excited moves, a subset of its support called *excited peaks* and denoted by $\Lambda(\Gamma)$. For $[\lambda/\mu] \in \mathcal{NI}(\lambda/\mu)$ the set of excited peaks is $\Lambda([\lambda/\mu]) = \varnothing$. If Γ is a tuple in $\mathcal{NI}(\lambda/\mu)$ with an active cell $u = (i,j) \in [\lambda] \setminus \Gamma$ then the excited peaks of $\alpha_u(\Gamma)$ are

$$\Lambda(\alpha_u(\Gamma)) := \Big(\Lambda(\Gamma) - \{(i,j+1),(i+1,j)\}\Big) \cup \{u\}.$$

That is, the excited peaks of $\alpha_u(\Gamma)$ are obtained from those of Γ by adding the new peak (i,j) and removing $(i,j+1)$ and $(i+1,j)$ if any of the two are peaks in $\Lambda(\Gamma)$:



.

Finally, let $\operatorname{expk}(\Gamma) := |\Lambda(\Gamma)|$ be the number of excited peaks of Γ. Given a set $\mathcal{S}$, let $2^{\mathcal{S}}$ denote the subsets of $\mathcal{S}$.

Theorem 4.3 ([MPP1]). *For a skew shape λ/μ we have*

$$\mathcal{P}(\lambda/\mu) = \bigcup_{\Gamma\in\mathcal{NI}(\lambda/\mu)} \Big(\Lambda(\Gamma) \times 2^{\Gamma\setminus\Lambda(\Gamma)}\Big).$$

Thus

$$p(\lambda/\mu) = \sum_{\Gamma\in\mathcal{NI}(\lambda/\mu)} 2^{|\lambda/\mu|-\operatorname{expk}(\Gamma)}.$$

Example 4.4. For the shape $\lambda/\mu = (5441/21)$, Figure 2 has its eight non-intersecting paths in $\mathcal{NI}(\lambda/\mu)$, each with its excited peaks marked by . Thus

$$p(5441/21) = 2^{11} + 2\cdot 2^{10} + 2\cdot 2^{9} + 2^{8} + 2^{10} + 2^{9} = 6912.$$

4.2. **Border strip decomposition formula for pleasant diagrams.** By Theorem 4.3, the number of pleasant diagrams are given by a weighted sum over non-intersecting paths in $\mathcal{NI}(\lambda/\mu)$. Since the number of such paths $|\mathcal{NI}(\lambda/\mu)| = e(\lambda/\mu)$ is given by a Lascoux–Pragacz type determinant (Theorem 3.3), one could ask if there also a similar determinantal identity for $p(\lambda/\mu)$. The following example gives negative evidence. Later, we will see that Conjecture 9.3 suggests that in some cases there might be such a formula.

Example 4.5. For $\lambda/\mu = (5441/21)$ we showed that $p(5441/21) = 6912$, but $p(5441/33)p(22/1) - p(322/11)p(441/3) = 4352$ and the ratio of these two numbers is $27/17$.

Remark 4.6. One difficulty in applying the Lindström–Gessel–Viennot Lemma (Lemma 3.8) in order to write $\mathcal{P}(\lambda/\mu)$ as a determinant of $\mathcal{P}(\theta_i\#\theta_j)$ is that the non-intersecting paths corresponding to a pleasant diagram have excited-peaks that depend on the structure of the path and not just on its support. In the proof of the Lemma, the sign-reversing involution of switching the paths that intersect will not respect these *local* excited peaks.

5. Combinatorial proofs of the NHLF and first q-NHLF

The goal of this section is to give a combinatorial proof of the NHLF. The proof is split into two parts: first, we reduce the claim from all skew shapes to the border strips. We then give two elementary proofs of the NHLF in the border strip case, in the two sections that follow.

5.1. **NHLF for border strips.** In this case the NHLF is more elegant and can be stated as follows.

Lemma 5.1 (NHLF for border strips). *For a border strip $\theta = \lambda/\mu$ with endpoints (a,b) and (c,d) we have*

$$\frac{f^{\theta}}{|\theta|!} = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{1}{h(i,j)}, \tag{5.1}$$

where $h(i,j) = \lambda_i - i + \lambda'_j - j + 1$.

Since the endpoints (a,b) and (c,d) are on the boundary of λ without loss of generality we assume that $(a,b)=(\lambda_1',1)$ and $(c,d)=(1,\lambda_1)$. The proof is based on an identity of the following multivariate function. For a border strip λ/μ let

$$F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y}) = F_{\lambda/\mu}(x_1,x_2,\ldots,x_d \mid y_1,y_2\ldots,y_{n-d}) := \sum_{\substack{\gamma:(\lambda_1',1)\to(1,\lambda_1),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{1}{x_i-y_j}.$$

Note if we evaluate $F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y})$ at $x_i=\lambda_i+d-i+1$ and $y_j=d+j-\lambda_j'$ we obtain the RHS of (5.1),

$$F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y})\big|_{\substack{x_i=\lambda_i+d-i+1,\\ y_j=d+j-\lambda_j'}} = \sum_{\substack{\gamma:(\lambda_1',1)\to(1,\lambda_1),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{1}{h(i,j)}. \tag{5.2}$$

5.2. **From border trips to all skew shapes.** We need the analogue of Theorem 2.1 for $f^{\lambda/\mu}$.

Lemma 5.2 (Lascoux–Pragacz)**.** *If $(\theta_1,\ldots,\theta_k)$ is a Lascoux–Pragacz decomposition of λ/μ, then*

$$\frac{f^{\lambda/\mu}}{|\lambda/\mu|!} = \det\left[\frac{f^{\theta_i\#\theta_j}}{|\theta_i\#\theta_j|!}\right]_{i,j=1}^{k}. \tag{5.3}$$

where $f^{\varnothing}=1$ and $f^{\phi[p,q]}=0$ if $\phi[p,q]$ is undefined.

Proof. The result follows by doing the principal specialization in (2.2), using the theory of P-partitions [S4, Thm. 3.15.7] and letting $q\to 1$. ☐

Proof of Theorem 1.2. Combining Theorem 5.1 and Lemma 5.2 we have

$$f^{\lambda/\mu} = |\lambda/\mu|!\cdot\det\left[\sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}} \prod_{(r,s)\in\gamma}\frac{1}{h(r,s)}\right]_{i,j=1}^{k}. \tag{5.4}$$

Note that the weight $1/h(r,s)$ of each step in the path only depends on the coordinate (r,s) and the fixed partition λ. By the weighted Lindström–Gessel–Viennot lemma (Lemma 3.8), with $y_{r,s}=1/h(r,s)$, we rewrite the RHS of (5.4) as a weighted sum over k-tuples non-intersecting paths Γ in $\mathcal{NI}(\lambda/\mu)$. That is,

$$f^{\lambda/\mu} = |\lambda/\mu|!\cdot\sum_{(\gamma_1,\ldots,\gamma_k)\in\mathcal{NI}(\lambda/\mu)}\ \prod_{(r,s)\in(\gamma_1,\ldots,\gamma_k)}\frac{1}{h(r,s)}, \tag{5.5}$$

Finally, by Proposition 3.5 the supports of these non-intersecting paths are precisely the complements of excited diagrams of λ/μ. This finished the proof of NHLF. ☐

5.3. **Proof of the first q-NHLF.** In this case too the SSYT q-analogue of NHLF is elegant and can be stated as follows.

Lemma 5.3. *For a border strip $\theta=\lambda/\mu$ with end points (a,b) and (c,d) we have*

$$s_\theta(1,q,q^2,\ldots,) = \sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}}\prod_{(i,j)\in\gamma}\frac{q^{\lambda_j'-i}}{1-q^{h(i,j)}}. \tag{5.6}$$

The proof is postponed to Section 7.4.

Lemma 5.4 (Lascoux–Pragacz)**.**

$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \det\left[s_{\theta_i\#\theta_j}(1,q,q^2,\ldots)\right]_{i,j=1}^{k}, \tag{5.7}$$

where $s_\varnothing=1$ and $s_{\phi[p,q]}=0$ if $\theta[p,q]$ is undefined.

Proof. The result follows by doing a principal specialization in (2.2). ☐

Proof of Theorem 1.3. Combining Lemma 5.4 and Lemma 5.3 we have

$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \det\left[\sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}} \prod_{(r,s)\in\gamma} \frac{q^{\lambda'_s - r}}{1-q^{h(r,s)}}\right]_{i,j=1}^{k}. \tag{5.8}$$

Note that the weight of each step (r,s) in the path is $q^{\lambda'_s-r}/(1-q^{h(r,s)})$ which only depends on the coordinate (r,s) and the fixed partition λ. By the weighted Lindström–Gessel–Viennot lemma (Lemma 3.8), with $y_{r,s} = q^{\lambda'_s-r}/(1-q^{h(r,s)})$, we rewrite the RHS of (5.8) as a weighted sum of k-tuples non-intersecting paths in $[\lambda]$. That is,

$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \sum_{(\gamma_1,\ldots,\gamma_k)\in\mathcal{NI}(\lambda/\mu)} \prod_{(r,s)\in(\gamma_1,\ldots,\gamma_k)} \frac{q^{\lambda'_s-r}}{1-q^{h(r,s)}}, \tag{5.9}$$

Finally, by Proposition 3.5 the supports of these non-intersecting paths are precisely the complements of excited diagrams of λ/μ. Thus we obtain the (first q-NHLF). □

6. First proof of NHLF for border strips

In this section we give a proof of the NHLF for border strips based on a multivariate identity of the weighted sum of paths $F_\theta(\mathbf{x}\mid\mathbf{y})$. We show that this weighted sum satisfies a recurrence from SYT.

6.1. **Multivarite lemma.** For any connected skew shape λ/μ, the entry 1 in a standard Young tableau T of shape λ/μ will be in an inner corner of λ/μ. The remaining entries $2,3,\ldots,n$ form a standard Young tableau T' of shape λ/ν where $\mu\to\nu$. Conversely, given a standard Young tableau T' of shape λ/ν where $\mu\to\nu$, by filling the new cell with 0 we obtain a standard Young tableau of shape λ/μ. Thus

$$f^{\lambda/\mu} = \sum_{\mu\to\nu} f^{\lambda/\nu}. \tag{6.1}$$

We show combinatorially that for border strips λ/μ the multivariate rational function $F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y})$ satisfies this type of relation.

Lemma 6.1 (Pieri–Chevalley formula for border strips)**.**

$$F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y}) = \frac{1}{(x_1-y_1)}\sum_{\mu\to\nu} F_{\lambda/\nu}(\mathbf{x}\mid\mathbf{y}). \tag{6.2}$$

Remark 6.2. A very similar multivariate relation holds for general skew shapes (the only difference is a different linear factor on the RHS of (6.2)), a fact proved by Ikeda and Naruse [IN] algebraically and combinatorially by Konvalinka [Kon]. Our proof for border strips is different than these two proofs. See Section 10.2 for more details.

6.2. **Proof of multivarite lemma.** The rest of the section is devoted to the proof of Lemma 6.1. We start with some notation that will help us in the proof.

