VANISHING OF SCHUBERT COEFFICIENTS IN PROBABILISTIC POLYNOMIAL TIME

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Abstract. The *Schubert vanishing problem* asks whether Schubert structure constants are zero. We give a complete solution of the problem from an algorithmic point of view, by showing that Schubert vanishing can be decided in probabilistic polynomial time.

1. Introduction

1.1. **Vanishing of Schubert coefficients.** Determining Schubert structure constants (*Schubert coefficients*) is one of the oldest and most celebrated problems in enumerative geometry, going back to Schubert's original work in 1870s, see [Sch79]. Motivated in part by *Hilbert's 15th Problem* aiming to make Schubert's work rigorous (see [Kle76]), the area of Schubert calculus has exploded and developed rich connections with representation theory and algebraic combinatorics (see e.g. [AF24, BGP25, Knu22]).

In this paper we study the *Schubert vanishing problem* which asks whether Schubert coefficients are zero. This problem has remained a major challenge for decades and remained unresolved despite significant study (see §1.2, §1.3 and §5.1). We resolve it algorithmically, by showing that deciding Schubert vanishing can be done in probabilistic polynomial time. This is an ultimate result of a series of our previous papers [PR24a, PR24b, PR25b].

We start with a general setup, see e.g. [BH95, AF24] for the background. Let G be a simply connected semisimple complex Lie group. Take $\mathsf{B} \subset \mathsf{G}$ and $\mathsf{B}_- \subset \mathsf{G}$ to be the Borel subgroup and opposite Borel subgroup, respectively. The *torus subgroup* is defined as $\mathsf{T} = \mathsf{B} \cap \mathsf{B}_-$. The *Weyl group* is defined as the normalizer $\mathcal{W} \cong N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$. The *Bruhat decomposition* states that

$$\mathsf{G} \, = \, \bigsqcup_{w \in \mathcal{W}} \mathsf{B}_- \, \dot{w} \, \mathsf{B},$$

where $\dot{w}$ is the preimage of w in the normalizer $N_{\mathsf{G}}(\mathsf{T})$.

The *generalized flag variety* is defined as G/B. Recall that G/B has finitely many orbits under the left action of B_-. These are called *Schubert cells* and denoted Ω_w. Schubert cells are indexed by the Weyl group elements $w \in \mathcal{W}$.

The *Schubert varieties* X_w are the Zariski closures of Schubert cells Ω_w. The *Schubert classes* $\{\sigma_w\}_{w \in \mathcal{W}}$ are the Poincaré duals of Schubert varieties. These form a $\mathbb{Z}$–linear basis of the cohomology ring $H^*(\mathsf{G}/\mathsf{B})$. The *Schubert coefficients* $c^w_{u,v}$ are defined as structure constants:

$$\sigma_u \smile \sigma_v \, = \, \sum_{w \in \mathcal{W}} c^w_{u,v} \, \sigma_w \, . \tag{1.1}$$

Thus $c^w_{u,v} = [\sigma_{\mathrm{id}}] \, \sigma_u \smile \sigma_v \smile \sigma_{w_\circ w}$, where $w_\circ$ is the *long word* in $\mathcal{W}$. These are a special case of

$$c(u_1, u_2, \ldots, u_k) \, := \, [\sigma_{\mathrm{id}}] \, \sigma_{u_1} \smile \sigma_{u_2} \smile \cdots \smile \sigma_{u_k} \, , \tag{1.2}$$

where $k \geq 3$. In particular, we have $c^w_{u,v} = c(u, v, w_\circ w)$. By commutativity of $H^*(\mathsf{G}/\mathsf{B})$, Schubert coefficients $c(u_1, \ldots, u_k)$ exhibit S_k-symmetry.

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By *Kleiman transversality* [Kle74], the coefficients $c(u_1,\ldots,u_k)$ count the number of points in the intersection of generically translated Schubert varieties:

$$c(u_1,\ldots,u_k) \,=\, \#\big\{\, X_{u_1}\big(F^{(1)}_\bullet\big)\cap\cdots\cap X_{u_k}\big(F^{(k)}_\bullet\big)\,\big\}, \tag{1.3}$$

where $F^{(i)}_\bullet$ are generic flags. In particular, we have $c(u_1,\ldots,u_k)\in\mathbb{N}$. The *Schubert vanishing problem* is the decision problem

$$\textsc{SchubertVanishing} := \big\{c(u_1,\ldots,u_k) =^? 0\big\},$$

where $u_1,\ldots,u_k\in\mathcal{W}$. We consider the problem only for classical types $Y\in\{A,B,C,D\}$, and use notation $\textsc{SchubertVanishing}(Y)$ to denote the Schubert vanishing problem in type Y.[1] These correspond to considering groups $\mathsf{G}\in\{\mathsf{SL}_n(\mathbb{C}),\mathsf{SO}_{2n+1}(\mathbb{C}),\mathsf{Sp}_{2n}(\mathbb{C}),\mathsf{SO}_{2n}(\mathbb{C})\}$, respectively. We will use $Y_n\in\{A_n,B_n,C_n,D_n\}$, where $n=n(\mathsf{G})$ is the *rank* of Lie group G. Recall that $c(u_1,\ldots,u_k)=0$ for $k>\ell(w_\circ)$, assuming each $u_i\neq\mathrm{id}$. Thus we consider only the case when $k\le\ell(w_\circ)$, where ℓ denotes the *length function* in $\mathcal{W}$.

