Dedicated to Askold Khovanskii on the occasion of his 60th birthday

THE KAUFFMAN BRACKET OF VIRTUAL LINKS AND THE BOLLOBÁS-RIORDAN POLYNOMIAL

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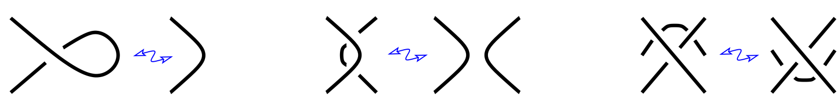
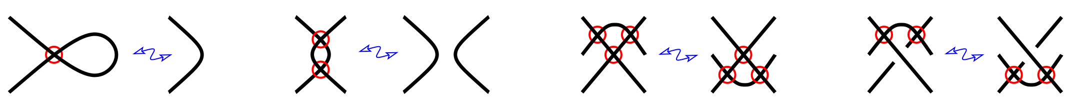
Abstract. We show that the Kauffman bracket $[L]$ of a checkerboard colorable virtual link L is an evaluation of the Bollobás-Riordan polynomial R_{G_L} of a ribbon graph associated with L. This result generalizes the celebrated relation between the Kauffman bracket and the Tutte polynomial of planar graphs.

Introduction

The theory of *virtual links* was discovered independently by L. Kauffman [K3] and M. Goussarov, M. Polyak, and O. Viro [GPV]. Virtual links are represented by their diagrams which differ from ordinary knot diagrams by presence of *virtual crossings*, which should be understood not as crossings but rather as defects of our two-dimensional picture. They should be treated in the same way as the extra crossings appearing in planar pictures of non-planar graphs. Virtual link diagrams are considered modulo the *classical* Reidemeister moves

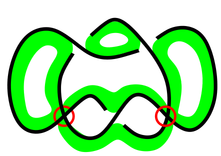


and the *virtual* Reidemeister moves

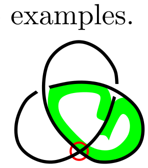


Here the virtual crossings are encircled for the emphasis.

N. Kamada introduced [Ka1, Ka2] the notion of a *checkerboard coloring* of a virtual link diagram. This is a coloring of one side of the diagram in its small neighborhood, such that near a classical crossing it alternates like on a checkerboard, and near a virtual crossing the colorings go through without noticing the crossing strand and its coloring. Not every virtual link is checkerboard colorable. Here are two examples.



checkerboard colorable



not checkerboard colorable

A similar notion was introduced and explored by V. Manturov (see [M] and the references therein), who called them *atoms*.

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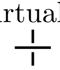
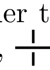
Note that the left virtual knot diagram above is alternating in a sense that classical overcrossings and undercrossing alternate, while the right virtual diagram is not alternating. It was proved in [Ka1] that a virtual link diagram is checkerboard colorable if and only if it can be made alternating by a number of classical crossing changes. In particular, every classical link diagram is checkerboard colorable.

In [Ka1, Ka2] N. Kamada showed that many classical results on knots and links can be extended to checkerboard colorable virtual links. This paper is devoted to a new result in this direction. Namely, we generalize the celebrated theorem of M. Thistlethwaite [Th] (see also [K1, K2]), which established a connection between the *Jones polynomial* for links and knots and the *Tutte polynomial* for graphs. Formally, Thistlethwaite showed that, up to a sign and a power of t, the Jones polynomial $V_L(t)$ of an alternating link L is equal to the specialization of the Tutte polynomial $T_{\Gamma_L}(-t,-t^{-1})$ of the corresponding graph Γ_L. For virtual links, the graph Γ_L is naturally embedded into a surface rather than into the plane, i.e. it becomes a *ribbon graph*. In this case, instead of the Tutte polynomial we should consider its generalization, the *Bollobás-Riordan polynomial*. Interestingly, the Bollobás-Riordan polynomial was introduced with (very different) knot theoretic applications in mind [BR2, BR3].