For cells $A,B\in[\lambda]$ such that B is NW of A, let

$$F(A\to B) := \sum_{\gamma:A\to B,\gamma\subseteq[\lambda]} \prod_{(i,j)\in\gamma} \frac{1}{x_i-y_j},$$

so that $F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y}) = F((\lambda'_1,1)\to(1,\lambda_1))$. For a given path γ let

$$H(\gamma) := \prod_{(i,j)\in\gamma} \frac{1}{(x_i-y_j)}$$

be its multivariate weight. Let $F(A^*,B)$ and $F(A,B^*)$ denote similar rational functions where we omit the term $x_i - y_j$ corresponding to A and B respectively. By abuse of notation $F(A \to C^* \to B)$ denotes the product $F(A \to C^*)F(C^* \to B)$. Let $\overline{C}$ and $\underline{C}$ denote the boxes in the Young diagram $[\lambda]$ that are right above and right below C, respectively. Let $R_k(\lambda)$ denote the kth row of the Young diagram of λ.

We will show by induction on the total length of the path between A and B that

$$F(A \to B) = \frac{1}{x_1 - y_1} \sum_C F(A \to C^* \to B), \tag{6.3}$$

where the sum is over inner corners C of λ/μ. This relation implies the desired result.

For the base case $\lambda = (2,2)$ and $\mu = (1)$, the shape $(2,2)/(1)$ has inner corners $(1,2)$ and $(2,1)$. We have

$$\begin{aligned} F((2,1) \to (1,2)) &= \frac{1}{(x_2-y_1)(x_2-y_2)(x_1-y_2)} + \frac{1}{(x_2-y_1)(x_1-y_1)(x_1-y_2)} \\ &= \frac{x_1 - y_1 + x_2 - y_2}{(x_1-y_1)(x_2-y_1)(x_1-y_2)(x_2-y_2)} \\ &= \frac{1}{x_1-y_1}\left(\frac{1}{(x_2-y_1)(x_2-y_2)} + \frac{1}{(x_2-y_2)(x_1-y_2)}\right), \end{aligned}$$

which equals $\left[F((1,2) \to (2,1)^*) + F((1,2)^* \to (2,1))\right]/(x_1 - y_1)$, thus proving the relation.

The next sublemma will be useful in the inductive step later.

Lemma 6.3. *For cells $A = (d,r)$ and $B = (1,s)$ in $[\lambda]$ with $r \le s$, we have*

$$(x_1 - x_d)F(A \to B) = \sum_C F(A \to C^* \to B),$$

where the sum is over inner corners C of λ/μ.

Proof. We can write $x_k - x_{k-1} = (x_k - y_j) - (x_{k-1} - y_j)$ for any j. Let γ be a path from A to B, and suppose that it crosses from row k to row $k-1$ in column j for some j. Then both points $(k,j) \in \gamma$ and $(k-1,j) \in \gamma$

$$(x_k - x_{k-1})H(\gamma) = (x_k - y_j)H(\gamma) - (x_{k-1} - y_j)H(\gamma) = H(\gamma \setminus (k,j)) - H(\gamma \setminus (k-1,j)). \tag{6.4}$$

Since every path from A to B crosses from row k to row $k-1$ at some cell, denoted by $C = (k,j)$, by (6.4) we have the following:

$$\begin{aligned} (x_k - x_{k-1})F(A \to B) \;&=\; \sum_{C \in R_k(\lambda)} \Big(F(A \to C^*)F(\overline{C} \to B) - F(A \to C)F(\overline{C}^* \to B)\Big) \\ &= \sum_{C \in R_k(\lambda)} F(A \to C^* \to \overline{C} \to B) - \sum_{C_1 \in R_{k-1}(\lambda)} F(A \to \underline{C_1} \to C_1^* \to B) \;=\; \circledast \end{aligned}$$

where in the last line we denote $C_1 = \overline{C}$ – a box in row $k-1$, and we note that the existence of the boxes below and above is implicit in the specified path functions F.

Let us now rewrite the RHS. in the last equation in a different way. Note that the paths $A \to C^* \to \overline{C} \to B$ can be thought as paths from A to B without their outer corner on row k, and, likewise, the paths $A \to \underline{C_1} \to C_1^* \to B$ are paths $A \to B$ without the inner corner on row $k-1$. However, they can both be thought as composed of two paths, $A \to A_1$ and $B_1 \to B$, where A_1 is the last box on row k (or row $k+1$ if C was the only cell on row k), B_1 is the first box on row $k-1$ (or the box above $\overline{C}$, in row $k-2$) and A_1's top right vertex is the same as B_1's bottom left (i.e. the boxes have that common vertex), or as in the second case B_1 is one box above A_1. In the case of $A \to C^* \to \overline{C} \to B = A \to A_1, B_1 \to B$, we must have that A_1 is not the last box in the row (for C

to exist), and for $A \to \underline{C_1} \to C_1^* \to B = A \to A_1, B_1 \to B$ there are no restrictions. Thus

$$\begin{aligned}
&(x_k - x_{k-1})F(A \to B) \,=\, \circledast \\
&= \sum_{A_1 \neq (k,\lambda_k), B_1} F(A \to A_1)F(B_1 \to B) \,-\, \sum_{A_1,B_1} F(A \to A_1)F(B_1 \to B) \\
&= \sum_{j:\substack{A_1=(k+1,j),\\ B_1=(k-1,j)}} F(A \to A_1)F(B_1 \to B) -, \sum_{j:\substack{A_1=(k,j),\\ B_1=(k-2,j)}} \Big(F(A \to A_1)F(B_1 \to B) - F(A \to D_k^* \to B)\Big),
\end{aligned}$$

where all terms cancel except for the cases where A_1, C, B_1 are in the same column, and when C is an outer corner of λ on row k, denoted by D_k (if such corner exists).

Finally, since $x_d - x_1 = \sum_k (x_k - x_{k-1})$, we have

$$\begin{aligned}
&(x_d - x_1)F(A \to B) \,=\, \sum_k (x_k - x_{k-1})F(A \to B) \\
&= \sum_{k,j:\substack{A_1=(k+1,j),\\ B_1=(k-1,j)}} F(A \to A_1)F(B_1 \to B) \,-\, \sum_{k,j:\substack{A_1=(k,j),\\ B_1=(k-2,j)}} \Big(F(A \to A_1)F(B_1 \to B) - F(A \to D_k^* \to B)\Big) \\
&= -\sum_k F(A \to D_k^* \to B),
\end{aligned}$$

since all other terms cancel across the various values for k, and we obtain the desired identity. $\square$

We continue with the proof of Lemma 6.1. In a path $\gamma : A \to B$ the first step from A is either right to cell A_r or up to cell A_u. Note that in the first case A is then an inner corner of λ/μ. Thus

$$F(A \to B) \,=\, \frac{1}{x_d - y_1}\,\big(F(A_r \to B) + F(A_u \to B)\big).$$

By induction the term $F(A_u \to B)$ becomes

$$F(A \to B) \,=\, \frac{1}{x_d - y_1}\left(F(A_r \to B) \,+\, \frac{1}{x_1 - y_1}\sum_C F(A_u \to C^* \to B)\right). \tag{6.5}$$

On the other hand, since a step to A_r indicates that A is an inner corner then the RHS of (6.3) equals

$$\frac{1}{x_1 - y_1}\sum_C F(A \to C^* \to B) \,=\, \frac{1}{x_1 - y_1}\left[F(A_r \to B) \,+\, \frac{1}{x_d - y_1}\sum_C F(A^* \to C^* \to B)\right].$$

Again, depending on the first step of the paths we split $F(A^* \to C^* \to B)$ into $F(A_r \to C^* \to B)$ and $F(A_u \to C^* \to B)$ so the above equation becomes

$$\begin{aligned}
&\frac{1}{x_1 - y_1}\sum_C F(A \to C^* \to B) \\
&\quad = \frac{1}{x_1 - y_1}\left[F(A_r \to B) \,+\, \frac{1}{x_d - y_1}\sum_C \Big(F(A_r \to C^* \to B) \,+\, F(A_u \to C^* \to B)\Big)\right].
\end{aligned} \tag{6.6}$$

Finally, by (6.5) and (6.6) if we subtract the LHS and RHS of (6.3) the terms with $A_u \to C^* \to B$ cancel. Collecting the terms with $A_r \to B$ we obtain

$$\begin{aligned}
&F(A \to B) \,-\, \frac{1}{x_1 - y_1}\sum_C F(A \to C^* \to B) \\
&\quad = \frac{x_1 - x_d}{(x_1 - y_1)(x_d - y_1)}\,F(A_r \to B) \,-\, \frac{1}{(x_1 - y_1)(x_d - y_1)}\sum_C F(A_r \to C^* \to B_1).
\end{aligned} \tag{6.7}$$

Lastly, the RHS above is zero since by Lemma 6.3 we have

$$(x_1 - x_d)F(A_r \to B) \;=\; \sum_{C} F(A_r \to C^* \to B).$$

Thus the desired relation (6.3) follows.

6.3. **Proof of NHLF for border strips.** In this section we use Lemma 6.1 to prove Theorem 5.1.
Let $H_{\lambda/\mu}$ denote the RHS of (5.2). We prove by induction on $n = |\lambda/\mu|$ that $f^{\lambda/\mu} = n! \cdot H_{\lambda/\mu}$.
We start with (6.2) and evaluate $x_i = \lambda_i + d - i + 1$ and $y_j = d + j - \lambda'_j$, by (5.2) we obtain

$$n \cdot H_{\lambda/\mu} = \sum_{\mu \to \nu} H_{\lambda/\nu}.$$

Multiplying both sides by $(n-1)!$ and using induction we obtain

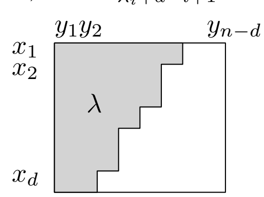
$$n! \cdot H_{\lambda/\mu} = \sum_{\mu \to \nu} f^{\lambda/\nu}.$$

By (6.1) the result follows.

7. Second proof of NHLF for border strips

In this section we give another proof of the NHLF for border strips based on another multivariate identity involving *factorial Schur functions*. The proof consists of two steps. First we show that a ratio of an evaluation of factorial Schur functions equals the weighted sum of paths $F_\theta(\mathbf{x} \mid \mathbf{y})$. Second we show how the ratio of factorial Schur functions properly specialized equals $f^{\lambda/\mu}$ and $s_{\lambda/\mu}(1, q, q^2, \ldots)$.

