HOW TO CUT OUT A CONVEX POLYHEDRON

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Abstract. It is known that one can fold a convex polyhedron from a non-overlapping face unfolding, but the complexity of the algorithm in [MP] remains an open problem. In this paper we show that every convex polyhedron $P \subset \mathbb{R}^d$ can be obtained in polynomial time, by starting with a cube which contains P and sequentially cutting out the extra parts of the surface.

Our main tool is of independent interest. We prove that given a convex polytope P in $\mathbb{R}^d$ and a facet F of P, then F is contained in the union $\cup_{G \neq F} A_G$. Here the union is over all the facets G of P different from F, and A_G is the set obtained from G by rotating towards F the hyperplane spanned by G about the intersection of it with the hyperplane spanned by F.

1. Introduction

The study of non-overlapping *face unfolding* of convex polyhedra was initiated in [ShS] in $\mathbb{R}^3$ and was extended to higher dimension in [MP]. This is in contrast with the better known and still open *edge unfolding* problem, sometimes called the *Dürer conjecture* (see [O2] for an elegant introduction). To appreciate the difference, consider three unfoldings of a cube in Figure 1, where the first is the edge unfolding and the next two are face unfoldings. Note however, that finding and verifying a non-overlapping unfolding is not an easy task, and may not be the most practical thing to do. In this paper we present a new approach to making the surface of a convex polytope, by sequentially cutting it out of a cube (see below).

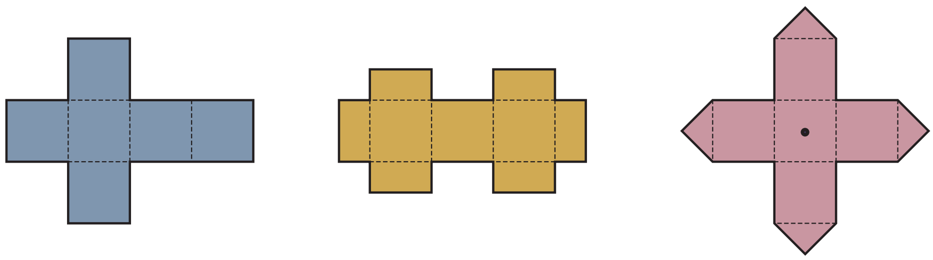

Figure 1. Foldouts of three unfoldings of a cube.

There are two kinds of non-overlapping unfolding of a convex polyhedron $P \subset \mathbb{R}^3$ known in the literature. First, there is a *source unfolding* defined as a cut locus (or

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geodesic Voronoi diagram) of a point on the surface. Second, there is an *Alexandrov* (also known as star) unfolding constructed by cutting along shortest paths to all vertices from a generic point. Both unfolding can be constructed in polynomial time and have various properties which are well studied (see [AAOS, AO, ShS]), especially in connection to the *discrete geodesic problem* [MMP]. Most recently, the source unfolding was extended to higher dimensions in [MP], and there seem to be an evidence that the Alexandrov unfolding does not extend (see [MP, §8.4]). Unfortunately, the algorithm proposed in [MP] is yet to be fully analyzed and only conjectured to be polynomial. In other words, as of now there is no proven polynomial time algorithm for a non-overlapping unfolding in dimension $d \geq 4$.

Let us now think of the surface of a convex polyhedron in $\mathbb{R}^3$ as made out of carton. Then making a polyhedron out of a physical foldout would involve a rather involved continuous folding procedure which requires non-intersection at all times. Whether this is possible is a well known open problem due to Connelly and in the strong form is usually called the *blooming conjecture*; we discuss it at length in Section 5. Given these difficulties, from a practical point of view, an alternative approach can prove useful.

Start with a cube B which contains a given convex polyhedron $P \subset \mathbb{R}^3$. Intersect B with a plane H. Cut along the edges of B on one side of H. Cut out some portions in each of the flaps, place them the onto H, and glue them together (see Figure 2). Repeat the procedure. The question is whether one can obtain every polyhedron after a finite sequence of such "cutouts" (see a formal definition in the next section).