Theorem 1.1 (Main theorem)**.** *For each $Y\in\{A,B,C,D\}$, the problem* $\textsc{SchubertVanishing}(Y)$ *can be decided in probabilistic polynomials time. More precisely, for all $k\ge 3$ and $\varepsilon>0$, there is a probabilistic algorithm which inputs elements $u_1,\ldots,u_k\in Y_n$ and after $O\big(kn^{8.75}\log\frac{1}{\varepsilon}\big)$ arithmetic operations outputs either :*

- *$c(u_1,\ldots,u_k)>0$, which holds with probability $P=1$, or*
- *$c(u_1,\ldots,u_k)=0$, which holds with probability $P>1-\varepsilon$.*

The proof is based on Purbhoo's criterion for Schubert vanishing [Pur06, Cor. 2.6]. We show that the criterion is equivalent to the degeneracy of certain determinant with polynomial entries. It can then be tested in polynomial time whether this determinant is identically zero by a random substitution of variables. This explains the one-sided error in the algorithm, since finding a nonzero evaluation of the determinant *guarantees* positivity of the Schubert coefficient.

1.2. **Geometric background and motivation.** The literature on Schubert calculus is much too large to be reviewed here. We refer to [AF24, Ful97, Knu16, Man01] for geometric and combinatorial introductions, and to [Knu22] for a recent overview. Let us single out [BS00] and [BK06], where Schubert vanishing was studied in the context of representation theory and *Horn's inequalities* describing possible spectra of three Hermitian matrices satisfying $A+B=C$.

Now, recall the *Grassmannian structure coefficients*, in which the Schubert classes are pulled back from Grassmannians G/P, where P is a maximal parabolic subgroup. As with the full flag varieties, these Grassmannians have decompositions into Schubert cells $[X_\lambda]$. Taking the pullback by the projection $\pi:\mathsf{G}/\mathsf{B}\twoheadrightarrow\mathsf{G}/\mathsf{P}$ embeds $H^*(\mathsf{G}/\mathsf{P})$ as a subring of $H^*(\mathsf{G}/\mathsf{B})$. Thus by specializing our algorithm to the appropriate Grassmannian elements, we obtain a probabilistic poly-time algorithm to decide the vanishing of ordinary and maximal isotropic Grassmannian structure constants. We use notation $c^\nu_{\lambda,\mu}(Y)$ to denote these constants in type Y for $k=3$.

In type A, we have $\mathsf{G}/\mathsf{P}\simeq\mathsf{Gr}_{k,n}$ is the ordinary Grassmannian, the space of k-dimensional planes in $\mathbb{C}^n$. Here, the Schubert structure constants $c^\nu_{\lambda\mu}=c^\nu_{\lambda\mu}(A)$ are the *Littlewood–Richardson* (LR) *coefficients*, which are extremely well studied in the literature, see e.g. [Ful97, Sta99]. Famously, LR coefficients are the structure constants of Schur polynomials and have several combinatorial interpretations, see a long list in [Pak24, §11.4].

In a major breakthrough, Knutson and Tao [KT99] established the *saturation conjecture* in type A:

$$c^\lambda_{\mu\nu}(A)>0 \quad\Longleftrightarrow\quad c^{t\lambda}_{t\mu,t\nu}(A)>0 \ \ \textit{for any}\ \ t\ge 1. \tag{1.4}$$

[1]For non-classical types E_6, E_7, E_8, F_4 and G_2, there is only a finite number of Schubert coefficients, so the problem is uninteresting from the computational complexity point of view.

In [DM06, MNS12], the authors independently observed that the saturation property (1.4) implies that the vanishing of LR–coefficients can be solved by a linear program. This gives:

Theorem 1.2 ([KT99, DM06, MNS12]). *The vanishing of LR-coefficients* $\{c^{\lambda}_{\mu\nu}(A) =^? 0\}$ *can be decided in deterministic polynomials time.*

This result is exceptional and despite numerous conjectural generalizations (see e.g. [Kir04]), it does not seem to extend much beyond this narrow setting, see §5.3. Below we give a complexity comparison of our probabilistic approach and the deterministic approach as in the theorem.

The case of types B–D is also quite interesting and extensively studied. We refer to [BH95] for detailed overview of Schubert calculus in these types, to [PS08] for recursive Horn type formulas for the vanishing problem, and to [Pur06] for detailed combinatorial investigations of the vanishing. See also [Sea16] for discussion on the combinatorial formulas for the structure coefficients for other choices of non-maximal isotropic Grassmannians in types B–D.

In this case, we highlight those G/P that are maximal isotropic Grassmannians with respect to the appropriate skew–symmetric or symmetric bilinear form. In type C, we have $c^{\nu}_{\lambda,\mu}(C)$ as the structure constants of Q-Schur polynomials Q_λ [Pra91]. Similarly, in types B/D, we have $c^{\nu}_{\lambda,\mu}(B)$ and $c^{\nu}_{\lambda,\mu}(D)$ as the structure constants of P-Schur polynomials P_λ, proven in [Pra91].

As noted in [RYY22, Remark 7.7], the type B/C structure coefficients do not satisfy saturation.[2] Thus the arguments of [DM06, MNS12] may not be mirrored directly. In fact, our Theorem 1.1 gives the first poly-time algorithm for the vanishing of LR–coefficients in other types:

Corollary 1.3. *For types* $Y \in \{A, B, C, D\}$*, the problem*

$$\text{LRVanishing}(Y) \ := \ \{c^{\lambda}_{\mu\nu}(Y) =^? 0\} \tag{1.5}$$

can be decided in probabilistic polynomials time when the input λ, μ, ν *is in unary. More precisely, for* Y_n*, there is a probabilistic algorithm with a one-sided error, and* $O(n^{8.75})$ *expected time.*

The corollary addresses Problem 7.8 in [RYY22] and gets close to completely resolving it in the positive, see below. The unary input comes from the translation of Schubert problems into Grassmannian notation; in Theorem 1.2 the usual (binary) input is used, see §5.2 for further details.

