CONCRETE POLYTOPES MAY NOT TILE THE SPACE

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Abstract. Brandolini et al. conjectured in [BCRT] that all *concrete lattice polytopes* can multitile the space. We disprove this conjecture in a strong form, by constructing an infinite family of counterexamples in $\mathbb{R}^3$.

1. Introduction

The study of *integer points in convex polytopes* is so challenging because it combines the analytic difficulty of number theory with hardness of imagination typical to high dimensional geometry and the computational complexity of integer programming. Consequently, whenever a new conjecture is posed it is a joyful occasion, as it suggests an order in an otherwise disordered universe. When a conjecture is occasionally disproved, this adds another layer of mystery to the subject.

In this paper we study the conjecture by Brandolini et al. [BCRT, Conj. 5] that all concrete lattice polytopes can multitile the space. This conjecture was restated and further investigated in [MR, Conj. 8.6] from a different point of view. Here we disprove the conjecture by constructing a series of explicit counterexamples. In fact, our main result is stronger as it holds under more general notion of tileability. Our tools involve McMullen's theory of *valuations of lattice polyhedra* and *Dehn's invariant*. We conclude with final remarks and open problems.

A convex polytope $P \subset \mathbb{R}^d$ is called a *lattice polytope* if all its vertices are in $\mathbb{Z}^d$. Denote by

$$\omega_P(x) := \frac{\operatorname{vol}\big(B_\varepsilon(x) \cap P\big)}{\operatorname{vol} B_\varepsilon(x)}$$

the *solid angle* at point x, where $B_\varepsilon(x)$ is a ball of radius ε centered at x, and $\varepsilon > 0$ sufficiently small.

Define the (*regularised*) *discrete volume* of P as the sum of solid angles over all integer points:

$$\chi(P) := \sum_{x \in P \cap \mathbb{Z}^d} \omega_P(x),$$

cf. [Bar, BR]. *Pick's theorem* says that $\chi(P) = \operatorname{vol}(P)$ for all lattice polygons $P \subset \mathbb{R}^2$. In an attempt to extend the theorem, Brandolini et al. [BCRT] call a lattice polytope *concrete* if $\chi(P) = \operatorname{vol}(P)$. They made the following curious conjecture.

Conjecture 1.1 ([BCRT, MR])**.** *Every concrete lattice polytope $P \subset \mathbb{R}^d$* multitiles *$\mathbb{R}^d$ with parallel translations and finitely many reflections.*

Here we say that P *multitiles* $\mathbb{R}^d$ if there is an integer $k \geq 1$ and an infinite family $\mathcal{P}$ of congruent copies of P such that every generic point $x \in \mathbb{R}^d$ belongs to exactly k polytopes in $\mathcal{P}$, see [GRS]. For $k = 1$ this is the usual tiling of the space, see e.g. [GS]. In the conjecture, only $P' \in \mathcal{P}$ are allowed if they are obtained from P with parallel translations and finitely many reflections. We disprove a stronger claim in the main result of this paper:

Theorem 1.2. *There is a concrete lattice polytope $P \subset \mathbb{R}^3$ which does not multitile $\mathbb{R}^3$. Moreover, for all N, there is such a polytope P with more than N vertices.*

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2. Brief background and the countexample idea

Let us first expound on the background and the motivation behind the conjecture. The problem of classifying polytopes which can tile (tesselate) the space is classical. It goes back to the works of Fëdorov, Minkowski, Voronoi, Delone and Alexandrov, and was featured in *Hilbert's 18-th Problem*, see [GS]. For tilings with parallel translations much more is known; notably that in $\mathbb{R}^3$ all such polytopes must be *zonotopes* (polytopes with centrally-symmetric faces of all dimensions). In higher dimensions, or for larger discrete groups of translations and reflections, other polytopes appear to tile the space, e.g. the *24-cell* in $\mathbb{R}^4$.

For the lattice polytopes, the tilings are also heavily constrained and can be studied using analytic tools [BCRT, GRS]. The notion of *multitiling* goes back to Furtwängler (1936), and many classical tiling results extend to this setting [GKRS]. It is known and easy to see that if a lattice polytope multitiles the space with parallel translations then it is concrete [BCRT] (see below). In particular, all lattice zonotopes multitile the space [GRS], and they are concrete because they can be partitioned into parallelepipeds, see e.g. [BP, §7] and [Zie, Ch. 7]. The conjecture can then be viewed as an attempt to say that the class of concrete lattice polytopes is very small and can be characterized via the large body of work towards characterization of tilings and multitilings.