The paper is structured as follows. In the first two sections we recall definitions of the *Kauffman bracket* of virtual links and the Bollobás-Riordan polynomial of ribbon graphs. In section 3 we construct a ribbon graph from a checkerboard colorable virtual link diagram and state the Main Theorem for alternating virtual links. As often appears in these cases, the proofs of the results about (generalizations of) the Tutte polynomial are quite straightforward. The proof of the Main Theorem is postponed until section 5. In section 4 we extend our results to signed ribbon graphs and derive the Jones polynomial of an arbitrary checkerboard colorable virtual link as an appropriate evaluation. We conclude with final remarks and an overview of the literature.

1. The Kauffman bracket of virtual links.

Let L be a virtual link diagram. Consider two ways of resolving a classical crossing. The *A-splitting*,  $\rightsquigarrow$ , is obtained by uniting the two regions swept out by the overcrossing arc under the counterclockwise rotation until the undercrossing arc. Similarly, the *B-splitting*,  $\rightsquigarrow$ , is obtained by uniting the other two regions. A *state* S of a link diagram L is a way of resolving each classical crossing of the diagram. Denote by $\mathcal{S}(L)$ the set of the states of L. Clearly, a diagram L with n crossings has $|\mathcal{S}(L)| = 2^n$ different states.

Denote by $\alpha(S)$ and $\beta(S)$ the number of A-splittings and B-splittings in a state S, respectively. Also, denote by $\delta(S)$ the number of components of the curve obtained from the link diagram L by all splittings according to the state $S \in \mathcal{S}(L)$.

Definition 1.1. *The* Kauffman bracket *of a diagram L is a polynomial in three variables A, B, d defined by the formula:*

$$[L](A,B,d) \ := \sum_{S\in\mathcal{S}(L)} A^{\alpha(S)}\, B^{\beta(S)}\, d^{\delta(S)-1}\,. \tag{1}$$

Note that $[L]$ is *not* a topological invariant of the link and in fact depends on the link diagram. However, it defines the *Jones polynomial* $J_L(t)$ by a simple substitution

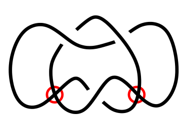
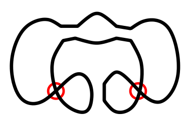
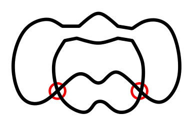
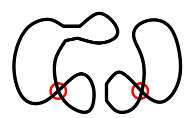
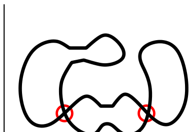
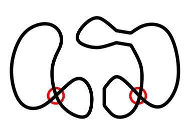
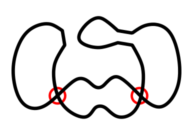
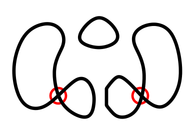
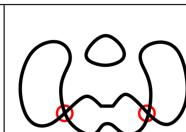
$A = t^{-1/4}$, $B = t^{1/4}$, $d = -t^{1/2} - t^{-1/2}$:

$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}) \ .$$

Here $w(L)$ denotes the *writhe*, determined by the orientation of L as the sum over the classical crossings of L of the following signs:

The Jones polynomial is a classical topological invariant (see e.g. [B, M]).

Example 1.2. Consider the virtual knot diagram L from the example above and shown on the left of the table below. It has two virtual and three classical crossings, so there are eight states for it, $|\mathcal{S}(L)| = 8$. The curves obtained by the splittings and the corresponding parameters $\alpha(S)$, $\beta(S)$, and $\delta(S)$ are shown in the remaining columns of the table.

				
(α, β, δ)	$(3,0,1)$	$(2,1,2)$	$(2,1,2)$	$(1,2,1)$
				
	$(2,1,2)$	$(1,2,1)$	$(1,2,3)$	$(0,3,2)$

In this case the Kauffman bracket of L is given by

$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d \ .$$

It is easy to check that $J_L(t) = 1$.