7.1. **Multivarite lemma.** We show combinatorially that the function $F_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y})$ is an evaluation of a factorial Schur function. Let $\mathbf{z}^\lambda$ be the word of length n of x's and y's by reading the horizontal and vertical steps of λ from $(d, 1)$ to $(1, n-d)$: i.e. $z^\lambda_{\lambda_i+d-i+1} = x_i$ and $z^\lambda_{\lambda'_j+n-d-j+1} = y_j$:



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Lemma 7.1 ([IN])**.** *For a border strip $\lambda/\mu \subseteq d \times (n-d)$ we have*

$$\frac{s^{(d)}_\mu(\mathbf{x} \mid \mathbf{z}^\lambda)}{s^{(d)}_\lambda(\mathbf{x} \mid \mathbf{z}^\lambda)} = F_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}). \tag{7.1}$$

Before we begin the proof we make a few definitions to simplify notation and a few observations to be used throughout. For any partition $\nu \subseteq d \times (n-d)$ and a set of variables $\mathbf{x}$ and $\mathbf{z}$ define

$$D(\nu) := \det[(x_i - z_1) \ldots (x_i - z_{\nu_j+d-j})]^d_{i,j=1},$$

so that

$$G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) := \frac{s^{(d)}_\mu(\mathbf{x} \mid \mathbf{z}^\lambda)}{s^{(d)}_\lambda(\mathbf{x} \mid \mathbf{z}^\lambda)} = \frac{D(\mu)}{D(\lambda)}. \tag{7.2}$$

Notice also that $z_{\lambda_j+1+d-j} = x_j$ and so $(x_i - z_1) \ldots (x_i - z_{\lambda_j+d-j}) = 0$ if $j < i$. So the matrix in $D(\lambda)$ is upper-triangular and

$$D(\lambda) = \prod_{i=1}^{d} (x_i - z_1) \cdots (x_i - z_{\lambda_i+d-i}). \tag{7.3}$$

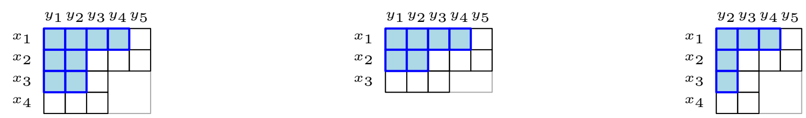
7.2. **Proof of multivariate lemma.** To prove Lemma 7.1 we verify that both sides of (7.1) satisfy the following trivial path identity. The first step of a path $\gamma : (\lambda'_1, 1) \to (1, \lambda_1)$ is either $(0,1)$ (up) or $(1,0)$ (right) provided $\lambda_d > 1$. So

$$(x_d - y_1) F_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) = F_{\lambda-\lambda_d/\mu-\mu_{d-1}}(x_1, \ldots, x_{d-1} \mid \mathbf{y}) + F_{\lambda-\mathbf{1}/\mu-\mathbf{1}}(\mathbf{x} \mid y_2, \ldots, y_{n-d}), \tag{7.4}$$

where the second term on the RHS vanishes if $\lambda_d = 1$.

Example 7.2. For the border strip $\lambda/\mu = (5533/422)$, we have

$$(x_4 - y_1) F_{(5533/422)}(x_1, \ldots, x_4 \mid y_1, \ldots, y_5) =$$
$$F_{(553/42)}(x_1, x_2, x_3 \mid y_1, \ldots, y_5) + F^{(4422/311)}(x_1, \ldots, x_4 \mid y_2, \ldots, y_5),$$



Next we show that the following ratio of factorial Schur functions, satisfies the same relation,

$$G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) := \frac{s^{(d)}_\mu(\mathbf{x} \mid \mathbf{z}^\lambda)}{s^{(d)}_\lambda(\mathbf{x} \mid \mathbf{z}^\lambda)}.$$

Lemma 7.3. *We have:*

$$(x_d - y_1) G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) \,=\, G_{\lambda-\lambda_d/\mu-\mu_{d-1}}(x_1, \ldots, x_{d-1} \mid \mathbf{y}) \,+\, G_{\lambda-\mathbf{1}/\mu-\mathbf{1}}(\mathbf{x} \mid y_2, \ldots, y_{n-d}),$$

where the second term on the RHS vanishes if $\lambda_d = 1$.

Proof of Lemma 7.1. We proceed by induction. For the base case $\lambda = (1)$ and $\mu = \varnothing$, we directly check that

$$F_{(1)/\varnothing}(\mathbf{x} \mid \mathbf{y}) \,=\, G_{(1)/\varnothing}(\mathbf{x} \mid \mathbf{y}) = \frac{1}{x_d - y_1}.$$

Then by (7.4) and Lemma 7.3 we have $F_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y})$ and $G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y})$ satisfy the same recurrence. Therefore, we have $F_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) = G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y})$ as desired. □

In the rest of the section we prove Lemma 7.3.

Proof of Lemma 7.3. We denote the shape $\lambda - \lambda_d/\mu - \mu_{d-1}$ by $\overline{\lambda}/\overline{\mu}$. Removing the first column of λ yields

$$\mathbf{z}^{\lambda-\mathbf{1}} = y_2, \ldots = z^\lambda_2, \ldots,$$

removing the last of λ yields $\mathbf{z}^{\bar{\lambda}} = y_1, y_2, \ldots, \widehat{x_d}, \ldots = z_1, \ldots, z_{\lambda_d}, z_{\lambda_d+2}, \ldots$, i.e. $\mathbf{z}^\lambda$ with the entry x_d omitted.

Assume $\lambda_d \neq 0$ and $\mu_d = 0$, the other case is trivially reduced. If λ/μ is a border strip $\mu_j = \lambda_{j+1} - 1$ for $j = 1, \ldots, d-1$. Hence in the ratio of determinants in (7.2) we have that the first $d-1$ columns of the determinant from $s^{(d)}_\mu(\cdot \mid \cdot)$ are the last $d-1$ columns from the determinant for $s^{(d)}_\lambda(\cdot \mid \cdot)$, and the dth column from $s^{(d)}_\mu(\cdot \mid \cdot)$ is all ones, since $\mu_d + d - d = 0$. Thus in (7.2), upon shifting the dth column to the first column in the determinant $D(\mu)$ in the numerator, we obtain

$$G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) = \frac{(-1)^{d-1}}{D(\lambda)} \det \left[\begin{cases} 1, & j = 1 \\ (x_i - z_1) \ldots (x_i - z_{\lambda_j + d - j}), & j > 1 \end{cases} \right]_{i,j=1}^{d}. \tag{7.5}$$

Next we have two cases depending on whether $\lambda_d = 1$ or $\lambda_d > 1$.

Case $\lambda_d = 1$: For λ we have $z^\lambda_1 = y_1$ and $z^\lambda_2 = x_d$. The (d,d) entry of the upper triangular matrix of the determinant in $D(\lambda)$ is $x_d - y_1$, so by doing a cofactor expansion on this row we get

$$D(\lambda) \,=\, (x_d - y_1) \det\bigl[(x_i - x_d)(x_i - y_1)(x_i - z_3) \cdots (x_i - z_{\lambda_j + d - j})\bigr]_{i,j=1}^{d-1}.$$

By factoring $x_i - z_2 = x_i - x_d$ from each row above we get

$$D(\lambda) \,=\, (x_d - y_1)\,\det\big[(x_i - y_1)(x_i - z_3)\cdots(x_i - z_{\lambda_j+d-j})\big]_{i,j=1}^{d-1}\prod_{i=1}^{d-1}(x_i - x_d).$$

Since $z_1^{\overline{\lambda}} = y_1$ and $z_j^{\overline{\lambda}} = z_{j+1}^{\lambda}$ for $j = 2, \ldots, d-1$, then by relabelling we get

$$D(\lambda) \,=\, (x_d - y_1)\,D(\overline{\lambda})\,\prod_{i=1}^{d-1}(x_i - x_d). \tag{7.6}$$

For μ we have $\mu_d = \mu_{d-1} = 0$ so the matrix in $D(\mu)$ has a dth column of ones

$$D(\mu) = \det\begin{bmatrix} \ldots & (x_1 - z_1)\cdots(x_1 - z_{\mu_j+d-j}) & \cdots & (x_1 - y_1) & 1 \\ \ldots & (x_2 - z_1)\cdots(x_2 - z_{\mu_j+d-j}) & \cdots & (x_2 - y_1) & 1 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & (x_d - y_1) & 1 \end{bmatrix}$$

Then, by adding $(x_d - y_1)$ to each entry in the $(d-1)$th column, the determinant remains unchanged but the last row becomes $0\ldots01$,

$$D(\mu) = \det\left[\begin{cases} 1, & j = d \\ x_i - x_d, & j = d-1 \\ (x_i - z_1)\ldots(x_i - z_{\mu_j+d-j}), & j < d-1 \end{cases}\right]_{i,j=1}^{d}$$

Next, we do a cofactor expansion on the last row and then we factor $x_i - z_2 = x_i - x_d$ from each row,

$$\begin{aligned} D(\mu) &= \det\left[\begin{cases} x_i - x_d, & j = d-1 \\ (x_i - z_1)(x_i - z_2)\ldots(x_i - z_{\mu_j+d-j}), & j < d-1 \end{cases}\right]_{i,j=1}^{d-1} \\ &= \det\left[\begin{cases} 1, & j = d-1 \\ (x_i - z_1)(\widehat{x_i - x_d})\ldots(x_i - z_{\mu_j+d-j}), & j < d-1 \end{cases}\right]_{i,j=1}^{d-1}\prod_{i=1}^{d-1}(x_i - x_d). \end{aligned}$$

Again, since $z_1^{\bar{\lambda}} = y_1$ and $z_j^{\bar{\lambda}} = z_{j+1}^{\lambda}$ for $j = 2, \ldots, d-1$, we have by relabeling

$$D(\mu) = D(\bar{\mu})\prod_{i=1}^{d-1}(x_i - x_d). \tag{7.7}$$

We now combine (7.6) and (7.7) in $(x_d - y_1)G_{\lambda/\mu}(\cdot \mid \cdot)$,

$$(x_d - y_1)G_{\lambda/\mu}(\mathbf{x} \mid \mathbf{y}) = (x_d - y_1)\frac{D(\bar{\mu})\prod_{i=1}^{d-1}(x_i - x_d)}{(x_d - y_1)D(\bar{\lambda})\prod_{i=1}^{d-1}(x_i - x_d)} = G_{\bar{\lambda}/\bar{\mu}}(x_1, \ldots, x_{d-1} \mid \mathbf{y}),$$

confirming the desired identity in this case as well since the term for $\lambda - \mathbf{1}/\mu - \mathbf{1}$ is vacuously zero.

Case $\lambda_d > 1$*:* Using $z_1 = y_1$ we have

$$\begin{aligned} G_{\lambda-\mathbf{1}/\mu-\mathbf{1}}(\mathbf{x} \mid y_2, \ldots, y_{n-d}) &= \frac{(-1)^{d-1}}{D(\lambda - \mathbf{1})}\det\left[\begin{cases} 1, & j = 1 \\ (x_i - z_2)\ldots(x_i - z_{\lambda_j+d-j}), & j > 1 \end{cases}\right]_{i,j=1}^{d} \\ &= \frac{(-1)^{d-1}}{D(\lambda)}\det\left[\begin{cases} 1, & j = 1 \\ (x_i - z_2)\ldots(x_i - z_{\lambda_j+d-j}), & j > 1 \end{cases}\right]_{i,j=1}^{d} \cdot \prod_{i=1}^{d}(x_i - y_1) \\ &= \frac{(-1)^{d-1}}{D(\lambda)}\det\left[\begin{cases} (x_i - y_1), & j = 1 \\ (x_i - z_1)\ldots(x_i - z_{\lambda_j+d-j}), & j > 1 \end{cases}\right]_{i,j=1}^{d} \end{aligned}$$

Similarly, we have:

$$(7.8)\quad G_{\bar{\lambda}/\bar{\mu}}(x_1,\ldots,x_{d-1}\mid \mathbf{y}) = \frac{(-1)^{d-2}}{D(\lambda)}\det\left[\begin{cases}(x_i-x_d), & j=1\\ (x_i-z_1)\ldots(x_i-z_{\lambda_{j-1}+d-j-1}), & 2\le j\le d-1\end{cases}\right]_{i,j=1}^{d-1}\cdot\prod_{j=1}^{\lambda_d}(x_d-y_j)$$