From this point on, we restrict ourselves to convex polytopes $P \subset \mathbb{R}^3$. For the clarity, observe that $\omega_P(x) = 1$ for x in relative interior of P, and $\omega_P(x) = \frac{1}{2}$ for x in relative interior of a face. Similarly, $\omega_P(x) = \alpha(e)$ for x in relative interior of an edge e of P, where $\alpha(e)$ is the dihedral angle at e, and $\omega_P(x)$ is the usual solid angle for a vertex x of P.

There is a way to understand both the conjecture and our theorem as part of the same asymptotic argument. For a polytope $P \subset \mathbb{R}^3$, define the (*lattice*) *volume defect* by

$$\delta(P) \, := \, \chi(P) \, - \, \mathrm{vol}(P) \, \in \mathbb{R}.$$

Similarly, the *Dehn invariant* is given by

$$\mathbb{D}(P) \, := \sum_{e \in E(P)} \ell(e) \otimes \alpha(e) \, \in \, \mathbb{R} \otimes_{\mathbb{Z}} \big(\mathbb{R}/\pi\mathbb{Z}\big),$$

where $E(P)$ is the set of edges in P, $\ell(e)$ is the length of edge e, and $\alpha(e)$ is the dihedral angle at e, see e.g. [Bol, Dup].

Theorem 2.1. *Let $P \subset \mathbb{R}^3$ be a convex polytope which multitiles the space. Then $\mathbb{D}(P) = 0$. Similarly, let P be a lattice convex polytope which multitiles the space with parallel translations. Then $\delta(P) = 0$, i.e. P is concrete.*

Versions of this result and its various generalizations have been repeatedly rediscovered, often with the same asymptotic argument which goes back to Debrunner (1980) and Mürner (1975). We refer to [LM] for generalizations to higher dimensions and further references (see also §5.3).

Proof outline. First, suppose the multitiling is the usual tiling. Let $\mathcal{P}$ be the set of copies of P which define the usual tiling of $\mathbb{R}^3$, and let $\mathcal{P}_R \subset \mathcal{P}$ be the set of copies of P which intersect a ball $B_R(O)$ of radius R around the origin. Denote by $\Gamma \subset \mathbb{R}^3$ the region covered with tiles in $\mathcal{P}_R$. On the one hand, both the volume defect and the Dehn invariant are additive, so $\delta(\Gamma) = |\mathcal{P}_R| \, \delta(P) = \Theta(R^3) \, \delta(P)$, and $\mathbb{D}(\Gamma) = |\mathcal{P}_R| \, \mathbb{D}(P) = \Theta(R^3) \, \mathbb{D}(P)$. On the other hand, both the volume defect and the Dehn invariant depend only on the boundary $\partial\Gamma$, which is within a constant distance from $\partial B_R(O)$. Thus both grow at most quadratically: $\delta(\Gamma), \mathbb{D}(P) = O(R^2)$. For the defect this is clear, and for the Dehn invariant this can be made precise by using Kagan functions f (see below), which extend to ring homomorphisms $\mathbb{R} \otimes_{\mathbb{Z}} \big(\mathbb{R}/\pi\mathbb{Z}\big) \to \mathbb{R}$, so the resulting function of R can then be viewed analytically. Comparing the lower and upper asymptotic bounds, this implies the result. For general k-tilings, the same argument work verbatim, as the changes are straightforward.[1] $\square$

[1]We are implicitly using the fact that $\delta(P) = \delta(P')$ for all copies of $P' \in \mathcal{P}$. This is not true when reflections are allowed, see §5.5.

Now, the idea of a counterexample to Conjecture 1.1 is very clear. We use the lattice valuation theory to construct a lattice polytope $P \subset \mathbb{R}^3$ whose volume defect $\delta(P) = 0$, while the Dehn invariant $\mathbb{D}(P) \neq 0$. By Theorem 2.1, this implies that P cannot multitile the space.