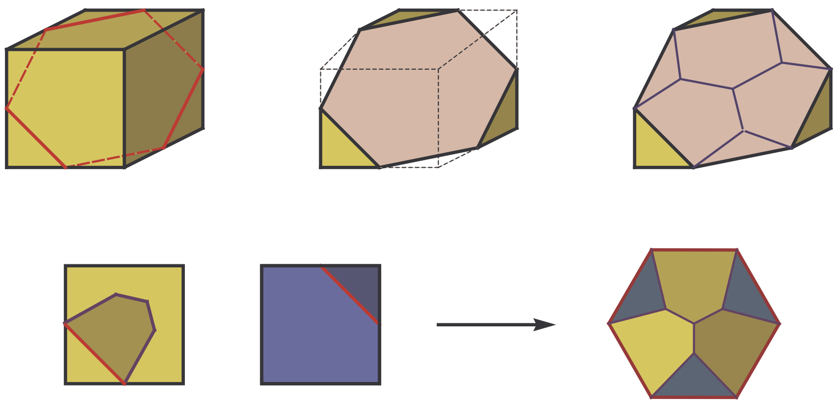

FIGURE 2. Cutting out the surface of a heptahedron from the surface of a cube. Dark right triangles are cut out of the back three faces, and the pentagons from the front three faces. Colored hexagon shows how these cutouts are folded onto a hexagonal face and fit together.

Of course, when the cube is sufficiently large, one can hope that its faces are large enough so that eventually all faces of P can be cut out of them (whether this can be done sequentially is less clear). We prove a much stronger claim, that this can be done for any cube which contains P. Moreover, this can be done efficiently in a much greater generality and in any dimension. Here is out main result:

Theorem 1.1. *Let $P, Q \subset \mathbb{R}^d$ be two convex polytopes, such that $P \subset Q$. Then the surface of P can be cut out from the surface of Q in time $N^{O(d)}$, where N denotes the total number of faces of P and Q.*

In particular, every convex polyhedron $P \subset \mathbb{R}^d$ with m facets can be cut out from a sufficiently large cube in time $m^{O(d)}$. A warning: we are counting the number of cuts that are made, not the number of pieces into which the eventual surface of P is divided. In fact, we conjecture (see Subsection 6.4) that the number of cut pieces can grow exponentially even in $\mathbb{R}^3$. In this sense, one can view our main result as a variation on a "one cut problem" in computational geometry (see e.g. [O1] for an introduction).

Another way to think of the theorem is to observe that it gives a direct way to see that $\operatorname{area}(P) < \operatorname{area}(Q)$ for all $P \subset Q$, in the spirit of the *Hilbert third problem* (see e.g. [Bol, Pak]). Of course, in $\mathbb{R}^3$ there is likely to be a more straightforward proof of this corollary, in the spirit of a proof of the Bolyai–Gerwien theorem (cf. Subsection 6.5).

The proof of the theorem is a direct consequence of a somewhat delicate collapsing walls lemma (Lemma 2.1), which is of independent interest. We present the definitions and the lemma in the next section. In Section 3 we present the proof of the collapsing walls lemma, and in Section 4 the proof of the main result (Theorem 1.1). We then outline a connection with Connelly's blooming conjecture in Section 5, and conclude with final remarks in Section 6.

2. Formal definitions and the collapsing walls lemma

We begin with some preliminary definitions. Let P be a convex polytope in $\mathbb{R}^d$, and let ∂P denotes its surface. For a facet F of P, we denote by H_F the hyperplane supporting P at the facet F. For two non-parallel facets F and G of P, we denote by $\Phi_{F,G}$ the affine transformation which is the rotation about $H_F \cap H_G$ of H_G onto H_F. The rotation is performed in the direction dictated by P, so that throughout the rotation H_G intersects the interior of P. For convenience, when H_F is parallel to F_G, denote by $\Phi_{F,G}$ the projection of G onto F.