2. The Bollobás-Riordan polynomial.

Let $\Gamma = (V, E)$ be a undirected graph with the set of vertices V and the set of edges E (loops and multiple edges are allowed). Suppose in each vertex $v \in V$ there is a fixed cyclic order on edges adjacent to v (loops are counted twice). We call this combinatorial structure a *ribbon graph*, and denote it by G. One can represent G by making vertices into 'discs' and connecting them by 'ribbons' as prescribed by the cyclic orders (see Example 2.2 below). This defines a 2-dimensional surface with boundary, which by a slight abuse of notation we also denote by G.

Formally, G is the surface with boundary represented as the union of two sets of closed topological disks, corresponding to vertices $v \in V$ and edges $e \in E$, satisfying the following conditions:

- these discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge,
- every edge contains exactly two such line segments.

It will be clear from the context whether by G we mean the ribbon graph or its underlying surface. In this paper we restrict ourselves to oriented surfaces G. We refer to [GT] for other definitions and references.

For a ribbon graph G, let $v(G) = |V|$ denote the number of vertices, $e(G) = |E|$ denote the number of edges, and $k(G)$ denote the number of connected components of G. Also, let $r(G) = v(G) - k(G)$ be the *rank* of G, and $n(G) = e(G) - r(G)$ be the *nullity* of G. Finally, let $\mathrm{bc}(G)$ be the number of connected components of the boundary of the surface G.

A spanning subgraph of a ribbon graph G is defined as a subgraph which contains all the vertices, and a subset of the edges. Let $\mathcal{F}(G)$ denote the set of the spanning subgraphs of G. Clearly, $|\mathcal{F}(G)| = 2^{e(G)}$.

Definition 2.1. *The* Bollobás-Riordan polynomial $R_G(x, y, z)$ *of a ribbon graph* G *is defined by the formula*

$$R_G(x, y, z) \;:=\; \sum_{F \in \mathcal{F}(G)} x^{r(G)-r(F)} y^{n(F)} z^{k(F)-\mathrm{bc}(F)+n(F)} . \tag{2}$$

This version of the polynomial is obtained from the original one [BR2, BR3] by a simple substitution. Note that for a planar ribbon graph G (i.e. when the surface G has genus zero) the Euler's formula gives $k(F) - \mathrm{bc}(F) + n(F) = 0$ for all $F \subseteq G$. Therefore, the Bollobás-Riordan polynomial R_G does not contain powers of z. In fact, in this case it is essentially equal to the classical Tutte polynomial $T_\Gamma(x, y)$ of the (abstract) core graph Γ of G:

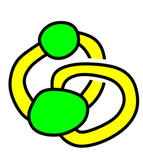
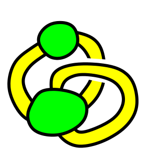
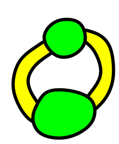
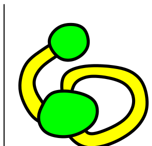
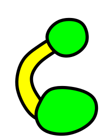
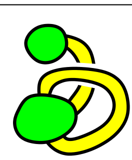
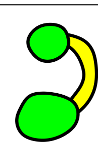
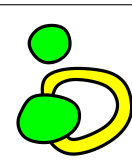
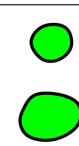
$$R_G(x - 1, y - 1, z) = T_\Gamma(x, y) .$$

Similarly, a specialization $z = 1$ of the Bollobás-Riordan polynomial of an arbitrary ribbon graph G, gives the Tutte polynomial once again:

$$R_G(x - 1, y - 1, 1) = T_\Gamma(x, y) .$$

We refer to [BR2, BR3] for proofs of these formulas and to [B, W] for general background on the Tutte polynomial.

Example 2.2. Consider the ribbon graph G shown on the left in the table below. The other columns show eight possible spanning subgraphs F and the corresponding values of $k(F)$, $r(F)$, $n(F)$ and $\mathrm{bc}(F)$.