1.3. **Complexity background and implications.** The algorithmic and complexity aspects of the Schubert vanishing problem have also been heavily studied, both explicitly in the computer algebra literature and implicitly in the algebraic combinatorics literature. In fact, the computational hardness even for the well-studied *2-step flag variety* setting remains challenging, see e.g. [ARY19, Question 4.2]. We refer to our extensive overview in [PR24a, v2, §1.6], to [SY22, §5] for a combinatorial introduction to Schubert vanishing tests in type A, and to [BV08, §5.2] for some motivating comments.

Below we give a brief discussion of prior complexity work on the *Schubert coefficient problem* (the problem of computing Schubert coefficients) and the Schubert vanishing problem. We assume the reader is familiar with standard complexity classes, which can be found, e.g., in [AB09, Gol08].

The Schubert coefficient problem is known to be in $\mathsf{GapP} = \#\mathsf{P} - \#\mathsf{P}$, see [Pak24, Prop. 10.2] for type A (see also [PR25c]) and [PR25a, Cor. 4.2] for other types.[3] Whether the Schubert coefficient problem is in $\#\mathsf{P}$ is a major open problem in the area, see e.g. [Sta00, Problem 11], [Pak24, Conj. 10.1] and [BGP25, O.P. 3.129].[4] This problem has been resolved in a number of special cases (see an overview in [Knu22, PR24a]), implying that Schubert vanishing is in coNP when restricted to each such case.

[2] We were unable to find in the literature a counterexample in type D.

[3] In a combinatorial language, this means that there are *signed rules* for Schubert coefficients.

[4] In a combinatorial language, this problem asks for a *combinatorial interpretation* (rule) for Schubert coefficients.

Notably, the classical *Littlewood–Richardson rule* (see e.g. [Sta99, §A1.3]) and the *shifted LR rules* of Worley, Sagan and Stembridge (see e.g. [CNO14]), show that $\text{LRVANISHING}(Y) \in \mathsf{coNP}$ for $Y \in \{A, B, C, D\}$.[5] Additionally, in the language of *root games*, Purbhoo showed that Schubert vanishing is in NP in some special cases [Pur06].

In [ARY19, Question 4.3], the authors asked if $\text{SCHUBERTVANISHING}(A)$ is NP-hard. In the opposite direction, the authors conjectured that the problem is coNP-hard [PR24a, Conj. 1.6]. In a major advance [PR25b, Thm 1.1], we showed that

$$(*) \qquad \text{SCHUBERTVANISHING}(Y) \in \mathsf{AM} \cap \mathsf{coAM} \quad \text{for all} \quad Y \in \{A, B, C, D\},$$

assuming the *Generalized Riemann Hypothesis* (GRH). Prior to [PR24a, PR25b], it was believed that the problem is not in PH. In fact, unconditionally (without the GRH assumption), prior to this work, the best known upper bound for was $\text{SCHUBERTVANISHING}(Y) \in \mathsf{PSPACE}$, even when restricting to type A. The inclusion is already nontrivial and follows from the GapP formulas mentioned above.

In complexity theoretic language, our Main Theorem 1.1 proves (unconditionally) that

$$(**) \qquad \text{SCHUBERTVANISHING}(Y) \in \mathsf{coRP} \quad \text{for all} \quad Y \in \{A, B, C, D\}.$$

This is very low in the polynomial hierarchy, and we remind the reader of standard inclusions:

$$\mathsf{P} \subseteq \mathsf{coRP} \subseteq \mathsf{BPP} \cap \mathsf{coNP} \subseteq \mathsf{NP} \cap \mathsf{coNP} \subseteq \mathsf{AM} \cap \mathsf{coAM} \subseteq \Sigma_2^{\mathsf{p}} \cap \Pi_2^{\mathsf{p}} \subseteq \mathsf{PH} \subseteq \mathsf{PSPACE}.$$

Corollary 1.3 similarly gives $\text{LRVANISHING}(Y) \in \mathsf{coRP}$ for the unary input, but the result is new only for $Y \in \{B, C, D\}$ as Theorem 1.2 gives $\text{LRVANISHING}(A) \in \mathsf{P}$.

In conclusion, we mention that $\mathsf{P} = \mathsf{RP} = \mathsf{coRP} = \mathsf{BPP}$ under standard derandomization assumptions [IW97], see also a discussion in [PR24b]. In the opposite direction, recall that Main Theorem 1.1 is proved via *Polynomial Identity Testing* (PIT), one of the main obstacles for derandomization [SY09]. It is thus unlikely that our approach can be used to show that Schubert vanishing is in P.

2. Preliminaries

2.1. **Notation.** We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$. Unless stated otherwise, the underlying field is always $\mathbb{C}$. Let $e_1, \ldots, e_n$ denote the standard basis in $\mathbb{C}^n$. We use bold letters $\boldsymbol{x} = (x_1, x_2, \ldots)$ for collections of variables, and $\overrightarrow{x}$ for their evaluations. We write $f \equiv g$ for the equality of polynomials $f, g \in \mathbb{C}[\boldsymbol{x}]$.