Next, we add the two determinants in (7.5) and (7.8) using the multilinearity property on the first column to obtain

$$(x_d-y_1)G_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y}) - G_{\lambda-\mathbf{1}/\mu-\mathbf{1}}(\mathbf{x}\mid y_2,\ldots,y_{n-d}) = \frac{(-1)^{d-1}}{D(\lambda)}\times$$

$$\left(\det\left[\begin{cases}(x_d-y_1), & j=1\\ (x_i-z_1)\ldots(x_i-z_{\lambda_j+d-j}), & j>1\end{cases}\right]_{i,j=1}^{d} - \det\left[\begin{cases}(x_i-y_1), & j=1\\ (x_i-z_1)\ldots(x_i-z_{\lambda_j+d-j}), & j>1\end{cases}\right]_{i,j=1}^{d}\right)$$

$$= \frac{(-1)^{d-1}}{D(\lambda)}\det\left[\begin{cases}x_d-x_i, & j=1\\ (x_i-z_1)\ldots(x_i-z_{\lambda_j+d-j}), & j>1\end{cases}\right]_{i,j=1}^{d} =: (*)$$

Consider the row $i=d$ in the last determinant. The entries there are all 0, except when $j=d$: when $j=1$ we have $x_d-x_i=0$ for $i=d$, when $j\in[2,d-1]$ we have $\lambda_j+d-j\ge\lambda_d+1$, and since $z_{\lambda_d+1}=x_d$ we have $\prod_{r=1}^{\lambda_j+d-j}(x_d-z_r)=0$. Using the cofactor expansion we compute the determinant in the last equation as the principal minor of the matrix times the (d,d) entry:

$$(*) = \frac{(-1)^{d-1}}{D(\lambda)}\det\left[\begin{cases}x_d-x_i, & j=1\\ (x_i-z_1)\ldots(x_i-z_{\lambda_j+d-j}), & j>1\end{cases}\right]_{i,j=1}^{d-1}(x_d-z_1)\cdots(x_d-z_{\lambda_d}).$$

We now compare this with equation (7.8), realizing that $z_1,\ldots,z_{\lambda_d}=y_1,\ldots,y_{\lambda_d}$, so the last expression coincides with $G_{\bar{\lambda}/\bar{\mu}}(\mathbf{x}\mid\mathbf{y})$ as desired. Notice also that if $j<i$, we have $\lambda_j+d-j\ge\lambda_i+d-i+1$, and since $x_i=z_{\lambda_i+d-i+1}$, the terms above are 0 when $j<i$ and $j\ne 1$. □

7.3. Proof of NHLF for border strips.

Lemma 7.4. *Let $\mu\subset\lambda$ be two partitions with d parts. Then*

$$\left.\frac{s^{(d)}_\mu(\mathbf{x}\mid\mathbf{z}^\lambda)}{s^{(d)}_\lambda(\mathbf{x}\mid\mathbf{z}^\lambda)}\right|_{\substack{x_i=\lambda_i+d-i+1,\\ y_i=d+j-\lambda'_j}} = \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}.$$

This statement also appears in [Naru] (see [MPP1, §8.4]) with a different proof.

Proof. Let $x_i=\lambda_i+d-i+1$ and $y_j=d+j-\lambda'_j$, then notice that x_i and y_j are exactly the numbers on the horizontal/vertical steps at row i/column j of the lattice path determined by λ when writing the numbers $1,2,\ldots$ along the path from the bottom left to the top right end. Thus $z_\lambda=1,2,3\ldots$, and so

$$(x_i-z_1)\cdots(x_i-z_{\mu_j+d-j}) = (\lambda_i+d-i)\cdots(\lambda_i+d-i+1-(\mu_j+d-j)) = \frac{(\lambda_i+d-i)!}{(\lambda_i-i-\mu_j+j)!}$$

whenever $\lambda_i-i\ge\mu_j-j$ and 0 otherwise. When $\mu=\lambda$ and $i=j$, we have $(x_i-z_1)\cdots(x_i-z_{\lambda_i+d-i}) = (\lambda_i+d-i)!$ and by (7.3) we have

$$D(\lambda)\Big|_{\substack{x_i=\lambda_i+d-i+1,\\ y_j=d+j-\lambda'_j}} = \prod_{i=1}^{d}(\lambda_i+d-i)!$$

Then, by definition and (7.2), we have

$$G_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y})\Big|_{\substack{x_i=\lambda_i+d-i+1,\\ y_j=d+j-\lambda'_j}} = \frac{D(\mu)}{D(\lambda)}\Big|_{\substack{x_i=\lambda_i+d-i+1,\\ y_j=d+j-\lambda'_j}} = \frac{\det\big[(\lambda_i+d-i)!/(\lambda_i-i-\mu_j+j)!\big]_{i,j=1}^d}{\prod_{i=1}^d(\lambda_i+d-i)!}$$
$$= \det\left[\frac{1}{(\lambda_i-i-\mu_j+j)!}\right]_{i,j=1}^d$$

Multiplying the last determinant by $|\lambda/\mu|!$, we recognize the exponential specialization of the Jacobi-Trudi identity for the ordinary $s_{\lambda/\mu}$ giving $f^{\lambda/\mu}$ (a formula due to Aitken, see e.g. [S4, Cor. 7.16.3]). Hence we get the desired formula. $\square$

Second proof of Theorem 5.1. We start with the relation from Lemma 7.1 and evaluate $x_i=\lambda_i+d-i+1$ and $y_j=d+j-\lambda'_j$. In the RHS by (5.2) we immediately obtain the RHS of (5.1).

Next, we do the same evaluation on the ratio of factorial Schur functions applying Lemma 7.4 that gives the ratio of factorial Schurs as $f^{\lambda/\mu}/|\lambda/\mu|!$. $\square$

7.4. **SSYT q-analogue for border strips.** To wrap up the section we show how the tools developed to prove Theorem 5.1 also yield the SSYT q-analogue for border strips.

Corollary 7.5 ((first q-NHLF) for border strips)**.** *For a border strip $\theta=\lambda/\mu$ with end points (a,b) and (c,d) we have*

$$s_\theta(1,q,q^2,\ldots,) = \sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}}. \tag{7.9}$$

Proof. We start with (7.1) from Lemma 7.1 and evaluate both sides at $x_i=q^{\lambda_i+d-i+1}$ and $y_j=q^{d+j-\lambda'_j}$. The path series $F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y})$ gives the RHS of (7.9)

$$F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y})\big|_{\substack{x_i=q^{\lambda_i+d-i+1},\\ y_j=q^{d+j-\lambda'_j}}} = \sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}}.$$

Next, by [MPP1, §4] the evaluation of the ratio of the factorial Schur functions gives the principal specialization of the Schur function, the LHS of (7.9)

$$\frac{s_\mu^{(d)}(\mathbf{x}\mid\mathbf{z}^\lambda)}{s_\lambda^{(d)}(\mathbf{x}\mid\mathbf{z}^\lambda)}\Bigg|_{\substack{x_i=q^{\lambda_i+d-i+1},\\ y_j=q^{d+j-\lambda'_j}}} = s_\theta(1,q,q^2,\ldots).$$

$\square$

7.5. **Lascoux–Pragacz identity for factorial Schur functions.** Lemma 7.1 holds for connected skew shape λ/μ in terms of non-intersecting paths $\Gamma=(\gamma_1,\ldots,\gamma_k)$ in $\mathcal{NI}(\lambda/\mu)$ (i.e. complements of excited diagrams).

$$F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y}) := \sum_{\Gamma\in\mathcal{NI}(\lambda/\mu)} \prod_{(r,s)\in\Gamma} \frac{1}{x_r-y_s} = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(r,s)\in[\lambda]\setminus D} \frac{1}{x_r-y_s}.$$

Ikeda and Naruse [IN] showed algebraically the following identity that we call the *multivariate NHLF.*

Theorem 7.6 ([IN])**.** *For a connected skew shape $\lambda/\mu\subseteq d\times(n-d)$ we have*

$$\frac{s_\mu^{(d)}(\mathbf{x}\mid\mathbf{z}^\lambda)}{s_\lambda^{(d)}(\mathbf{x}\mid\mathbf{z}^\lambda)} = F_{\lambda/\mu}(\mathbf{x}\mid\mathbf{y}). \tag{7.10}$$

In Lemma 7.1 we proved combinatorially this result for border strips. We can use the approach from the previous subsections in reverse to obtain a Lascoux–Pragacz type identity for evaluations of factorial Schur functions.

Corollary 7.7. *If $(\theta_1,\ldots,\theta_k)$ is a Lascoux–Pragacz decpomposition of $\lambda/\mu \subset d\times(n-d)$, then*

$$s_\mu^{(d)}(\mathbf{x}\mid \mathbf{z}^\lambda)\cdot s_\lambda^{(d)}(\mathbf{x}\mid \mathbf{z}^\lambda)^{k-1} = \det\left[s^{(d)}_{\lambda\setminus\,\theta_i\#\theta_j}(\mathbf{x}\mid \mathbf{z}^\lambda)\right]_{i,j=1}^k \tag{7.11}$$

where $\lambda\setminus\theta_i\#\theta_j$ denotes the partition obtained by removing from λ the outer substrip $\theta_i\#\theta_j$.

Proof. By the weighted Lindström-Gessel-Viennot lemma (Lemma 3.8) with $y_{r,s}=1/(x_r-y_s)$, we rewrite the RHS of (7.10) as a determinant.

$$\frac{s_\mu^{(d)}(\mathbf{x}\mid \mathbf{z}^\lambda)}{s_\lambda^{(d)}(\mathbf{x}\mid \mathbf{z}^\lambda)} = \det\left[\sum_{\substack{\gamma:(a_j,b_j)\to(c_i,d_i),\\ \gamma\subseteq\lambda}}\ \prod_{(r,s)\in\gamma}\frac{1}{x_r-y_s}\right]_{i,j=1}^k = \det\left[F_{\theta_i\#\theta_j}(\mathbf{x}\mid\mathbf{y})\right]_{i,j=1}^k.$$

Finally, by Lemma 7.1 each entry of the matrix can be written as the quotient of $s^{(d)}_{\lambda\setminus\,\theta_i\#\theta_j}(\mathbf{x}\mid \mathbf{z}_\lambda)$ and $s_\lambda^{(d)}(\mathbf{x}\mid\mathbf{z}_\lambda)$. By factoring the denominators out of the matrix we obtain the result. □

Calculations suggest that an analogue of (7.11) holds for general factorial Schur functions $s_\mu^{(d)}(\mathbf{x}\mid\mathbf{y})$ and not just for the evaluation $\mathbf{y}=\mathbf{z}^\lambda$.

Conjecture 7.8. *If $(\theta_1,\ldots,\theta_k)$ is a Lascoux–Pragacz decpomposition of $\lambda/\mu \subset d\times(n-d)$, then*

$$s_\mu^{(d)}(\mathbf{x}\mid \mathbf{y})\cdot s_\lambda^{(d)}(\mathbf{x}\mid \mathbf{y})^{k-1} = \det\left[s^{(d)}_{\lambda\setminus\,\theta_i\#\theta_j}(\mathbf{x}\mid \mathbf{y})\right]_{i,j=1}^k \tag{7.12}$$

where $\lambda\setminus\theta_i\#\theta_j$ denotes the partition obtained by removing from λ the outer substrip $\theta_i\#\theta_j$.