We are now ready to give a formal definition of the *cutting out* procedure described in the introduction in dimension three. Let $Q \subset \mathbb{R}^d$ be a convex polytope and let H be a hyperplane separating Q into polytopes Q_+ and Q_-. Denote by F the facet of Q_- which lies in H, and by $G_1, \ldots, G_m$ the remaining facets of Q_-. Suppose there exist convex polytopes $R_1, \ldots, R_m$ such that $R_i \subset G_i$, for all $1 \le i \le m$, and such that $X_i = \Phi_{F,G_i}(R_i) \subset F$ are non-intersecting and $X_1 \cup \ldots \cup X_m = F$. We then say that the surface ∂Q_+ is *cut out* of ∂Q in one step. More generally, we say that ∂P can be cut out from ∂Q if there is a finite sequence of polytopes

$$P = P_0 \subset P_1 \subset \ldots \subset P_\ell = Q,$$

such that ∂P_{i-1} can be cut out of ∂P_i in one step, for all $1 \le i \le \ell$.

Our main tool in the proof of Theorem 1.1 is the following result:

Lemma 2.1 (Collapsing walls lemma)**.** *Let $P \subset \mathbb{R}^d$ be a convex polytope and let F be fixed facet of P. Then*

$$F \subseteq \bigcup_{G \neq F} \Phi_{F,G}(G),$$

where the union is over all facets G of P, different from F.

In other words, if the walls of a (polyhedral) cage are collapsed onto the floor of the cage, they cover the whole floor. At first, this may seem obvious, but if the walls have non-right dihedral angles with the base, this is a rather delicate technical result (see Figure 3). In fact, we believe it is a new result of independent interest even for pyramids in $\mathbb{R}^3$.

FIGURE 3. An impossible configuration of four collapsing walls (sides of a pyramid cannot collapse on the base and leave an uncovered hole in the base).

Let us mention that the lemma by itself is insufficient to prove of the main theorem as the same wall can be used to cover the base in several possibly disconnected places (see Section 4). To avoid this, we use technical details from the proof of the lemma presented in the next section.

3. Proof of Lemma 2.1

Consider $\mathbb{R}^d$ endowed with the standard Cartesian coordinate system. We denote these coordinates by $x_1, \ldots, x_d$. Without loss of generality assume that H_F is the hyperplane $x_d = 0$, and P is contained in the half-space $x_d \geq 0$. Denote by $A_G = \Phi_{F,G}(G)$ the rotation of the facet G of P onto F. We need to show that every point in F lies in $\cup_{G \neq F} A_G$. Without loss of generality we can take this point to be the origin O.

Denote by $G_1, \ldots, G_m$ the facets of P different from F. To simplify the notation, let $A_i = A_{G_i}$, $H_i = H_{G_i}$, and $\Phi_i = \Phi_{F,G_i}$, for all $1 \leq i \leq m$. Denote by L_i the intersection of hyperplanes $H_i \cap H_F$. Let r_i be the distance from the origin to L_i, and let α_i be the dihedral angle of the cone between H_F and H_i, and which contains P.

Suppose now G_1 is such that

$$\tau_i = r_i \cdot \tan \frac{\alpha_i}{2} \quad \text{is minimized at} \quad \tau_1.$$

We will show that the origin O is contained in A_1. In other words, we prove that if $O \notin A_1$, then $\tau_i < \tau_1$ for some $i > 1$.