For a Weyl group $\mathcal{W}$, we use $\ell(w)$ to denote the *length* of the element $w \in \mathcal{W}$. The *long word* is an element $w_\circ \in \mathcal{W}$ of maximal length. We assume the reader is familiar with standard notation of barred and unbarred elements of the Weyl group $\mathcal{W} \simeq S_n \ltimes \mathbb{Z}_2^n$ in types B and C. We view this Weyl group as the group of *signed permutations* of $[n]$, see e.g. [AF24, §14.1.1] for further details.

Recall the following standard notation for almost simple algebraic groups. We have the *special linear group* $\mathsf{SL}_n(\mathbb{C})$, the *odd special orthogonal group* $\mathsf{SO}_{2n+1}(\mathbb{C})$, the *symplectic group* $\mathsf{Sp}_{2n}(\mathbb{C})$ and the *even special orthogonal group* $\mathsf{SO}_{2n}(\mathbb{C})$. These groups correspond to *root systems* A_n, B_n, C_n and D_n, and are called *groups of type* A, B, C and D, respectively.

To distinguish the types, we use parentheses or subscripts in LR and Schubert coefficients, e.g. $c^\lambda_{\mu\nu}(A)$ and $c_{\langle A \rangle}(u, v, w)$. We omit the dependence on the type when it is clear from the context.

[5] Although usually stated for the unary input, the result extends to the binary input, cf. [Pan24, §5.2].

2.2. **Polynomial identity testing.** In this paper we use the following textbook result:

Lemma 2.1 (Schwarz–Zippel Lemma). *For a field $\mathbb{F}$, let $Q \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ be a non-zero polynomial with degree $d \geq 0$ over $\mathbb{F}$. Take $S \subset \mathbb{F}$ be a finite set. Then:*

$$\mathbf{P}\big[Q(c_1, c_2, \ldots, c_n) = 0\big] \ \leq \ \frac{d}{|S|}\,,$$

where the probability is over random, independent and uniform choices of $c_1, c_2, \ldots, c_n \in S$.

This lemma is frequently used to test whether a polynomial given by an arithmetic circuit is identically zero. We refer to [SY09] for an extensive overview of complexity applications and many references.

2.3. **Types B and C.** We will need the following well-known result relating the Schubert vanishing in two types:

Proposition 2.2. SchubertVanishing(B) *coincides with* SchubertVanishing(C).

Proof. First we note that both $\mathsf{SO}_{2n+1}(\mathbb{C})$ and $\mathsf{Sp}_{2n}(\mathbb{C})$ share the hyperoctahedral group as their Weyl group $\mathcal{W}$. We interpret $\mathcal{W}$ as signed permutations of $[n]$. Let $\zeta(\pi)$ count the number of sign changes in the signed permutation $\pi \in \mathcal{W}$. It follows from [BH95, Thm 3], that

$$c_{\langle B \rangle}(u_1, \ldots, u_k) \ = \ 2^a \, c_{\langle C \rangle}(u_1, \ldots, u_k), \tag{2.1}$$

where

$$a \, := \, \zeta(w_\circ u_k) \, - \, \zeta(u_1) \, - \, \ldots \, - \, \zeta(u_{k-1}).$$

This implies the result. ∎

2.4. **Purbhoo's criterion.** Take type $Y \in \{A, B, C, D\}$ and let $\mathsf{G} = \mathsf{G}_Y$ be the simply connected semisimple complex Lie group of type Y. Here G is a matrix group inside an ambient vector space V. Let N be the subgroup of unipotent matrices, giving

$$\mathsf{N} \subset \mathsf{B} \subset \mathsf{G} \subset V.$$

Let $\mathfrak{n}$ denote the Lie algebra of N. Again, we view $\mathfrak{n}$ as a subspace of V. Finally, for a Weyl group element $w \in \mathcal{W}$, let $Z_w := \mathfrak{n} \cap (w \mathsf{B}_- w^{-1})$.

Lemma 2.3 (Purbhoo's criterion [Pur06, Cor. 2.6]). *For generic $\rho_1, \ldots, \rho_k \in \mathsf{N} \subset \mathsf{G}$ and $u_1, \ldots, u_k \in \mathcal{W}$, we have:*

$$c(u_1, \ldots, u_k) \, > \, 0 \quad \Longleftrightarrow \quad \rho_1 R_{u_1} \rho_1^{-1} \, + \, \ldots \, + \, \rho_k R_{u_k} \rho_k^{-1} = \rho_1 R_{u_1} \rho_1^{-1} \, \oplus \, \ldots \, \oplus \, \rho_k R_{u_k} \rho_k^{-1}.$$

Generalizing the number of inversions condition, the *dimension condition* says that

$$c(u_1, \ldots, u_k) \, = \, 0 \quad \text{if} \quad \ell(u_1) + \cdots + \ell(u_k) \, \neq \, \dim(\mathfrak{n}). \tag{2.2}$$

Thus it suffices to restrict to the case $\ell(u_1) + \ldots + \ell(u_k) = \dim(\mathfrak{n})$. Then we consider the following specialization of Lemma 2.3:

Corollary 2.4. *For generic $\rho_1, \ldots, \rho_k \in \mathsf{N} \subset \mathsf{G}$ and $u_1, \ldots, u_k \in \mathcal{W}$ such that $\ell(u_1) + \ldots + \ell(u_k) = \dim(\mathfrak{n})$, we have:*

$$c(u_1, \ldots, u_k) \, > \, 0 \quad \Longleftrightarrow \quad \rho_1 R_{u_1} \rho_1^{-1} \, + \, \ldots \, + \, \rho_k R_{u_k} \rho_k^{-1} = \mathfrak{n}.$$