Since factorial Schur functions reduce to Schur functions when $\mathbf{y}=\mathbf{0}$, this conjecture implies an identity of Schur functions.

Proposition 7.9. *Conjecture 7.8 implies the Schur function identity*

$$s_\mu(\mathbf{x})\cdot s_\lambda(\mathbf{x})^{k-1} = \det\left[s_{\lambda\setminus\,\theta_i\#\theta_j}(\mathbf{x})\right]_{i,j=1}^k,$$

where $(\theta_1,\ldots,\theta_k)$ is a Lascoux–Pragacz decpomposition of λ/μ.

Example 7.10. From the example in the right of Figure 2.4 we obtain the identity

$$s_{(2,1)}s_{(5,4^2,1)} = s_{(3^2)}s_{(5,3,2,1)} - s_{(3^2,2,1)}s_{(5,3)}.$$

Remark 7.11. Note that instead of reversing the approach in Section 5, having a combinatorial proof of the identity in Corollary 7.7 would show that the multivariate NHLF (Theorem 7.6) for skew shapes is equivalent to the multivariate NHLF for border srips (Lemma 7.1).

8. Excited diagrams and SSYT of border strips and thick strips

In the next two sections we focus on the case of the *thick strip* δ_{n+2k}/δ_n where δ_n denotes the staircase shape $(n-1,n-2,\ldots,2,1)$. We study the excited diagrams $\mathcal{E}(\delta_{n+2k}/\delta_n)$ using the results from Section 3.3 and the number of SYT of this shape combining the NHLF, its SSYT q-analogue (Theorem 1.3) and the Lascoux–Pragacz identity.

8.1. **Excited diagrams and Catalan numbers.** We start enumerating the excited diagrams of the shape δ_{n+2k}/δ_n.

Corollary 8.1. *We have:* $e(\delta_{n+2}/\delta_n) = C_n, \quad e(\delta_{n+4}/\delta_n) = C_n C_{n+2} - C_{n+1}^2,$

$$e(\delta_{n+2k}/\delta_n) \;=\; \det[C_{n-2+i+j}]_{i,j=1}^{k} \;=\; \prod_{1\le i<j\le n} \frac{2k+i+j-1}{i+j-1}\,. \tag{8.1}$$

Proof. We start with the case $k = 1$ for the zigzag border strip δ_{n+2}/δ_n. By Proposition 3.5 the complement of excited diagrams of δ_{n+2}/δ_n are paths $\gamma : (n+1, 1) \to (1, n+1)$, $\gamma \subseteq \delta_{n+2}$. By rotating these paths 45° clockwise one obtain the Dyck paths in $\mathsf{Dyck}(n)$ as illustrated in Figure 5. Thus $e(\delta_{n+2}/\delta_n) = C_n$.

For general k, the shape δ_{n+2k}/δ_n has a Lascoux–Pragacz decomposition into k maximal border strips $(\theta_1, \ldots, \theta_k)$ where θ_i is the zigzag strip from $(n+2k-2i-1, 1)$ to $(1, n+2k-2i-1)$ (see Figure 6: Left). Then by Theorem 3.3 we have

$$e(\delta_{n+2k}/\delta_n) = \det\big[\, e(\theta_i \# \theta_j)\,\big]_{i,j=1}^{k}.$$

The cutting strip τ of the decomposition of δ_{n+2k}/δ_n is the zigzag θ_1. The strips $\theta_i\#\theta_j$ in the determinant, being substrips of θ_1, are themselves zigzags. The strip $\theta_i\#\theta_j$ in θ_1 consists of the cells with content from $2+2j-n-2k$ to $n+2k-2i-2$. So the strip is a zigzag δ_{m+2}/δ_m of size $2m+1$ where $m = n+2k+i+j+2$. Since we already know shapes δ_{m+2}/δ_m have C_m excited diagrams then the above determinant becomes

$$e(\delta_{n+2k}/\delta_n) = \det\big[\, C_{n+2k-i-j-2}\,\big]_{i,j=1}^{k} = \det\big[\, C_{n+i+j-2}\,\big]_{i,j=1}^{k},$$

where the last equality is obtained by relabelling the matrix. This proves the first equality.

To prove the second equality we use the characterization of excited diagrams as flagged tableaux. By [MPP1, Prop. 3.6], excited diagrams in $\mathcal{E}(\delta_{n+2k}/\delta_n)$ are in bijection with flagged tableaux of shape δ_n with flag $(k+1, k+2, \ldots, k+n-1)$. By subtracting i to all entries in row i, these tableaux are equivalent to reverse plane partitions of shape δ_n with entries $\le k$ which are counted by the given product formula due to Proctor (unpublished research announcement 1984; see [FK]). ☐

Remark 8.2. Note that by [MPP1, Prop. 3.6], excited diagrams in $\mathcal{E}(\delta_{n+2k+1}/\delta_n)$ are in correspondence with flagged tableaux of shape δ_n with flag $(k+1, k+2, \ldots, k+n-1)$, thus $|\mathcal{E}(\delta_{n+2k}/\delta_n)| = |\mathcal{E}(\delta_{n+2k+1}/\delta_n)|$. In what follows the formulas for the even case δ_{n+2k} are simpler than those of the odd case so we omit the latter.

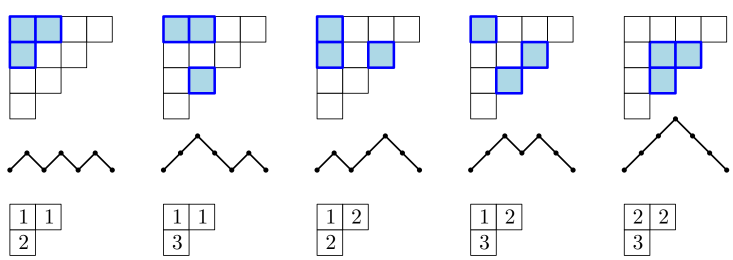


FIGURE 5. Correspondence between excited diagrams in δ_5/δ_3, Dyck paths in $\mathsf{Dyck}(3)$ and flagged tableaux of shape δ_3 with flag $(2, 3)$.

From the first determinantal formula for $e(\lambda/\mu)$ (Proposition 3.1) we easily obtain the following curious determinantal identity (see also §10.4).

Corollary 8.3. *We have:*

$$\det\left[\binom{n-i+j}{i}\right]_{i,j=1}^{n-1} \;=\; C_n\,.$$

Proof. By Corollary 8.1, we have $|\mathcal{E}(\delta_{n+2}/\delta_n)| = C_n$. We apply Proposition 3.1 to the shape δ_{n+2}/δ_n, where the vector $\mathbf{f}^{\delta_{n+2}/\delta_n} = (2, 3, \ldots, n)$, see §3.2. This expresses $|\mathcal{E}(\delta_{n+2}/\delta_n)|$ as the given determinant, and the identity follows. $\square$

Next we give a description of the excited diagrams of the shape δ_{n+2k}/δ_n. Let $\mathsf{FanDyck}(k, n)$ be the set of tuples $(p_1, \ldots, p_k)$ of k noncrossing Dyck paths from $(0, 0)$ to $(2n, 0)$ (see Figure 6: Right). We call such tuples *k-fans of Dyck paths*. It is known [SV] that fans of Dyck pahts are counted by the determinant of Catalan numbers and the product formula in (8.1).

Corollary 8.4. *We have $e(\delta_{n+2k}/\delta_n) = |\mathsf{FanDyck}(k, n)|$ and the complements of the excited diagrams correspond to k-fans of paths in $\mathsf{FanDyck}(k, n)$.*

Proof. By Proposition 3.5 the complements of excited diagrams in $\mathcal{E}(\delta_{n+2k}/\delta_n)$ correspond to k-tuples of nonintersecting paths in $\mathcal{NI}(\delta_{n+2k}/\delta_n)$ (paths obtained via ladder moves from the original paths $(\gamma_1^*, \ldots, \gamma_k^*)$ of the Kreiman outer decomposition of δ_{n+2k}/δ_n).

The path γ_i^* consists of zigzag path p_i^* of $2n+1$ cells bookended by a vertical and horizontal segment of $k - i$ cells each (see Figure 6:Middle). Because the excited/ladder moves preserve the contents of the cells of δ_n, the path γ_i in $(\gamma_1, \ldots, \gamma_k) \in \mathcal{NI}(\delta_{n+2k}/\delta_n)$ will consist of a Dyck path p_i bookended by the same vertical and horizontal segments as in γ_i^*. Thus the map $(\gamma_1, \ldots, \gamma_k) \mapsto (p_1, \ldots, p_k)$ denoted by φ is a correspondence between $\mathcal{NI}(\delta_{n+2k}/\delta_n)$ and $\mathsf{FanDyck}(k, n)$. See Figure 6, right, for an example. $\square$

Remark 8.5. Fans of Dyck paths in $\mathsf{FanDyck}(k, n)$ are equinumerous with *k-triangulations* of an $(n + 2k)$-gon [Jon] (see also [S5, A12] and [SS] for a bijection for general k).

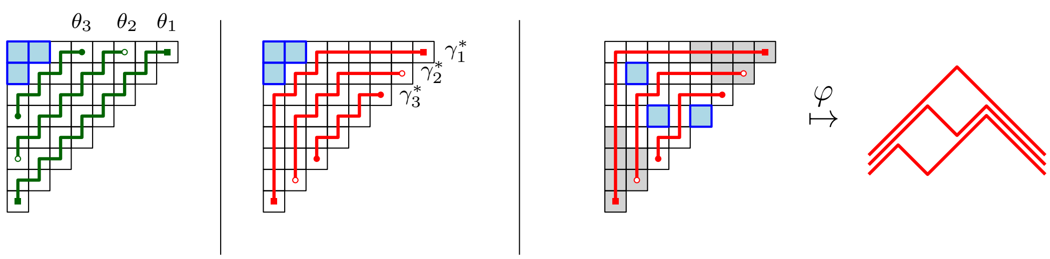


FIGURE 6. Left, Middle: the Lascoux–Pragacz and the Kreiman outer decompositions of the shape δ_{3+6}/δ_3. Right: the hook-lengths of an excited diagram of δ_{3+6}/δ_3 corresponding to the 3-fan of Dyck paths on the right. Each gray area has cells with product of hook-lengths $(3!! \cdot 7!!)$.

8.2. **Determinantal identity of Schur functions of thick strips.** Observe that SYT of shape δ_{n+2}/δ_n are in bijection with *alternating permutations* of size $2n + 1$. These permutations are counted by the odd Euler number E_{2n+1}. Thus,

$$f^{\delta_{n+2}/\delta_n} = E_{2n+1}\,.$$

Let $E_n(q)$ be as in the introduction, the q-analogue of Euler numbers.[1]

Example 8.6. We have: $E_1(q) = E_2(q) = 1$, $E_3(q) = q^2 + q$, $E_4(q) = q^4 + q^3 + 2q^2 + q$, and $E_5(q) = q^8 + 2q^7 + 3q^6 + 4q^5 + 3q^4 + 2q^3 + q^2$.

[1] In the survey [S3, §2], our $E_n(q)$ is denoted by $E_n^\star(q)$.

By the theory of (P,ω)-partitions, we have:

$$E_{2n+1}(q) \;=\; s_{\delta_{n+2}/\delta_n}(1,q,q^2,\ldots)\cdot \prod_{i=1}^{2n+1}(1-q^i)\,. \tag{8.2}$$

Next we apply the Lascoux–Pragacz identity to the shape δ_{n+2k}/δ_n.