Let $z \in H_1$ such that the rotation of z onto F is the origin: $\Phi_1(z) = O$. It suffices to show that $z \in G_1$. Let $\boldsymbol{v} = (v_1, \ldots, v_{d-1}, 0)$ be the unit vector that in a normal to L_1 in the hyperplane H_F. It is easy to see that

$$\overrightarrow{Oz} = r_1(1 - \cos\alpha_1)\,\boldsymbol{v} + (0, \ldots, 0, r_1 \sin\alpha_1).$$

Assume to the contrary that $z \notin G_1$. Then there exists a facet of P, say G_2, such that H_2 separates z from the origin. Denote by y the closest point to z on L_2, and by α' the angle between the line (zy) and the hyperplane H_F, where the angle is taken with the half-hyperplane of H_F which contains F (and thus the origin). In this notation, the above condition implies that $\alpha' > \alpha_2$.

Without loss of generality we may assume that L_2 is given by equations $x_d = 0$ and $x_{d-1} = r_2$. Then

$$y = (r_1(1 - \cos\alpha_1)v_1, \ldots, r_1(1 - \cos\alpha_1)v_{d-2}, r_2, 0),$$

and

$$\cos\alpha' = \cos\widehat{Oyz} = \frac{r_2 - r_1(1 - \cos\alpha_1)v_{d-1}}{\sqrt{r_1^2 \sin^2\alpha_1 + (r_2 - r_1(1 - \cos\alpha_1)v_{d-1})^2}}.$$

Note that the function $x/\sqrt{a^2 + x^2}$ is monotone increasing as a function of x, and that $v_{d-1} \le 1$. We get

$$\cos\alpha' \ge \frac{r_2 - r_1(1 - \cos\alpha_1)}{\sqrt{r_1^2 \sin^2\alpha_1 + (r_2 - r_1(1 - \cos\alpha_1))^2}}.$$

Applying $\cos\alpha' < \cos\alpha_2$, we conclude:

$$\frac{r_2 - r_1(1 - \cos\alpha_1)}{\sqrt{r_1^2 \sin^2\alpha_1 + (r_2 - r_1(1 - \cos\alpha_1))^2}} < \cos\alpha_2. \tag{1}$$

Recall the assumption that $\tau_1 \le \tau_2$. This gives $r_1 \tan\frac{\alpha_1}{2} \le r_2 \tan\frac{\alpha_2}{2}$, or

$$\frac{r_2}{r_1} \ge \frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\alpha_2}{2}}. \tag{2}$$

The rest of this section is dedicated to showing that both (1) and (2) are impossible. This gives a contradiction with our assumptions and proves the claim. We split the proof into two cases depending on whether the dihedral angle α_2 is acute or obtuse. In each case we repeatedly rewrite (1) and (2), eventually leading to a contradiction.

Case 1. Suppose $\frac{\pi}{2} < \alpha_2 < \pi$. In this case $\cos\alpha_2 < 0$, and (1) is equivalent to

$$1 + \frac{r_1^2 \sin^2\alpha_1}{(r_2 - r_1(1 - \cos\alpha_1))^2} < \frac{1}{\cos^2\alpha_2}, \tag{3}$$

and

$$\frac{r_1 \sin\alpha_1}{r_2 - r_1(1 - \cos\alpha_1)} > \tan\alpha_2\,. \tag{4}$$

This can be further rewritten as:

$$\frac{r_2}{r_1} < 1 - \cos\alpha_1 + \frac{\sin\alpha_1}{\tan\alpha_2}\,. \tag{5}$$

Now (5) and (2) together imply

$$\frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\alpha_2}{2}} \,<\, 1 - \cos\alpha_1 + \frac{\sin\alpha_1}{\tan\alpha_2}\,,$$

which is impossible. Indeed, suppose for some $0 < a, b < \pi$, we have

$$\frac{\tan\frac{a}{2}}{\tan\frac{b}{2}} \,<\, 1 - \cos a + \frac{\sin a}{\tan b}. \tag{6}$$

Dividing both sides by $(\tan\frac{a}{2})$, after some easy manipulations, we conclude that (6) is equivalent to

$$\frac{1}{\tan\frac{b}{2}} \,<\, \sin a + \frac{1 + \cos a}{\tan b}\,, \tag{7}$$

which in turn is equivalent to

$$\left(\frac{1}{\tan\frac{b}{2}} - \frac{1}{\tan b}\right)\sin b \,<\, \cos(a-b)\,. \tag{8}$$

Since the left hand side of (8) is equal to 1, we get a contradiction and complete the proof in Case 1.