Using Corollary 2.4, it suffices to determine the dimension of the vector space

$$H \, := \, \rho_1 R_{u_1} \rho_1^{-1} \, + \, \ldots \, + \, \rho_k R_{u_k} \rho_k^{-1} \quad \text{for generic} \quad \rho_i\,.$$

3. General setup

In the setting of Purbhoo's criterion, we describe how to construct bases for R_{u_i} in §3.1. Then, in §3.2, we describe how to construct matrices ρ_i. In §3.3, we combine these constructions to obtain bases for each summand $\rho_i R_{u_i} \rho_i^{-1}$. From these, we obtain vectors π_j which generate H.

3.1. **Root systems.** The Weyl group $\mathcal{W}$ is generated by reflections r_γ, indexed by roots γ in a *root system* Φ. This root system Φ may be partitioned in terms of its positive and negative roots: $\Phi = \Phi_+ \sqcup \Phi_-$. In Table 1, we describe the positive roots Φ_+ in term of vectors e_i. Here e_i denotes the i-th elementary basis vector in the appropriate $\mathbb{C}^m$.

G	Φ_+	$U(\mathsf{G})$
SL_n	$\{e_i - e_j \,:\, 1 \le i < j \le n\}$	$\{(i,j) \,:\, 1 \le i < j \le n\}$
SO_{2n+1}	$\{e_i \pm e_j \,:\, 1 \le i < j \le n\} \cup \{e_i \,:\, i \in [n]\}$	$\{(i,j) \,:\, 1 \le i < j \le 2n+1-i\}$
SO_{2n}	$\{e_i \pm e_j \,:\, 1 \le i < j \le n\}$	$\{(i,j) \,:\, 1 \le i < j \le 2n-i\}$

Table 1. Positive roots and corresponding matrix entries.

Define the integer $N(\mathsf{G})$, where

$$N(\mathsf{G}) = \begin{cases} n & \text{if } \mathsf{G} = \mathsf{SL}_n(\mathbb{C}), \\ 2n+1 & \text{if } \mathsf{G} = \mathsf{SO}_{2n+1}(\mathbb{C}), \\ 2n & \text{if } \mathsf{G} = \mathsf{SO}_{2n}(\mathbb{C}). \end{cases}$$

To each $\gamma \in \Phi_+$, we may associate an $m \times m$ matrix, where $m = N(\mathsf{G})$. We define a distinguished subset $U(\mathsf{G}) \subset [m] \times [m]$ as outlined in Table 1. We then construct a bijection $\phi : U(\mathsf{G}) \to \Phi_+$, detailed below.

(A) For SL_n take $\phi(i,j) := e_i - e_j$.

(B) For SO_{2n+1} take

$$\phi(i,j) := \begin{cases} e_i + e_j & \text{if } j \le n, \\ e_i - e_{2n+2-j} & \text{if } n+1 < j, \\ e_i & \text{if } j = n+1. \end{cases}$$

(D) For SO_{2n} take

$$\phi(i,j) := \begin{cases} e_i + e_j & \text{if } j \le n, \\ e_i - e_{2n+1-j} & \text{if } n < j. \end{cases}$$

For $\gamma \in \Phi_+$, set E'_γ to be the $m \times m$ matrix with a 1 in position $\phi^{-1}(\gamma)$ and 0 elsewhere. For SL_n take $\mathrm{E}_\gamma := \mathrm{E}'_\gamma$. For SO_{2n+1} and SO_{2n}, define $\mathrm{E}_\gamma := \mathrm{E}'_\gamma - \mathrm{D}_m (\mathrm{E}'_\gamma)^T \mathrm{D}_m$, where D_m is the antidiagonal matrix.

3.2. **Generic unipotent subgroup elements.** Let $m := N(\mathsf{G})$. We now describe how to construct an upper unitriangular $m \times m$ matrix K which lies in $\mathsf{N} \subset \mathsf{B} \subset \mathsf{G}$. Define:

$$\kappa_{ij} = \begin{cases} \alpha_{ij} & \text{if } i < j \text{ and } (i,j) \in U(\mathsf{G}), \\ z_{ij} & \text{if } i < j\,,\ (i,j) \notin U(\mathsf{G}), \text{ and } i + j \ne m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Here we treat α_{ij} as parameters set $z_{ij} = -\alpha_{(m+1-j)(m+1-i)}$. Let $\kappa = (\kappa_{ij})$.

For $\mathsf{G} = \mathsf{SL}_n(\mathbb{C})$, set $\mathrm{K} := \mathrm{I}_m + \kappa$. For $\mathsf{G} = \mathsf{SO}_{2n+1}(\mathbb{C})$ and $\mathsf{G} = \mathsf{SO}_{2n}(\mathbb{C})$, we use the Cayley transform to construct $\mathrm{K} \in \mathsf{G}$ from κ. In particular, set $\mathrm{K} = (\mathrm{I}_m + \kappa)^{-1}(\mathrm{I}_m - \kappa)$. By construction, K is upper unitriangular. It is straightforward to confirm $\mathrm{K} \in \mathsf{G}$ and that such elements are dense in N. To check $\mathrm{K} \in \mathsf{G}$, one need only confirm $\mathrm{K}^T \cdot \mathrm{D}_m \cdot \mathrm{K} = \mathrm{D}_m$, and note that $\det(\mathrm{K}) = 1$.