				
(k, r, n, bc)	$(1, 1, 2, 1)$	$(1, 1, 1, 2)$	$(1, 1, 1, 2)$	$(1, 1, 0, 1)$
				
	$(1, 1, 1, 2)$	$(1, 1, 0, 1)$	$(2, 0, 1, 3)$	$(2, 0, 0, 2)$

Now use the definition to compute the corresponding Bollobás-Riordan polynomial:

$$R_G(x, y, z) \;=\; y^2z^2 + 3y + 2 + xy + x$$

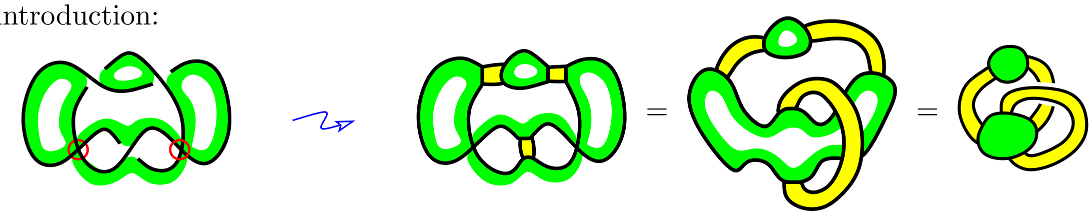
3. Ribbon graphs from virtual diagrams and the Main Theorem.

In this section we construct a ribbon graph G_L starting with an alternating virtual link diagram L.

It was shown in [Ka2] that every alternating link diagram L has a canonical checkerboard coloring which can be constructed in the following way. Near every classical crossing we color black the vertical angles swept out by the overcrossing arc under the counterclockwise rotation, i.e. angles that are glued together by the A-splitting. Since the diagram is alternating, it is evident that all these local colorings near classical crossings agree with each other and can be extended to a global checkerboard coloring. For example, a checkerboard coloring of the virtual knot in the introduction is canonical.

Topologically a coloring is represented by a bunch of annuli. Each annulus has two boundary circles, an exterior circle which goes along the link except small arcs near classical crossings where it jumps from one strand to another one, and an interior circle. In order to construct a ribbon graph from a (virtual) link diagram we replace every crossing by an edge-ribbon connecting the corresponding arcs of the exterior circles:

The ribbon graph G_L is obtained by gluing discs along the interior circles of the annuli of the coloring. Here is a ribbon graph constructed from a virtual link diagram in the introduction:



Note that here we ignore the way a ribbon graph is embedded into $\mathbb{R}^3$ and consider it as a 2-dimensional surface. Of course, every ribbon graph can be obtained in this way from an appropriate virtual link diagram.

Main Theorem 3.1. *Let L be an alternating virtual link diagram and G_L be the corresponding ribbon graph. Then*

$$[L](A, B, d) \;=\; A^{r(G)}B^{n(G)}d^{k(G)-1}\,R_{G_L}\left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{1}{d}\right).$$

We should warn the reader that although both the Kauffman bracket and the Bollobás-Riordan polynomial are polynomials in three variables, the former has only two free variables since $[L]$ is always homogeneous in A and B. This follows from the identity $\alpha(S) + \beta(S) = e(S)$ for all $S \in \mathcal{S}(L)$. Another way to see this is to note that the values at which the Bollobás-Riordan polynomial $R_G(x, y, z)$ is evaluated in the theorem satisfy the equation $xyz^2 = 1$. Thus, the situation here is different from the planar case where the Kauffman bracket $[L_G](A,B,d)$ and the Tutte polynomial $T_\Gamma(x,y)$ determine each other.