3. Minkowski additivity

3.1. **Volume defect.** By definition, the volume defect $\delta(P)$ is a translation invariant valuation on lattice polytopes, i.e. $\delta(P+x) = \delta(P)$ for all $x \in \mathbb{Z}^3$, and

$$\delta(P \cup Q) + \delta(P \cap Q) \,=\, \delta(P) + \delta(Q),$$

for all lattice polytopes $P, Q \subset \mathbb{R}^3$. In particular, the volume defect is additive under disjoint union (except at the boundary). We also need the following linearity property under *Minkowski addition* $P + Q = \{x + y \,|\, x \in P, y \in Q\}$ and *expansion* $cP = \{cx \,|\, x \in P\}$.

Lemma 3.1 (see §5.2)**.** *Let $P_1, \ldots, P_k$ be lattice convex polytopes in $\mathbb{R}^3$, and let $t_1, \ldots, t_k \in \mathbb{N}$. Then*

$$\delta\big(t_1 P_1 \,+\, \ldots \,+\, t_k P_k\big) \,=\, t_1\, \delta(P_1) \,+\, \ldots \,+\, t_k\, \delta(P_k)\,.$$

Proof outline. Let $P \subset \mathbb{R}^3$ be a lattice polytope, and let $t \in \mathbb{N}$ be a variable. Both $\chi(tP)$ and $\mathrm{vol}(tP)$ are cubic polynomials, see [M2, p. 127] (see also [BL1, Thm 7.9] and [Joc, Thm 2.1] for surveys and further references). Moreover, $\chi(tP)$ and $\mathrm{vol}(tP)$ are *odd* cubic polynomials, see [Mac, Thm 4.8] (see also [M1]), with the same leading coefficient. Thus, $\delta(P) = \chi(P) - \mathrm{vol}(P)$ is linear. The polynomiality under Minkowski addition follows from McMullen's *homogeneous decomposition* [M1] (see also e.g. [Joc, Thm 4.1]). Again, the cubic terms cancel, and the same argument proves multilinearity as in the lemma. □

3.2. **Dehn invariant.** For the clarity of exposition, we follow [Pak, §17] (see also [Bol, Dup]). Fix an *additive function* $f : \mathbb{R} \longrightarrow \mathbb{R}$, s.t. $f(a + b) = f(a) + f(b)$ for all $a, b \in \mathbb{R}$. Additive function f s.t. $f(\pi) = 0$ is called a *Kagan function*, after [Kag].

For a convex polytope $P \subset \mathbb{R}^3$ and a Kagan function f, denote

$$D_f(P) \,:=\, \sum_{e \in E(P)} \ell(e)\, f(\alpha_e)\,.$$

Observe that D_f is a translation invariant valuation, and that $D_f(cP) = cD_f(P)$.

Lemma 3.2 (see §5.2)**.** *Let $P_1, \ldots, P_k$ be convex polytopes in $\mathbb{R}^3$, and let $t_1, \ldots, t_k \in \mathbb{R}_+$. Then:*

$$D_f\big(t_1 P_1 \,+\, \ldots \,+\, t_k P_k\big) \,=\, t_1\, D_f(P_1) \,+\, \ldots \,+\, t_k\, D_f(P_k)\,,$$

for every Kagan function f.

Proof outline. For $k = 2$, the result follows immediately from the homogeneous decomposition again and the additivity of the Dehn invariant under disjoint union. For larger k, proceed by induction.

Alternatively, recall that the Dehn invariant is a simple, translation-invariant valuation (see §5.2), so $D_f\big(t_1 P_1 \,+\, \ldots \,+\, t_k P_k\big)$ is a polynomial in $t_1, \ldots, t_k$ of degree at most 3. Now, the restriction of that polynomial onto any ray from the origin is a linear function, which implies that this polynomial is linear. The details of both arguments are straightforward. □

4. Counterexample construction

Consider the following three tetrahedra:

$$\begin{aligned} T_1 &:= \operatorname{conv}\{(0,0,0),\,(1,1,0),\,(1,0,1),\,(0,1,1)\},\\ T_2 &:= \operatorname{conv}\{(0,0,0),\,(2,2,-1),\,(2,-1,2),\,(1,-2,-2)\},\\ T_3 &:= \operatorname{conv}\{(0,0,0),\,(2,2,-1),\,(3,0,-3),\,(5,-1,-1)\}.\end{aligned}$$

In notation of [Pak, §16], tetrahedron T_1 is *regular* with edge length $\sqrt{2}$, tetrahedron T_2 is *standard* with three pairwise orthogonal edges of length 3, and T_3 is an *orthoscheme* (also called *path simplex* and *Hill tetrahedron*), with three edge lengths 3.