Case 2. Suppose now that $0 < \alpha_2 \le \frac{\pi}{2}$. Then $\cos\alpha_2 \ge 0$, and $0 < \tan\frac{\alpha_2}{2} \le 1$. Let us first show that the numerator of (1) is nonnegative, i.e. that $r_2 \ge r_1(1 - \cos\alpha_1)$. From the contrary, otherwise $r_2/r_1 < (1 - \cos\alpha_1)$. Together with (2), this implies:

$$1 - \cos\alpha_1 \,>\, \frac{r_2}{r_1} \,\ge\, \frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\alpha_2}{2}} \,\ge\, \tan\frac{\alpha_1}{2}\,,$$

which is impossible for all $0 < \alpha_1 < \pi$.

From above, we can now exclude the case $\alpha_2 = \frac{\pi}{2}$, for else the l.h.s. of (1) is non-negative, while r.h.s. is equal to zero. Thus, $\cos\alpha_2 > 0$. Therefore, the inequality (1) in this case can be rewritten as

$$1 + \frac{r_1^2\sin^2\alpha_1}{(r_2 - r_1(1-\cos\alpha_1))^2} \,>\, \frac{1}{\cos^2\alpha_2}\,, \tag{9}$$

and

$$\frac{r_1\sin\alpha_1}{r_2 - r_1(1-\cos\alpha_1)} \,>\, \tan\alpha_2\,. \tag{10}$$

Note now that (10) coincides with (4). Since (6) holds for all $0 \le a, b < 1$, we obtain the contradiction as in Case 1. This completes Case 2 and finishes the proof of the lemma. $\square$

4. Proof of Theorem 1.1

We cut polytope Q with hyperplanes containing facets of P, one by one. Start with a hyperplane H spanned by a facet in P, and let $F = Q \cap H$. Apply the collapsing wall lemma (Lemma 2.1) to the polytope Q_-. Let us show that the proof of the lemma implies that Q_+ can be cut out from Q in one step.

As in the proof of the lemma, denote by $G_1, \ldots, G_m$ the facets of Q_- other than F. For a point $z \in F$, define $\tau_i = |zA_i| \tan \frac{\alpha_i}{2}$, where $A_i = F \cap G_i$ is a facet of F, and $|zA_i|$ is the distance to a subspace spanned by A_i from z. Let $X_i \subset F$ be a subset of the facet F on which $\tau_i \leq \tau_j$, for all $j \neq i$. Let us first prove that X_i is a convex polytope. Define $\eta = \min_i \tau_i$. Clearly, η is a convex piecewise linear function which is zero on the boundary $\partial F = \cup_i A_i$. Then each R_i is a projection of a facet of a cone $\{(z,t) \mid z \in F, t \leq \eta(z)\}$, and thus a convex polytope. Therefore, convex polytopes $R_i = \Phi^{-1}(X_i) \subset G_i$ give the desired cutout of Q_+.

It remains to compute the complexity of the algorithm. There are $O(N)$ iterations of the cut out procedure. Each time, function η can be computed in time $N^{O(d)}$, and thus so are all R_i (see e.g. [PS]). The details are straightforward. $\square$

5. Connection to continuous blooming

Consider the process of folding the boundary of a convex polyhedron: if someone provides a polyhedral nonoverlapping foldout made of hinged steel, is it always possible to glue its corresponding edges together? Because steel is rigid, we need not only a nonoverlapping property on the foldout as it lies flat on the ground, but also a nonintersecting property as we continuously fold it up to be glued (see Figure 4). The reverse process was introduced by Bob Connelly (with some extra conditions), who called it *continuous blooming*, and proposed the following conjecture:

Conjecture 5.1 (Connelly's continuous blooming conjecture)**.** *Every convex polyhedral boundary has a continuous blooming.*

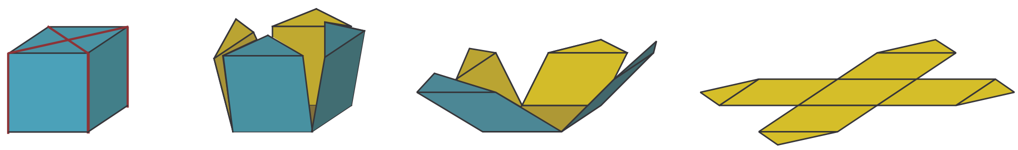

Figure 4. An example of a continuous blooming of the surface of the cube.

We refer to [MP, §9] for the background, precise definitions, and a formal statement of the conjecture. In this section we discuss the connections between out cutting out procedure and the the continuous blooming.

First, observe that in $\mathbb{R}^3$, in notation of the proof of the Theorem 1.1, we have convex polygons $X_i \subset F$ which are attached along the edges to the faces adjacent to F. Therefore, here is an attractive but incorrect scheme on how the blooming can be obtained. Think of a polytope $P \subset \mathbb{R}^3$ defined as the intersection of halfspaces corresponding to its facets. Removing a halfspace corresponding to the facet F of P gives a bigger convex polytope Q.[1] Sequentially rotate polygons X_i away from F to the surrounding walls to obtain a partial blooming. It is easy to see that this can be done without intersection. Now repeat until a tetrahedron is obtained, which can be then easily bloomed onto a plane.

The problem with this scheme is that while the first step is indeed an honest partial blooming, the subsequent steps can fail as the new subdivision of the faces can give disconnected pieces. The next two examples show both positive and negative cases of this connection.

Example 5.2 (Archimedean solids)**.** It is well known and easy to see that *Platonic solids* (regular polytopes in $\mathbb{R}^3$) have a continuous blooming (see e.g. [Pol]). Similarly, it is easy to see that every *Archimedean solid* can be obtained from a Platonic solid by a sequence of truncations. One can use the symmetry and check directly that the above unfolding procedure gives a correct continuous blooming in each case. We leave the details to the reader.

Example 5.3 (Disconnected cutout)**.** Consider a decahedron shown in Figure 5. Removing the top face produced a nonahedron, whose top view is as in the middle figure. The resulting blooming cuts out the original surface to two pentagons and two hexagons, located symmetrically on the sides. Removing one of the large quadrilateral faces makes the hexagonal unfolding disconnected as in the figure. As a result, the the next blooming iteration cannot be made.

In the opposite direction, start with a brick circumscribing the nonahedron, and apply the cutting out procedure first for the four side quadrilaterals, and then for the top face. It is then easy to see that the resulting cutout on the decahedron will consist of disconnected pieces from the surface of the brick.

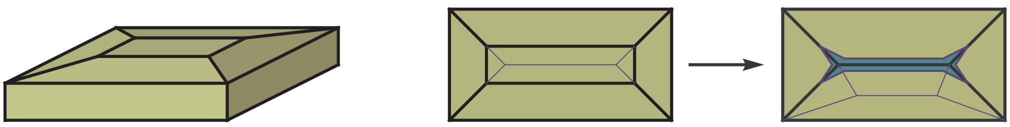

FIGURE 5. An example of polyhedron with a disconnected cutout.

6. Final remarks

6.1. The title of the paper may seem similar to that of a classical paper [Tve] by Tverberg. This resemblance is intensional, since the underlying idea is indeed similar. It has been known for centuries that a convex polytope can be dissected into simplices. Tverberg shows that such a dissection can be obtained sequentially, by cutting a

[1] In general, an unbounded convex polyhedron is also possible, but in fact for polytopes P whose facets lie in hyperplanes in general position, one can always avoid this by choosing an appropriate F. This is an easy corollary of the Helly theorem (see e.g. [Pak, §1]).

polytope with hyperplanes. In a similar manner we show that the unfolding which can be obtained sequentially, by cutting out extra pieces every time a new hyperplane is introduced.