Corollary 8.7 (Lascoux–Pragacz for δ_{n+2k}/δ_n)**.** *We have:*

$$s_{\delta_{n+2k}/\delta_n}(\mathbf{x}) \;=\; \det\left[s_{\delta_{n+i+j}/\delta_{n-2+i+j}}(\mathbf{x})\right]_{i,j=1}^{k}\,.$$

Proof. By Theorem 2.1 for the shape δ_{n+2k}/δ_n we have

$$s_{\delta_{n+2k}/\delta_n}(\mathbf{x}) = \det\left[\, s_{\theta_i\#\theta_j}(\mathbf{x})\,\right]_{i,j=1}^{k},$$

where $(\theta_1,\ldots,\theta_k)$ is the decomposition of the shape δ_{n+2k}/δ_n into k maximal border strips. As in the proof of Corollary 8.1, the strip $\theta_i\#\theta_j$ has shape δ_{m+2}/δ_m for $m = n+2k-i-j+2$. Thus, after relabelling the matrix, the above equation becomes the desired expression. □

Corollary 8.8. *We have:*

$$s_{\delta_{n+2k}/\delta_n}(1,q,q^2,\ldots) \;=\; \det\left[\widetilde{E}_{2(n+i+j)-3}(q)\right]_{i,j=1}^{k}\,,$$

where

$$\widetilde{E}_n(q) \;:=\; \frac{E_n(q)}{(1-q)(1-q^2)\cdots(1-q^n)}\,.$$

Proof. The result follows from Corollary 8.7 and equation (8.2). □

Taking the limit $q\to 1$ in Corollary 8.8 we get corresponding identities for $f^{\delta_{n+2k}/\delta_n}$.

Corollary 8.9. *We have:*

$$\frac{f^{\delta_{n+2k}/\delta_n}}{|\delta_{n+2k}/\delta_n|!} \;=\; \det\left[\widehat{E}_{2(n+i+j)-3}\right]_{i,j=1}^{k}\,,\quad \textit{where}\;\; \widehat{E}_n \;:=\; \frac{E_n}{n!}\,.$$

Remark 8.10. Baryshnikov and Romik [BR] gave similar determinantal formulas for the number of standard Young tableaux of skew shape $(n+m-1,n+m-2,\ldots,m)/(n-1,n-2,\ldots,1)$, extending the method of Elkies (see e.g. [AR, Ch. 14]).

In a different direction, one can use Corollary 8.9 when $n=1,2$ to obtain the following determinant formulas for Euler numbers in terms of for $f^{\delta_{2k+1}}$ and $f^{\delta_{2k}}$, which of course can be computed by a HLF (cf. [OEIS, A005118]).

Corollary 8.11. *We have:*

$$\det\left[\widehat{E}_{2(i+j)-1}\right]_{i,j=1}^{k} \;=\; \frac{f^{\delta_{2k+1}}}{\binom{2k+1}{2}!}\,,\qquad \det\left[\widehat{E}_{2(i+j)+1}\right]_{i,j=1}^{k} \;=\; \frac{f^{\delta_{2k}}}{\left(\binom{2k}{2}-1\right)!}\,.$$

8.3. **SYT and Euler numbers.** We use the NHLF to obtain an expression for $f^{\delta_{n+2}/\delta_n} = E_{2n+1}$ in terms of Dyck paths.

Proof of Corollary 1.6. By the NHLF, we have

$$f^{\delta_{n+2}/\delta_n} \;=\; |\delta_{n+2}/\delta_n|!\sum_{D\in\mathcal{E}(\delta_{n+2}/\delta_n)}\;\prod_{u\in\overline{D}}\frac{1}{h(u)}\,, \tag{8.3}$$

where $\overline{D} = [\delta_{n+2}/\delta_n]\setminus D$. Now $|\delta_{n+2}/\delta_n| = (2n+1)!$ and by Corollary 8.1 (complements of) excited diagrams D of δ_{n+2}/δ_n correspond to Dyck paths γ in $\mathsf{Dyck}(n)$. In this correspondence, if $u\in\overline{D}$ corresponds to point (a,b) in γ then $h(u)=2b+1$ (see Figure 5). Translating from excited diagrams to Dyck paths, (8.3) becomes the desired Equation (EC). □

Equation (EC) can be generalized to thick strips δ_{n+2k}/δ_n.

Corollary 8.12. *We have:*

$$\sum_{\substack{(\mathsf{p}_1,\ldots,\mathsf{p}_k)\in\mathsf{Dyck}(n)^k\\ noncrossing}} \prod_{r=1}^{k} \prod_{(a,b)\in\mathsf{p}_r} \frac{1}{2b+4r-3} \;=\; \left[\prod_{r=1}^{k-1}(4r-1)!!\right]^2 \det\left[\widehat{E}_{2(n+i+j)-3}\right]_{i,j=1}^{k}, \tag{8.4}$$

where $\widehat{E}_n = E_n/n!$ and $(a,b)\in\mathsf{p}$ denotes a point of the Dyck path p.

Proof. For the RHS we use Corollary 8.9 to express $f^{\delta_{n+2k}/\delta_n}$ in terms of Euler numbers. For the LHS, we first use the NHLF to write $f^{\delta_{n+2k}/\delta_n}$ as a sum over excited diagrams $\mathcal{E}(\delta_{n+2k}/\delta_n)$:

$$f^{\delta_{n+2k}/\delta_n} \;=\; |\delta_{n+2k}/\delta_n|! \sum_{D\in\mathcal{E}(\delta_{n+2k}/\delta_n)} \prod_{u\in\overline{D}} \frac{1}{h(u)},$$

where $\overline{D} = [\delta_{n+2k}/\delta_n]\setminus D$. By Corollary 8.4, excited diagrams of δ_{n+2k}/δ_n correspond to k-tuples of noncrossing Dyck paths in $\mathsf{FanDyck}(k,n)$ via the map φ. Finally, one can check (see Figure 6 right) that if $\varphi: D\mapsto(\mathsf{p}_1,\ldots,\mathsf{p}_k)$ then

$$\prod_{u\in\overline{D}} h(u) \;=\; \left[\prod_{r=1}^{k-1}(4r-1)!!\right]^2 \prod_{(a,b)\in\mathsf{p}_r} (2b+4r-3)\,,$$

which gives the desired RHS. □

8.4. **Probabilistic variant of** (EC)**.** Here we present a new identity (8.6) which a close relative of the curious identity (EC) we proved above.

Let $\mathcal{BT}(n)$ be the set of *plane full binary trees* τ with $2n+1$ vertices, i.e. plane binary trees where every vertex is a leaf or has two descendants. These trees are counted by $|\mathcal{BT}(n)| = C_n$ (see e.g. [S5, §2]). Given a vertex v in a tree $\tau\in\mathcal{BT}(n)$, $h(v)$ denotes the number of descendants of v (including itself). An *increasing* labelling of τ is a labelling $\omega(\cdot)$ of the vertices of τ with $\{1,2,\ldots,2n+1\}$ such that if u is a descendant of v then $\omega(v)\le\omega(u)$. By abuse of notation, let f^τ is the number of increasing labelings of τ. By the HLF for trees (see e.g. [Sag3]), we have:

$$f^\tau \;=\; \frac{(2n+1)!}{\prod_{v\in\tau} h(v)}\,. \tag{8.5}$$

Proposition 8.13. *We have:*

$$\sum_{\tau\in\mathcal{BT}(n)} \prod_{v\in\tau} \frac{1}{h(v)} \;=\; \frac{E_{2n+1}}{(2n+1)!}\,. \tag{8.6}$$

Proof. The RHS of (8.6) gives the probability $E_{2n+1}/(2n+1)!$ that a permutation $w\in S_{2n+1}$ is alternating. We use the representation of a permutation w as an *increasing binary tree* $T(w)$ with $2n+1$ vertices (see e.g. [S4, §1.5]). It is well-known that w is an *down-up* permutation (equinumerous with up-down/alternating permutations) if and only if $T(w)$ is an increasing full binary tree [S4, Prop. 1.5.3]. See Figure 7 for an example. We conclude that the probability p that an increasing binary tree is a full binary tree is given by $p = E_{2n+1}/(2n+1)!$.

On the other hand, we have:

$$p \;=\; \sum_{\tau\in\mathcal{BT}(n)} \frac{f^\tau}{(2n+1)!}\,,$$

where $f^\tau/(2n+1)!$ is the probability that a labelling of a full binary tree τ is increasing. By (8.5), the result follows. □

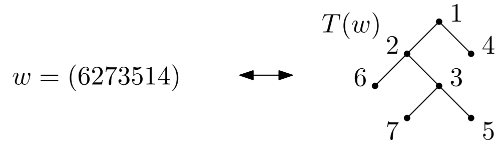


FIGURE 7. The full binary tree corresponding to the alternating permutation $w = (6273514)$.

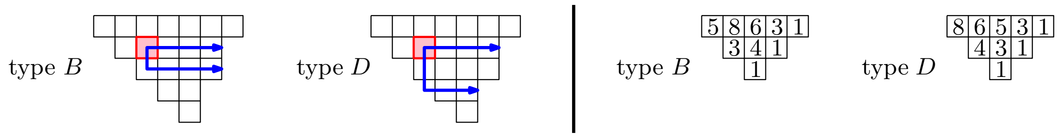


FIGURE 8. Left: examples of the type B and type D hook of a cell (i, j) of λ of lengths 9 (cell $(3, 3)$ is counted twice) and 7 respectively. Right: the type B and D hook-lengths of the cells of the shifted shape $(5, 3, 1)$.

Remark 8.14. Note the similarities between (8.6) and (EC). They have the same RHS, both are sums over the same number C_n of Catalan objects of products of n terms, and both are variations on the (usual) (HLF) for other posets. As the next example shows, these equations are quite different.

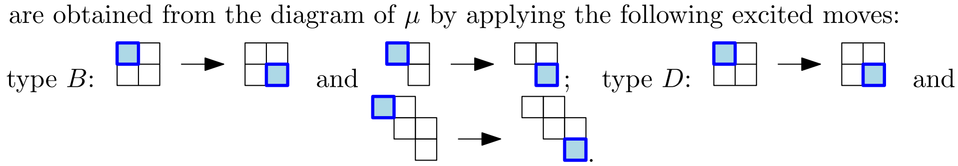
Example 8.15. For $n = 2$ there are $C_2 = 2$ full binary trees with 5 vertices and $E_5 = 16$. By Equation (8.6)

$$\frac{1}{3 \cdot 5} + \frac{1}{3 \cdot 5} = \frac{16}{5!}.$$

On the other hand, for the two Dyck paths in $\mathsf{Dyck}(2)$, Equation (EC) gives

$$\frac{1}{3 \cdot 3} + \frac{1}{3 \cdot 3 \cdot 5} = \frac{16}{5!}.$$

8.5. **Formula** (EC) **for other types.** In this section λ and μ are partitions with distinct parts. We consider shifted diagrams of shape λ and skew shape λ/μ and standard tableaux of shifted shape λ/μ. Along with Theorem 1.2, Naruse also announced two formulas for the number $g^{\lambda/\mu}$ of standard tableaux of skew shifted shape λ/μ, in terms of type B and type D excited diagrams. These excited diagrams are obtained from the diagram of μ by applying the following excited moves:



We denote the set of type B (type D) excited diagrams of shifted skew shape λ/μ by $\mathcal{E}^B(\lambda/\mu)$ ($\mathcal{E}^D(\lambda/\mu)$). As in Section 3.2 or [MPP1, §3], type B excited diagrams of λ/μ are equivalent to certain flagged tableaux of shifted shape μ and to certain non-intersecting paths (see Figure 9).