3.3. **Main construction.** Let $m = N(\mathsf{G})$. With Corollary 2.4 in mind, consider the vector space

$$H \, = \, \rho_1 R_{u_1} \rho_1^{-1} \, + \, \ldots \, + \, \rho_k R_{u_k} \rho_k^{-1}.$$

Let $d := \dim \mathsf{G}/\mathsf{B}$ and note $\dim \mathfrak{n} = d$. Using the dimension condition, we assume

$$\ell(u_1) \, + \, \ldots \, + \, \ell(u_k) \, = \, d. \tag{3.1}$$

Additionally, we assume $\ell(u_i) \geq 1$ for each $i \in [k]$. Combining these assumptions gives $k \leq d$. We also assume $k \geq 3$, as taking $k \leq 2$ is trivial.

First recall the definition, $Z_w = \mathfrak{n} \cap (w\mathsf{B}_- w^{-1})$ for $w \in \mathcal{W}$. Alternatively, Z_w is the subspace of $\mathfrak{n}$ generated by basis elements E_γ (see §3.1) for $\gamma \in \Phi_+(w)$, where

$$\Phi_+(w) \, := \, \big\{ \beta \in \Phi_+ \, : \, w^{-1}\beta \notin \Phi_+ \big\}.$$

Now we construct bases for the spaces R_{u_i} for $i \in [k]$:

$$S_{u_i} \, := \, \big\{ x_{\gamma,i} \, \mathrm{E}_\gamma \, : \, \gamma \in \Phi_+(u_i) \big\}.$$

Note the number of positive roots $|\Phi_+| = d = O(m^2)$. Let $\boldsymbol{x} := \big\{ x_{\gamma,i} \big\}$ be the set of those variables appearing in the above collection. Since (2.2) holds, we have the following:

$$\sum_{i=1}^{k} \ell(u_i) \, = \, \sum_{i=1}^{k} \dim(R_{u_i}) \, = \, \sum_{i=1}^{k} \big| S_{u_i} \big| \, = \, d. \tag{3.2}$$

Then construct generic matrices $\rho_1, \ldots, \rho_k \in \mathsf{N}$ as outlined in §3.2, using the formal parameters $\alpha_{j\ell}^{(i)}$. Define $\boldsymbol{\alpha} := \big\{ \alpha_{j\ell}^{(i)} \big\}$ to be the set of parameters, respectively, appearing in some ρ_i, where $i \in [k]$. Then $|\boldsymbol{\alpha}| \leq k \cdot m^2 = O(m^4)$.

For each $i \in [k]$, construct bases for summands $\rho_i R_{u_i} {\rho_i}^{-1}$:

$$T_{u_i} \, := \, \rho_i S_{u_i} {\rho_i}^{-1} \, = \, \big\{ \rho_i \cdot g \cdot {\rho_i}^{-1} \, : \, g \in S_{u_i} \big\}.$$

Using (3.2), we find $|T_{u_1}| + \ldots + |T_{u_k}| = d$.

Let τ be the map on $m \times m$ matrices defined by restricting to the matrix entries in positions $U(\mathsf{G})$. By their definition, matrices in $\mathfrak{n}$ are determined by their entries in positions $U(\mathsf{G})$. Thus, $\dim(T_{u_i}) = \dim\big(\tau(T_{u_i})\big)$ for each $i \in [k]$. Take

$$T \, := \, \bigcup_{i \in [k]} \tau(T_{u_i}),$$

and let $T = \{\pi_i \, : \, i \in [d]\}$. Using the fact that $|U(\mathsf{G})| = d$, we view each $\pi_i \in T$ as a d-vector.

Finally, consider the $d \times d$ matrix M with column vectors π_j. Using Purbhoo's criterion, the Schubert vanishing problem reduces to determining if the matrix M is singular. Since M is a matrix with entries polynomials in $\mathbb{Z}[\boldsymbol{\alpha}, \boldsymbol{x}]$, we can apply the Schwarz–Zippel Lemma 2.1. We do this carefully in the next section. We also note that the construction above in similar to that in [PR25b].

4. Proof of the main theorem

By Proposition 2.2, it suffices to consider only types A, B and D. For clarity, we consider types A and types B/D separately.

4.1. **Algorithm for SL_n.** Considering the converse of Corollary 2.4, we let:

$$c(u_1,\ldots,u_k) \,=\, 0 \quad\iff\quad \rho_1 R_{u_1}\rho_1^{-1} \,+\, \ldots \,+\, \rho_k R_{u_k}\rho_k^{-1} \,\subsetneq\, \mathfrak{n}. \tag{4.1}$$

Construct M as in §3.3. Then the right-hand side of (4.1) holds if and only if $\det(M) \equiv 0$. Here, the condition $\det(M) \equiv 0$ indicates that $\det(M)$ is identically zero. We denote $D(\boldsymbol{\alpha},\boldsymbol{x}) := \det(M)$.