Note that six copies of T_3 tile a cube spanned by vectors $v_1 = (2,2,-1)$, $v_2 = (1,-2,-2)$, and $v_3 = (2,-1,2)$ starting at the origin O. Indeed, these six copies correspond to six permutations of $\{v_1, v_2, v_3\}$, and are spanned by the paths formed by these vectors. This implies that $D_f(T_3) = 0$ for every Kagan function f defined above.

Proposition 4.1. *Let $P := 5T_1 + 12T_2 + 19T_3$. Then $\delta(P) = 0$, and $D_f(P) \neq 0$ for some Kagan function f.*

Proof. Let $\alpha = \arccos\frac{1}{3}$, and recall that $\frac{\alpha}{\pi} \notin \mathbb{Q}$, see e.g. [Pak, §41.3] and [Bol]. Thus, there is a Kagan function f which satisfies $f(\alpha) \neq 0$, and, moreover, $f(\alpha) \notin \overline{\mathbb{Q}}$, see [Pak, Ex. 17.8].

Observe that all dihedral angles of T_1 are equal to α. Dihedral angles of T_2 are equal to $\frac{\pi}{2}$ for the three edges at the origin, and to $\beta := \arccos\frac{\sqrt{3}}{3} = \frac{\pi-\alpha}{2}$ for the three other edges. Finally, all dihedral angles of T_3 are rational multiples of π. The values of the volume defect and the Dehn invariant for all three tetrahedra are given in Table 1 below.

	T_1	T_2	T_3
$\delta(\cdot)$	$\frac{3\alpha}{\pi} - \frac{4}{3}$	$-\frac{5\alpha}{4\pi} - \frac{1}{2}$	$\frac{2}{3}$
$D_f(\cdot)$	$6\sqrt{2}\,f(\alpha)$	$-\frac{9}{\sqrt{2}}\,f(\alpha)$	0

Table 1. Values of the volume defect and the Dehn invariant.

Using values from the table, Lemmas 3.1 and 3.2 imply:

$$\begin{aligned} \delta(P) &= 5\,\delta(T_1) + 12\,\delta(T_2) + 19\,\delta(T_3) = 0,\\ D_f(P) &= 5\,D_f(T_1) + 12\,D_f(T_2) + 19\,D_f(T_3) = -24\sqrt{2}\,f(\alpha) \neq 0,\end{aligned}$$

as desired.[2] □

Proof of Theorem 1.2. The polytope $P \subset \mathbb{R}^3$ constructed in the proof of Proposition 4.1 is concrete, but has a non-zero Dehn invariant. Thus, by Theorem 2.1, it cannot multitile the space. This proves the first part of the theorem.

For the second part, take a lattice zonotope $Q \subset \mathbb{R}^3$ with at least N vertices. From the results in §2, we have $\delta(Q) = 0$, and $D_f(Q) = 0$ for all Kagan functions f. By Lemma 3.1, we have $\delta(P+Q) = \delta(P) = 0$, so $(P+Q)$ is concrete. On the other hand, by Lemma 3.2, we have $D_f(P+Q) = D_f(P) \neq 0$. Thus, by Theorem 2.1, polytope $(P+Q)$ cannot multitile $\mathbb{R}^3$. Finally, observe that $(P+Q)$ has at least N vertices, see e.g. [Zie, Prop. 7.12]. This completes the proof. □

[2] See [Pak, §16.4] for the algebraic approach, which implies $D_f(P) \neq 0$ without computing dihedral angles directly.

5. Final remarks and open problems

5.1. Curiously, the volume defect can be both very small or very large for general lattice polytopes in $\mathbb{R}^3$. Indeed, consider the following *wedge tetrahedron* and *flat square pyramid*:

$$W_n := \operatorname{conv}\{(0,0,0),\,(1,1,0),\,(1,0,n),\,(0,1,n)\},$$
$$V_n := \operatorname{conv}\{(0,0,0),\,(n,0,0),\,(0,n,0),\,(n,n,0),\,(0,0,1)\}.$$

As $n \to \infty$, we have:

$$\chi(W_n) = \Theta\left(\frac{1}{n}\right), \qquad \operatorname{vol}(W_n) = \frac{n}{3}, \qquad \text{and} \qquad \delta(W_n) \sim -\frac{n}{3}.$$