4. Extensions and applications.

Define a *signed ribbon graph* $\widehat{G}$ to be a ribbon graph G given by (V,E), and a *sign function* $\varepsilon : E \to \{\pm 1\}$. For a spanning subgraph $F \subset \widehat{G}$ denote by $e_-(F)$ the number of edges $e \in E$ with $\varepsilon(e) = -1$. Denote by $\overline{F} = \widehat{G} - F$ a complement to F in $\widehat{G}$, i.e. a spanning subgraph of $\widehat{G}$ with only those (signed) edges of G that do not belong to F. Finally, let

$$s(F) = \frac{e_-(F) - e_-(\overline{F})}{2}$$

We define the *signed Bollobás-Riordan polynomial* $R_{\widehat{G}}(x,y,z)$ as follows:

$$R_{\widehat{G}}(x,y,z) \ := \sum_{F\in\mathcal{F}(\widehat{G})} x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-\mathrm{bc}(F)+n(F)} . \tag{3}$$

Any checkerboard colorable virtual link diagram L can be made alternating $\widetilde{L}$ by switching some classical overcrossings to undercrossings [Ka1]. We can label the edges of $G_{\widetilde{L}}$ corresponding to the crossings where the switching was performed by -1, and the other edges by $+1$. The result is a signed ribbon graph denoted by $\widehat{G}_L$

Theorem 4.1. *Let L be a checkerboard colorable virtual link diagram, and $\widehat{G}_L$ be the corresponding signed ribbon graph. Then*

$$[L](A,B,d) \ = \ A^{r(G)} B^{n(G)} d^{k(G)-1}\, R_{\widehat{G}_L}\left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{1}{d}\right).$$

The proof follows verbatim the proof of the Main Theorem (see the next section). We leave the details to the reader.

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let $\widehat{G}$ be a signed ribbon graph corresponding to a checkerboard colorable virtual link diagram L endowed with an orientation. Then*

$$J_L(t) \ = \ (-1)^{w(L)}\, t^{\frac{3w(\widetilde{L})-r(\widehat{G})+n(\widehat{G})}{4}} \left(-t^{1/2} - t^{-1/2}\right)^{k(\widehat{G})-1} R_{\widehat{G}}\Big(-t-1, -t^{-1}-1, \frac{1}{-t^{1/2}-t^{-1/2}}\Big).$$

In particular, if $\widehat{G}$ is a planar ribbon graph with only positive edges and Γ is its core graph, we have the following well-known relation:

$$J_L(t) \ = \ (-1)^{w(L)} t^{\frac{3w(\widetilde{L})-r(\widehat{G})+n(\widehat{G})}{4}} \left(-t^{1/2} - t^{-1/2}\right)^{k(\widehat{G})-1} T_\Gamma(-t, -t^{-1}).$$

5. Proof of the Main Theorem.

The notation used in the definitions of the Kauffman bracket (Definition 1.1) and the Bollobás-Riordan polynomial (Definition 2.1) hints on how to prove the Main Theorem. Since the crossings of the diagram L correspond to the edges of $G = G_L$, there is a natural one-to-one correspondence $\varphi : \mathcal{S}(L) \to \mathcal{F}(G)$ between the states $S \in \mathcal{S}(L)$ and spanning subgraphs $F \subseteq G$. Namely, let an A-splitting of a crossing in S mean that we keep the corresponding edge in the spanning subgraph $F = \varphi(S)$. Similarly, let a B-splitting in S mean that we remove the edge from the subgraph $F = \varphi(S)$.

By definition, we have $\delta(S) = \mathrm{bc}(F)$, for all $F = \varphi(S)$. Furthermore, we easily obtain the following relation between the parameters:

$$e(F) = \alpha(S), \qquad e(G) - e(F) = \beta(S)\ ,$$

for all $S \in \mathcal{S}(L)$, and $F = \varphi(S)$. Now, for a spanning subgraph $F \in \mathcal{F}(G)$, consider the term $x^{r(G)-r(F)}y^{n(F)}z^{k(F)-\mathrm{bc}(F)+n(F)}$ of $R_G(x, y, z)$. After a substitution

$$x = \frac{Bd}{A}, \qquad y = \frac{Ad}{B}, \qquad z = \frac{1}{d}$$

and multiplication of this term by $A^{r(G)}B^{n(G)}d^{k(G)-1}$ as in the Main Theorem, we get