Given a shifted shape λ, the type B hook of a cell (i, i) in the diagonal is the cells in row i of λ. The hook of a cell (i, j) for $i \le j$ is the cells in row i right of (i, j), the cells in column j below (i, j), and if (j, j) is one these cells below then the hook also includes the cells in the jth row of λ (overall counting (j, j) twice). The type D hook is the usual shifted hook (e.g., see [Sag2, Ex. 3.21]) The hook-length of (i, j) is the size of the hook of (i, j) and is denoted by $h^B(i, j)$ ($h^D(i, j)$); see Figure 8.

The NHLF then extends verbatim.

Theorem 8.16 (Naruse [Naru])**.** *Let λ, μ be partitions with distinct parts, such that $\mu \subset \lambda$. We have*

$$g^{\lambda/\mu} = |\lambda/\mu|! \sum_{S \in \mathcal{E}^B(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus S} \frac{1}{h^B(i,j)}, \tag{8.7}$$

$$= |\lambda/\mu|! \sum_{S \in \mathcal{E}^D(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus S} \frac{1}{h^D(i,j)}, \tag{8.8}$$

where $h^B(i,j)$ and $h^D(i,j)$ are the shifted hook-lengths of type B and type D, respectively.

Example 8.17 (shifted thick zigzag strip)**.** The shifted analogue of the staircase is the *trapezoid* $\nabla_n = (2n-1, 2n-3, \ldots, 1)$. The analogue of the thick strip is the shifted skew shape ∇_{n+k}/∇_n. The number of type B excited diagrams of this shape has a product formula analogous to (8.1).

Proposition 8.18.

$$|\mathcal{E}^B(\nabla_{n+k}/\nabla_n)| = \prod_{h=1}^{k} \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{h+i+j-1}{h+i+j-2}. \tag{8.9}$$

Proof. As in the standard shape case, the type B excited diagrams correspond to shifted flagged tableaux of trapezoid shape ∇_n with entries in row $i \leq i+k$. By subtracting i to all entries in row i of such tableaux they are equivalent to plane partitions of trapezoid shape ∇_n with entries $\leq k$. By a result of Proctor [Pro], recently proved bijectively in [HPPW], these are equinumerous with plane partitions in a $n \times n \times k$ box (see also [HW]). Thus, by MacMahon's boxed plane partition formula the result follows. $\square$

In the case $k = 1$ we obtain $|\mathcal{E}^B(\nabla_{n+1}/\nabla_n)| = \binom{2n}{n}$ (see Figure 9). When $k = n$, $|\mathcal{E}^B(\nabla_{2n}/\nabla_n)|$ counts the number of plane partitions that fit inside the $n \times n \times n$ box (see e.g. [OEIS, A08793]).

The shape ∇_{n+1}/∇_n is a zigzag and so $g^{\nabla_{n+1}/\nabla_n} = E_{2n+1}$ (see Figure 9). Thus, as a corollary of (8.7), we obtain a type B variant of the Euler-Catalan identity (EC). Let $\mathsf{Dyck}^B(n)$ be the set of lattice paths p starting at $(0,0)$ with steps $(1,1)$ and $(1,-1)$ of length $2n$ that stay on or above the x-axis. Note that $|\mathsf{Dyck}^B(n)| = \binom{2n}{n}$, sometimes called the type B Catalan number [OEIS, A000984].

Corollary 8.19.

$$\sum_{\mathsf{p} \in \mathsf{Dyck}^B(n)} \prod_{(a,b) \in \mathsf{p}} \frac{1}{wt(a,b)} = \frac{E_{2n+1}}{(2n+1)!}, \quad \textit{where} \quad wt(a,b) = \begin{cases} 2b+1 & \textit{if } a \leq n, \\ 2b+2 & \textit{if } n < a < 2n, \\ b+1 & \textit{if } a = 2n. \end{cases} \tag{EC-B}$$

Example 8.20. Figure 9 shows the $\binom{4}{2}$ excited diagrams of shape ∇_3/∇_2. By taking their complements and reflecting vertically, we obtain the paths in $\mathsf{Dyck}^B(2)$. Either using $wt(a,b)$ on the paths or the hook-lengths for the shape ∇_3/∇_2 (see Figure 8 right), (EC-B) gives

$$\frac{1}{4 \cdot 3 \cdot 1^3} + \frac{1}{6 \cdot 4 \cdot 3 \cdot 1^2} + \frac{1}{4 \cdot 3^2 \cdot 1^2} + \frac{1}{6 \cdot 4 \cdot 3^2 \cdot 1} + \frac{1}{8 \cdot 6 \cdot 3^2 \cdot 1} + \frac{1}{8 \cdot 6 \cdot 5 \cdot 3 \cdot 1} = \frac{16}{5!}.$$

Remark 8.21. The complements of type D excited diagrams of the shape ∇_{n+1}/∇_n are just the Dyck paths in $\mathsf{Dyck}(n)$, thus $|\mathcal{E}^D(\nabla_{n+1}/\nabla_n)| = C_n$. In addition, one can see that (8.8) for ∇_{n+1}/∇_n is just (EC). It would be of interest to find a formula for $|\mathcal{E}^D(\nabla_{n+k}/\nabla_n)|$ similar to (8.9).

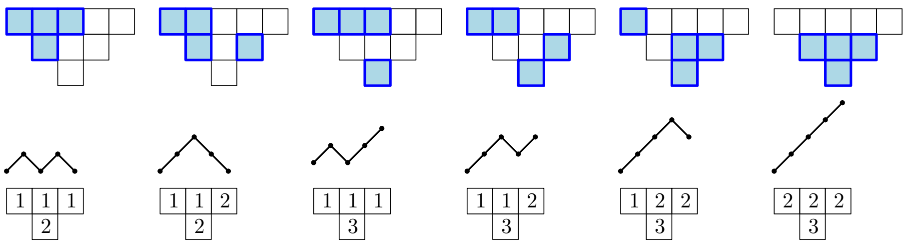


FIGURE 9. Correspondence between type B excited diagrams in ∇_3/∇_2, paths in $\mathsf{Dyck}^B(2)$ and flagged tableaux of shape ∇_2 with flag $(2,3)$.

8.6. **q-analogue of Euler numbers via SSYT.** We use our first q-analogue of NHLF (Theorem 1.3) to obtain identities for $s_{\delta_{n+2k}/\delta_n}(1,q,q^2,\ldots)$ in terms of Dyck paths.

Proof of Corollary 1.7. By Theorem 1.3 for the skew shape δ_{n+2}/δ_n and (8.2) we have

$$\frac{E_{2n+1}(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})} = \sum_{D\in\mathcal{E}(\delta_{n+2}/\delta_n)} \prod_{(i,j)\in[\delta_{n+2}]\setminus D} \frac{q^{\lambda'_j-i}}{1-q^{h(i,j)}}. \tag{8.10}$$

Let D in $\mathcal{E}(\delta_{n+2}/\delta_n)$ corresponds to the Dyck path p and cell (i,j) in D corresponds to point (a,b) in p then $h(i,j)=2b+1$ and $\lambda'_j-i=b$. Using this correspondence, the LHS of (8.10) becomes the LHS of the desired expression. $\square$

Corollary 8.22.

$$\sum_{\substack{(\mathsf{p}_1,\ldots,\mathsf{p}_k)\in\mathsf{Dyck}(n)^k\\ \textit{noncrossing}}} \prod_{r=1}^{k}\prod_{(a,b)\in\mathsf{p}_r} \frac{q^{b+2r-2}}{1-q^{2b+4r-3}} = \left(\prod_{r=1}^{k-1}[4r-1]!!\right)^2 \det\left[\widetilde{E}_{2(n+i+j)-3}(q)\right]_{i,j=1}^{k}$$

where $\widetilde{E}_n(q):=E_n(q)/(1-q)(1-q^2)\cdots(1-q^n)$ *and* $[2m-1]!!:=(1-q)(1-q^3)\cdots(1-q^{2m-1})$.

Proof. For the LHS, use Corollary 8.8 to express $s_{\delta_{n+2k}/\delta_n}(1,q,q^2,\ldots)$ in terms of q-Euler polynomials $\widetilde{E}_m(q)$. For the RHS, first use Theorem 1.3 for the skew shape δ_{n+2k}/δ_n and then follow the same argument as that of Corollary 8.12. $\square$

Remark 8.23. Combinatorial proofs of Corollary 1.6 and Corollary 1.7 are obtained from the proof in Section 5 of (NHLF). Similarly, combinatorial proofs of Corollary 8.12 and Corollary 8.22 are obtained from the proof in Section 5 of (NHLF) and the first q-NHLF for all shapes, or from Konvalinka's bijective proof of (NHLF) in [Kon] and the first q-NHLF for border strips.[2]

9. PLEASANT DIAGRAMS AND RPP OF BORDER STRIPS AND THICK STRIPS

In this section we study pleasant diagrams in $\mathcal{P}(\delta_{n+2}/\delta_n)$ and our second q-analogue of NHLF (Theorem 1.4) for RPP of shape δ_{n+2}/δ_n. Recall that $p(\lambda/\mu)$ denotes the number of pleasant diagrams of shape λ/μ.

9.1. **Pleasant diagrams and Schröder numbers.** Let s_n be the n-th *Schröder number* [OEIS, A001003] which counts lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$, and $(2,0)$ that never go below the x-axis and no steps $(2,0)$ on the x-axis.

Theorem 9.1. *We have:* $p(\delta_{n+2}/\delta_n)=2^{n+2}s_n$, *for all* $n\geq 1$.

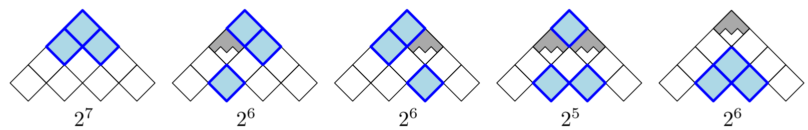


FIGURE 10. Each Dyck path p of size n with m excited peaks (denoted in gray) yields 2^{2n-m+2} pleasant diagrams. For $n=3$, we have $C_5 = 5$ and $s_3 = 11$. Thus, there are $|\mathcal{E}(\delta_{3+2}/\delta_3)| = C_3 = 5$ excited diagrams and $p(\delta_{3+2}/\delta_3) = 2^5 s_3 = 352$ pleasant diagrams.

The proof is based on the following corollary which is in turn a direct application of Theorem 4.3. A *high peak* of a Dyck path p is a peak of height strictly greater than one. We denote by $\mathcal{HP}(\mathsf{p})$ the set of high peaks of p, and by $\mathcal{NP}(\mathsf{p})$ the points of the path that are not high peaks. We use $2^{\mathcal{S}}$ denote the set of subsets of $\mathcal{S}$.