On the other hand,

$$\chi(V_n) = \frac{n^2}{2} - O(n), \qquad \operatorname{vol}(V_n) = \frac{n^2}{3}, \qquad \text{and} \qquad \delta(V_n) \sim \frac{n^2}{6}.$$

5.2. Lemma 3.1 follows from a more general result in the literature that every translation invariant valuation on $\mathbb{R}^d$ which is homogeneous of degree one is Minkowski additive (see [Sch, Rem. 6.3.3] and [BL2, Cor. 32]). We include a short proof outline both for simplicity, to remain as much self-contained as possible, and as a brief guide to the literature. While Lemma 3.2 is very natural, we could not find it stated in this form. As we explain above, its proof follows along steps similar to the proof of Lemma 3.1.

5.3. The asymptotic argument in the proof of Theorem 2.1 can also be applied in $\mathbb{R}^2$, where it is traditionally used to show that the plane cannot be tiled with congruent convex n-gons, for $n \geq 7$. See [Ale, Niv] for early versions of this result. See also [KPP, Thm D] for an advanced version of this argument, proving that strictly acute tetrahedra cannot tile $\mathbb{R}^4$, and for further references.

5.4. One can ask if the results of this paper can be further extended. First, we can always extend Theorem 1.2 to higher dimensions $d \geq 4$. By the argument in the proof of Theorem 2.1, every $P \subset \mathbb{R}^d$ which multitiles $\mathbb{R}^d$ has zero *Hadwiger* (generalized Dehn) *invariants* [LM]. Take an orthogonal prism $P \times [0,1]$ over the polytope $P \subset \mathbb{R}^3$ as in Proposition 4.1. The dihedral angles are either $\pi/2$ or the same as in P. Thus the corresponding codim-2 Hadwiger invariant is non-zero, giving a counterexample in $\mathbb{R}^4$. Proceed by induction; the details are straightforward.

Going one step further, we say that a lattice polytope $P \subset \mathbb{R}^d$ is *super concrete*, if it is concrete and *scissors congruent* to a d-cube. For $d = 3, 4$, by the *Sydler–Jessen theorem* this is equivalent to zero Dehn invariant [Bol, Dup]. For $d \geq 5$, scissors congruence with a d-cube implies and is conjecturally equivalent to zero Hadwiger invariants, see e.g. [Zak]. Also, every $P \subset \mathbb{R}^d$ which multitiles $\mathbb{R}^d$ with translations must be super concrete, see [LM]. So in the spirit of Conjecture 1.1, one can ask whether all super concrete lattice polytopes $P \subset \mathbb{R}^d$ can multitile the space.

We conjecture that the answer is negative for all $d \geq 3$. For example, for $d \geq 4$, the local structure of cones around a vertex can be constrained by a *spherical Dehn invariant*, see e.g. [Dup]. In principle, the concrete assumption is too weak and can allow "bad cones" which would locally not multitile the sphere $\mathbb{S}^{d-1}$. It would be interesting to make this precise. The above problem is even more interesting in $\mathbb{R}^3$. In principle, the cones around vertices can all have nontrivial geometry generating a non-discrete group of symmetries, cf. [MM]. Again, it would be interesting to give an explicit construction.

5.5. Let us mention that the proof of Proposition 4.1 hinges on the following curious geometric property: the orthoscheme T_3 tiles the lattice cube and thus the space, yet has $\delta(T_3) \neq 0$. In particular, this shows that Theorem 2.1 cannot be extended to allow reflections. This non-zero volume defect of T_3 has to do with the fact that the remaining five orthoschemes in the tiling of the cube are obtained from T_3 by reflections which do not preserve the lattice. Although congruent to T_3, these reflected orthoschemes have both negative and positive volume defect, giving zero in total for the lattice cube.

Note that the (primitive) lattice cubes which arise in the construction, correspond (up to parallel translation) to rational orthogonal matrices $M \in O(3,\mathbb{R})$. These matrices are enumerated in [Cre]. We conjecture that the corresponding orthoschemes have a nonzero volume defect with probability at least $\varepsilon > 0$, as the cube edge length $\ell \to \infty$. This would give further examples of polytopes as in the proposition, all with a bounded number of vertices.

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