In this case, $D(\boldsymbol{\alpha},\boldsymbol{x}) \in \mathbb{Z}(\boldsymbol{\alpha})[\boldsymbol{x}]$. In fact, we have $D(\boldsymbol{\alpha},\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{x}]$ since

$$(\rho_i^{-1})_{j_1,j_2} \,=\, \frac{1}{\det(\rho_i)}\, C^{(i)}(j_2,j_1), \tag{4.2}$$

where $C^{(i)}(j_2,j_1)$ is the cofactor of ρ_i for $j_1,j_2 \in [n]$. Further, by construction, we have $\det(\rho_i) = 1$, so $(\rho_i^{-1})_{j_1,j_2} \in \mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{x}]$. Note that $D(\boldsymbol{\alpha},\boldsymbol{x}) \equiv 0$ over $\mathbb{Q}(\boldsymbol{\alpha})$ if and only if $D(\boldsymbol{\alpha},\boldsymbol{x}) \equiv 0$ over $\mathbb{Q}$, now viewing $D(\boldsymbol{\alpha},\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{x}]$. Thus onwards, we take $\boldsymbol{\alpha}$ and $\boldsymbol{x}$ as variables.

Again, by (4.2), the expressions in $(\rho_i)^{-1}$ expanded in terms of α will be a polynomial of degree $n-1$. Thus matrix entries in M are polynomials of degree $n+1$. Thus $D(\boldsymbol{\alpha},\boldsymbol{x})$ has degree $d(n+1)$.

Pick values for $\overrightarrow{x}$, $\overrightarrow{\alpha}$ randomly over integers in $[p]$, where $p \in \mathbb{Z}_{>0}$. Let $\overrightarrow{\rho_i}$ denote ρ_i evaluated at $\overrightarrow{\alpha}$. Then compute $(\overrightarrow{\rho_i})^{-1}$. By the Schwartz–Zippel Lemma 2.1, if $D(\boldsymbol{\alpha},\boldsymbol{x}) \not\equiv 0$, we have:

$$\mathbf{P}\big[D(\overrightarrow{\alpha},\overrightarrow{x}) = 0\big] \,\le\, \frac{d(n+1)}{p}\,.$$

Note that if $D(\boldsymbol{\alpha},\boldsymbol{x}) \equiv 0$, we have:

$$\mathbf{P}\big[D(\overrightarrow{\alpha},\overrightarrow{x}) = 0\big] \,=\, 1.$$

We test $D(\overrightarrow{\alpha},\overrightarrow{x}) = 0$ in polynomial time for these sampled values in $[p]$. The probability of error is less than $\frac{1}{3}$ if we take $p > \frac{3}{2}n(n^2-1)$. Thus by Corollary 2.4, the algorithm for deciding SchubertVanishing(A) satisfies the bullet conditions in the Main Theorem 1.1 with $\varepsilon = \frac{1}{3}$.

4.2. **Algorithm for SO_m.** This case is similar to the case of SL_n, but we include the details for completeness. Considering the converse of Corollary 2.4, we examine the equation

$$c(u_1,\ldots,u_k) \,=\, 0 \quad\iff\quad \rho_1 R_{u_1}\rho_1^{-1} \,+\, \ldots \,+\, \rho_k R_{u_k}\rho_k^{-1} \,\subsetneq\, \mathfrak{n}. \tag{4.3}$$

Construct M as in §3.3. Let $m = N(\mathsf{G})$. Let κ_i be as in §3.2 such that $\rho_i = (\mathrm{I}_m+\kappa_i)^{-1}(\mathrm{I}_m-\kappa_i)$. Then the right-hand side of (4.3) holds if and only if $\det(M) \equiv 0$. We denote $D(\boldsymbol{\alpha},\boldsymbol{x}) := \det(M)$.

Again $D(\boldsymbol{\alpha},\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{x}]$ using (4.2) since again $\det(\rho_i) = 1$ for each $i \in [k]$. Note that $D(\boldsymbol{\alpha},\boldsymbol{x}) \equiv 0$ over $\mathbb{Q}(\boldsymbol{\alpha})$ if and only if $D(\boldsymbol{\alpha},\boldsymbol{x}) \equiv 0$ over $\mathbb{Q}$, now viewing $D(\boldsymbol{\alpha},\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{\alpha},\boldsymbol{x}]$. Thus going forward, we treat both $\boldsymbol{\alpha}$ and $\boldsymbol{x}$ as variables.

By (4.2), the expressions in ρ_i and $(\rho_i)^{-1}$ expanded in terms of α will be polynomials of degree $m-1$. Then matrix entries in M are polynomials of degree $2m+1$. Thus $D(\boldsymbol{\alpha},\boldsymbol{x})$ has degree $d(2m+1)$.

Pick values for $\overrightarrow{x}$, $\overrightarrow{\alpha}$ randomly over integers in $[p]$, where $p \in \mathbb{Z}_{>0}$. Then compute $\overrightarrow{\rho_i} = (\mathrm{I}_m + \overrightarrow{\kappa_i})^{-1}(\mathrm{I}_m - \overrightarrow{\kappa_i})$ and $(\overrightarrow{\rho_i})^{-1}$. By the Schwartz–Zippel Lemma 2.1, if $D(\boldsymbol{\alpha},\boldsymbol{x}) \not\equiv 0$, we have:

$$\mathbf{P}\big[D(\overrightarrow{\alpha},\overrightarrow{x}) = 0\big] \,\le\, \frac{d(2m+1)}{p}\,.$$

Note that if $D(\overrightarrow{\alpha},\overrightarrow{x}) \equiv 0$, we have:

$$\mathbf{P}\big[D(\overrightarrow{\alpha},\overrightarrow{x}) = 0\big] \,=\, 1.$$

We test $D(\overrightarrow{\alpha},\overrightarrow{x}) = 0$ in polynomial time for these sampled values in $[p]$. The probability of error is less than $\frac{1}{3}$ if we take $p > 3n^2(2m+1)$. Thus by Corollary 2.4, the algorithm for deciding both SchubertVanishing(B) and SchubertVanishing(D) corresponding to odd and even m, respectively, satisfy the bullet conditions in the Main Theorem 1.1 with $\varepsilon = \frac{1}{3}$.