6.2. An interesting interpretation of the main theorem was proposed by Ezra Miller.[2] Think of a convex polytope P given as a gift inside a box Q, which is a bigger polytope. Then one can gift wrap P in a paper in which Q was previously gift wrapped (presumably, to be regifted later).

6.3. Subdivision of the facet F into X_i given in the proof of the theorem is in fact a weighted analogue of the dual Voronoi subdivision (see [Aur, For]). As a consequence, computing this subdivision can be done more efficiently, both theoretically and practically.

6.4. Following the idea of Example 5.3, it would be interesting to find the upper bound for the number of disconnected pieces of a cutout of a polyhedron with n facets. It would be particularly interesting if this number is polynomial rather than exponential, at least in $\mathbb{R}^3$.

6.5. In the spirit of the remark under Theorem 1.1, it would be interesting to give a simple more direct proof of the monotonicity $\text{area}(P) < \text{area}(Q)$ for 3-dimensional convex polytopes $P \subset Q$. In fact, it would be interesting to investigate whether one can keep the number of pieces polynomial (in the total number of faces of P and Q). Note that although there is an elementary proof of the Bolyai–Gerwien theorem (see e.g. [Pak, §15]), it does not produce polynomial size dissections of polygons.

In a similar direction, for convex polyhedra $P \subset Q$, there is a classical monotonicity of the *total mean curvature*: $M(P) < M(Q)$ (see e.g. [BZ, §7] and [Pak, §28]). It would be interesting to prove this inequality in $\mathbb{R}^3$, by a direct argument in the spirit of Theorem 1.1.

6.6. The collapsing walls lemma (Lemma 2.1) in the special case of a pyramid in $\mathbb{R}^3$, was first proposed in a preprint [PP] by the authors, as a generalization of the following classical result: every point in a convex polytope has an orthogonal projection into the interior of a facet.

To see how the lemma implies the above result, suppose $P \subset \mathbb{R}^d$ is a pyramid over a convex polytope $F \subset H$ with a very large height. Then all facets $G \neq F$ in P are adjacent to F and are nearly orthogonal at $E_G = F \cap G$. The lemma then implies that every point $z \in F$ has an orthogonal projection into the interior of some facet E_G of F. We refer to [Pak, §9] for a simple proof and various extensions of this result.

6.7. In connection with Lemma 2.1, let us note the following much easier analogue. Suppose the walls of the polytope P collapse "outside" of the facet F. We claim that this gives pairwise disjoint (but not necessarily continuous) set of cutouts (see Figure 6).

[2]Personal communication.

Formally, let P be a convex polytope. Fix a facet F of P, and assume that no facet $G \neq F$ of P is parallel to F. For every facet G of P, different from F, denote by $\Phi^{out}_{F,G}$ the affine transformation which is the rotation about $H_F \cap H_G$ of H_G onto H_F. The rotation is performed in the direction dictated by P, so that throughout the rotation H_G *does not* intersect the interior of P. Then the sets $\Phi^{out}_{F,G}(G)$, for all facets G of P, have pairwise disjoint (relative) interiors. We leave the proof to the reader.

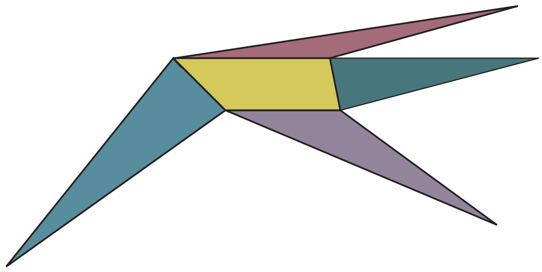

FIGURE 6. Walls of a pyramid collapsing outside the base do not intersect.

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