Corollary 9.2. *The pleasant diagrams in $\mathcal{P}(\delta_{n+2}/\delta_n)$ are in bijection with*

$$\bigcup_{\mathsf{p}\in \mathsf{Dyck}(n)} \left(\mathcal{HP}(\mathsf{p}) \times 2^{\mathcal{NP}(\mathsf{p})}\right).$$

Proof. By Corollary 8.4 for the zigzag strip δ_{n+2}/δ_n we have $\mathcal{NI}(\delta_{n+2}/\delta_n)$ is the set of Dyck paths $\mathsf{Dyck}(n)$. Then by Theorem 4.3, we have:

$$\mathcal{P}(\delta_{n+2}/\delta_n) = \bigcup_{\mathsf{p}\in \mathsf{Dyck}(n)} \left(\Lambda(\mathsf{p}) \times 2^{\mathsf{p}\setminus\Lambda(\mathsf{p})}\right).$$

Lastly, note that the excited peaks of a Dyck path are exactly the high peaks so $\Lambda(\mathsf{p}) = \mathcal{HP}(\mathsf{p})$ and $\mathsf{p}\setminus\Lambda(p) = \mathcal{NP}(\mathsf{p})$. ☐

Proof of Theorem 9.1. It is known (see [Deu]), that the number of Dyck paths of size n with $k-1$ high peaks equals the *Narayana number* $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. On the other hand, Schröder numbers s_n can be written as

$$s_n = \sum_{k=1}^{n} N(n,k)\, 2^{k-1} \tag{9.1}$$

(see e.g. [Sul]). By Lemma 9.2, we have:

$$p(\delta_{n+2}/\delta_n) = \sum_{\mathsf{p}\in \mathsf{Dyck}(n)} 2^{|\mathcal{NP}(\mathsf{p})|}. \tag{9.2}$$

Suppose Dyck path γ has $k-1$ peaks, $1 \le k \le n$. Then $|\mathcal{NP}(\gamma)| = 2n+1-(k-1)$. Therefore, equation (9.2) becomes

$$p(\delta_{n+2}/\delta_n) = 2^{n+2}\sum_{k=1}^{n} N(n,k)\, 2^{n-k} = 2^{n+2}\sum_{k=1}^{n} N(n,n-k+1)\, 2^{n-k} = 2^{n+2} s_n\,,$$

where we use the symmetry $N(n,k) = N(n,n-k+1)$ and (9.1). ☐

In the same way as $|\mathcal{E}(\delta_{n+2k}/\delta_n)|$ is given by a determinant of Catalan numbers, preliminary computations suggest that $p(\delta_{n+2k}/\delta_n)$ is given by a determinant of Schröder numbers.

Conjecture 9.3. *We have: $p(\delta_{n+4}/\delta_n) = 2^{2n+5}(s_n s_{n+2} - s_{n+1}^2)$. More generally, for all $k \ge 1$, we have:*

$$p(\delta_{n+2k}/\delta_n) = 2^{\binom{k}{2}} \det\left[\mathfrak{s}_{n-2+i+j}\right]_{i,j=1}^{k}, \quad \textit{where} \quad \mathfrak{s}_n = 2^{n+2} s_n\,.$$

[2] Jang Soo Kim has a direct proof of Corollary 1.7 using continued fractions and orthogonal polynomials (private communication).

Here we use $\mathfrak{s}_n = p(\delta_{n+2}/\delta_n)$ in place of s_n in the determinant to make the formula more elegant. In fact, the power of 2 can be factored out.

Remark 9.4. This conjecture is somewhat unexpected since different from excited diagrams, the number of pleasant diagrams does not appear to have a Lascoux–Pragacz-type identity (see Section 4.2).

9.2. **q-analogue of Euler numbers via RPP.** We use our second q-analogue of the NHLF (Theorem 1.4) and Lemma 9.2 to obtain identities for the generating function of RPP of shape δ_{n+2}/δ_n in terms of Dyck paths. Recall the definition of $E_n^*(q)$ from the introduction:

$$E_n^*(q) \;=\; \sum_{\sigma\in\mathsf{Alt}(n)} q^{\mathrm{maj}(\sigma^{-1}\kappa)}\,,$$

where $\kappa = (13254\ldots)$. Note that $\mathrm{maj}(\sigma\kappa)$ is the sum of the descents of $\sigma \in S_n$ not involving both $2i+1$ and $2i$.

Example 9.5. To complement Example 8.6, we have: $E_1^*(q) = E_2^*(q) = 1,\ E_3^*(q) \,=\, q+1,$ $E_4^*(q) \,=\, q^4+q^3+q^2+q+1,$ and $E_5^*(q) \,=\, q^7+2q^6+2q^5+3q^4+3q^3+2q^2+2q+1\,.$

Proof of Corollary 1.8. By the theory of P-partitions, the generating series of RPP of shape δ_{n+2}/δ_n equals

$$\sum_{\pi\in\mathrm{RPP}(\delta_{n+2}/\delta_n)} q^{|\pi|} \;=\; \frac{\sum_{u\in\mathcal{L}(P_{\delta_{n+2}/\delta_n})} q^{\mathrm{maj}(u)}}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,,$$

where the sum in the numerator is over linear extensions $\mathcal{L}(P_{\delta_{n+2}/\delta_n})$ of the *zigzag* poset $P_{\delta_{n+2}/\delta_n}$ with a natural labelling. These linear extensions are in bijection with alternating permutations of size $2n+1$ and

$$E_{2n+1}^*(q) \;=\; \sum_{\sigma\in\mathsf{Alt}_{2n+1}} q^{\mathrm{maj}(\sigma^{-1}\kappa)} \;=\; \sum_{u\in\mathcal{L}(P_{\delta_{n+2}/\delta_n})} q^{\mathrm{maj}(u)}\,.$$

Thus

$$\sum_{\pi\in\mathrm{RPP}(\delta_{n+2}/\delta_n)} q^{|\pi|} \;=\; \frac{E_{2n+1}^*(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,. \tag{9.3}$$

By Theorem 4.3 (see also [MPP1, S6.4]) for the skew shape δ_{n+2}/δ_n and (9.3), we have:

$$\sum_{D\in\mathcal{E}(\delta_{n+2}/\delta_n)} q^{a'(D)} \prod_{u\in[\lambda]\setminus D} \frac{1}{1-q^{h(u)}} \;=\; \frac{E_{2n+1}^*(q)}{(1-q)(1-q^2)\cdots(1-q^{2n+1})}\,, \tag{9.4}$$

where $a'(D) = \sum_{u\in\Lambda(D)} h(u)$. By the proof of Lemma 9.2, if $D \in \mathcal{E}(\delta_{n+2}/\delta_n)$ corresponds to the Dyck path p then excited peaks $u \in \Lambda(D)$ correspond to high peaks $(c,d) \in \mathcal{HP}(\mathsf{p})$ and $h(u) = 2d+1$. Using this correspondence, the LHS of (9.4) becomes the LHS of the desired expression. $\square$

Finally, preliminary computations suggest the following analogue of Corollary 8.8.

Conjecture 9.6. *We have:*

$$\sum_{\pi\in\mathrm{RPP}(\delta_{n+2k}/\delta_n)} q^{|\pi|} = q^{-N} \det\left[\widetilde{E}^*_{2(n+i+j)-3}(q)\right]_{i,j=1}^{k}\,,$$

where $N = k(k-1)(6n+8k-1)/6$ *and* $\widetilde{E}_k^*(q) = E_k^*(q)/(1-q)\cdots(1-q^k)$.

10. Final remarks

10.1. Other known formulas for $f^{\lambda/\mu}$ are the *Jacobi–Trudi identity*, the *Littlewood–Richardson rule*, and the Okounkov–Olshanski formula [OO]. We discuss these and other less known formulas for $f^{\lambda/\mu}$ coming from *equivariant Schubert structure constants* in [MPP1, §9].

The Jacobi-Trudi identity is one of the first nontrivial formulas to count $f^{\lambda/\mu}$. In this paper we have unveiled a strong relation between the Lascoux–Pragacz identity for Schur functions and the NHLF for $f^{\lambda/\mu}$. As mentioned in Section 7.5, Hamel and Goulden [HaG] unified these two identities into an exponential family of determinantal identities of Schur functions. Chen–Yan–Yang [CYY] gave a method to transform among these identities. It would be of interest if other formulas for $f^{\lambda/\mu}$, like the ones mentioned above, are related to special cases of Hamel–Goulden identities.

10.2. In [Kon], Konvalinka gives a new proof of the NHLF. Specifically, he presents a bumping algorithm on bicolored flagged tableaux of shape μ to prove the Pieri–Chevalley formula for general skew shapes (see [IN, §8.4] and [MPP2]). For the border strips this approach is different from our proof of Lemma 6.1. In fact, our proof uses the underlying single path in the excited diagrams of a border strip to perform cancelations. While Konvanlinka's proof is substraction-free, it involves an insertion on the inner partition μ that could be arbitrary even for a border strip.

It is worth noting that both proofs of Lemma 5.1 are quite technical. Initially, this came as a surprise to us, and our effort to understand the underlying multivariate algebraic identities led to [MPP2]. Let us also mention that in [IN], the authors use the Pieri–Chevalley formula for Kostant polynomials to prove the analogue of Lemma 6.1 for all skew shapes.

10.3. There is a very large literature on alternating permutations, Euler numbers, Dyck paths, Catalan and Schröder numbers, which are some of the classical combinatorial objects and sequences. We refer to [S3] for the survey on the first two, to [S5] for a thorough treatment of the last three, and to [GJ, OEIS, S4] for various generalizations, background and further references.

Finally, the first q-analogue $E_n(q)$ of Euler numbers we consider is standard in the literature and satisfies a number of natural properties, including a q-version of equation (1.1) (see e.g. [GJ, §4.2]). However, the second q-analogue $E_n^*(q)$ appears to be new. It would be interesting to see how it fits with the existing literature of multivariate Euler polynomials and statistics on alternating permutations.

10.4. The curious Catalan determinant in Corollary 8.3 appeared in the first `arXiv` version of [MPP1]. However, the proof here is more self contained as a direct application of Theorem 3.3. This Catalan determinant is both similar and related[3] to another Catalan determinant in [AL, proof of Lemma 1.1]. In fact, both determinants are special cases of more general counting results, and both can be proved by the Lindström–Gessel–Viennot lemma.

10.5. The connection between alternating permutations and symmetric functions of border strips goes back to Foulkes [Fou], and has been repeatedly generalized and explored ever since (see [S3]). It is perhaps surprising that Corollary 1.6 is so simple, since the other two positive formulas in Section 10.1 become quite involved. For the LR-coefficients, let partition $\nu \vdash 2n+1$ be such that $\nu_1, \ell(\nu) \leq n+1$. It is easy to see that in this case the corresponding LR-coefficient is nonzero: $c^{\delta_{n+2}}_{\delta_n\,\nu} > 0$, suggesting that summation over all such ν would can be hard to compute.

10.6. The proof in [MPP1] of the skew RPP q-analogue of Naruse (Theorem 1.4) is already bijective using the Hillman–Grassl correspondence. It would be interesting to see if for RPP the case for border strips implies the case for all connected skew shapes. Note that we do not know of a Lascoux–Pragacz analogue of (5.7) for skew RPP. Such an identity might not exists since the number of pleasant diagrams (the supports of arrays obtained from Hillman–Grassl applied to skew RPP) does not appear to have a Lascoux–Pragacz type identity as discussed in Section 4.2. On the other hand, Conjecture 9.3 suggests that there might be such a formula in some cases.

[3]The connection was found by T. Amdeberhan (personal communication).

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