$$\begin{aligned}
&A^{r(G)}B^{n(G)}d^{k(G)-1}(A^{-1}Bd)^{r(G)-r(F)}(AB^{-1}d)^{n(F)}d^{-k(F)+\mathrm{bc}(F)-n(F)}\\
&= A^{r(G)-r(G)+r(F)+n(F)}B^{n(G)+r(G)-r(F)-n(F)}d^{k(G)-1+r(G)-r(F)+n(F)-k(F)+\mathrm{bc}(F)-n(F)}\\
&= A^{r(F)+n(F)}B^{n(G)+r(G)-r(F)-n(F)}d^{k(G)-1+r(G)-r(F)-k(F)+\mathrm{bc}(F)}\ .
\end{aligned}$$

It is easy to see that $r(F)+n(F) = e(F)$, and $k(F)+r(F) = v(F) = v(G)$. Therefore, $k(G) - k(F) + r(G) - r(F) = 0$, and we can rewrite our term as

$$A^{e(F)}B^{e(G)-e(F)}d^{\mathrm{bc}(F)-1}\ .$$

In terms of the state $S = \varphi^{-1}(F) \in \mathcal{S}(L)$ this term is equal to

$$A^{\alpha(S)}\,B^{\beta(S)}\,d^{\delta(S)-1}\,,$$

which is precisely the term of $[L]$ corresponding to the state $S \in \mathcal{S}(L)$. This completes the proof. □

6. Final remarks and open problems.

1. Trivalent ribbon graphs are the main objects in the finite type invariant theory of knots, links and 3-manifolds, while general ribbon graphs appeared in the literature under a variety of different names (see e.g. [DKC, BR2, K1]). Embeddings of ribbon graphs into 3-space are studied in [RT].

2. The Bollobás-Riordan polynomial can be defined by recurrent contraction-deletion relations or by spanning tree expansion similar to those of the Tutte polynomial, except that deletion of a loop is not allowed. We refer to [BR2, BR3] for the details. We should note that [BR3] gives an extension to unorientable surfaces as well. One can also find the contraction-deletion relations and the spanning tree expansion for the signed Bollobás-Riordan polynomial defined by (3). For a planar signed ribbon graph $\widehat{G}$, the signed

Bollobás-Riordan polynomial $R_{\widehat{G}}$ is related to Kauffman's *signed Tutte polynomial* $Q[\widehat{G}]$ from [K2] by the formula

$$R_{\widehat{G}}(x,y,z) \,=\, x^{\frac{v(\widehat{G})+1}{2}-k(\widehat{G})}\, y^{\frac{-v(\widehat{G})+1}{2}}\, Q[\widehat{G}]\Big((y/x)^{1/2}, 1, (xy)^{1/2}\Big).$$

So, our version of $R_{\widehat{G}}$ may be considered as a generalization of the polynomial $Q[\widehat{G}]$ to signed ribbon graphs. If, besides planarity, all edges of $\widehat{G}$ are positive, then $R_{\widehat{G}}$ is related to the dichromatic polynomial $Z[\Gamma](q,v)$ (see [K2]) of the underlying graph Γ:

$$R_{\widehat{G}}(x,y,z) \,=\, x^{-k(\widehat{G})}\, y^{-v(\widehat{G})}\, Z[\Gamma](xy,y)\,.$$

3. It would be interesting to generalize the Bollobás-Riordan polynomial for colored ribbon graphs [BR1, Tr] and prove the corresponding relation with the Kauffman bracket. Let us also mention that in [J] (see also [Tr]), Jaeger found a different relation between links and graphs and proved that the whole Tutte polynomial, not just its specialization, can be obtained from the HOMFLY polynomial of the appropriate link. Extending these results to ribbon graphs is an important open problem.

Finally, recent results concerning combinatorial evaluations of the Tutte and Bollobás-Riordan polynomials [KP] leave an open problem of finding such evaluations for general values of the polynomial R_G. It would be interesting to use the Main Theorem to extend the results of [KP].

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