4.3. **Algorithm outline.** For clarity, and to ease the complexity analysis below, we give a concise outline of the algorithm in all types A, B/D. Note that type B and C are equivalent by Proposition 2.2.

Input: $u_1, \ldots, u_k \in \mathcal{W}$
Decide: $\big[c(u_1, \ldots, u_k) =^? 0\big]$

- Let
$$p \;:=\; \begin{cases} \frac{3}{2}\, n(n^2-1)+1 & \text{if} \quad \mathsf{G} = \mathsf{SL}_n \\ 3\left\lfloor \frac{m}{2}\right\rfloor^2 (2m+1)+1 & \text{if} \quad \mathsf{G} = \mathsf{SO}_m \end{cases}$$
- For all $i \in [k]$:
  - Generate strictly upper triangular matrices κ_i with random entries $\alpha_{j\ell} \in [p]$, see §3.2.
  - Compute K_i using κ_i and set $\rho_{u_i} := \mathrm{K}_i$, see §3.2.
  - Compute inverse matrices $\rho_{u_i}^{-1}$.
- For all $i \in [k]$ and $\gamma \in \Phi_+(u_i)$:
  - Compute matrices $\overrightarrow{x_{\gamma,i}}\,\mathrm{E}_\gamma$ with random values $\overrightarrow{x_{\gamma,i}} \in [p]$, see §3.1.
  - Compute matrices $T_{\gamma,i} = \rho_{u_i}\big(\overrightarrow{x_{\gamma,i}}\,\mathrm{E}_\gamma\big)\rho_{u_i}^{-1}$.
  - Record the entries of $T_{\gamma,i}$ in positions $U(\mathsf{G})$ as a vector $v_{\gamma,i}$.[6]
- Let M be the matrix with column vectors $v_{\gamma,i}$, over all $\gamma \in \Phi_+(u_i)$ and $i \in [k]$.

Output:
$$\begin{cases} c(u_1, \ldots, u_k) \;=\; 0 & \text{if} \quad \det(M) = 0, \\ c(u_1, \ldots, u_k) \;>\; 0 & \text{if} \quad \det(M) \neq 0. \end{cases}$$

4.4. **Algorithm analysis and proof of Theorem 1.1.** From the discussion in §4.1 and §4.2, the Algorithm above is always correct when it outputs $[c(u_1, \ldots, u_k) > 0]$, and has a probability of error $\leq \frac{1}{3}$ when it outputs $[c(u_1, \ldots, u_k) = 0]$. Repeating the algorithm s times reduces this probability to $\frac{1}{3^s} \leq \varepsilon$ for $s = \lceil \log_3 \frac{1}{\varepsilon} \rceil$.

Now, the algorithm runs over $i \in [k]$. For each i and γ, to compute $T_{\gamma,i}$ it multiplies and takes inverses four times in type A, of matrices of size d with integer entries in $[p]$. Note $d = O(n^2)$ and $p = O(n^3)$. In types B/D, computing ρ_i requires four additional multiplications/inversions. Note that these matrices are unitriangular, thus always invertible.

Recall that matrix multiplication and matrix inversion of $m \times m$ matrices with entries in $[q]$, has cost $O\big(m^\omega \log q \log\log q\big)$ of arithmetic operations, where $2 \leq \omega < 2.3728$ is the *matrix multiplication constant*, see e.g. [Bla13, §9.4]. Note also that by the *Hadamard inequality*, the inverse matrix has entries in absolute value at most $q^m m^{m/2}$, see e.g. [BB61, §2.11].

In summary, the second loop of the algorithm uses $O(kd)$ multiplications and inversions of $d \times d$ matrices, but the size of matrix entries in the inverse matrix is $a = O\big(p^d d^{d/2}\big) = n^{O(n^2)}$. Putting everything together, the total cost of this loop is at most

$$O\left(kd \cdot d^\omega \cdot \log a \log\log a\right) \;=\; O\left(kn^2 \cdot n^{2\omega} \cdot n^2 (\log n)^2\right) \;=\; O\big(kn^{4+2\omega} (\log n)^2\big) \;=\; O\big(kn^{8.75}\big)$$

arithmetic operations. In the final step, the algorithm then computes the determinant of a $d \times d$ matrix M with entry sizes polynomial in a. A similar calculation gives a $O\big(n^{7.75}\big)$ bound for the cost of this step; the details are straightforward.

Finally, recall that we repeat the algorithm $O(\log \frac{1}{\varepsilon})$ times. Thus for the total cost of deciding the Schubert vanishing as in the theorem, is $O(kn^{8.75} \log \frac{1}{\varepsilon})$ arithmetic operations. This completes the proof of Theorem 1.1. □

[6]Formally, we need to do this under a fixed order (e.g., lexicographic order on matrix positions).

5. Final remarks

5.1. Schubert vanishing is a major problem with connections across areas, such as representation theory, category theory, matroid theory and pole placement problem in linear systems theory (see [PR24a] for many references). Let us quote Knutson's ICM paper: *"For applications (including real-world engineering applications) it is more important to know that* [Schubert] *structure constant is positive, than it is to know its actual value"* [Knu22, §1.4].

More broadly, the vanishing of structure constants in Algebraic Combinatorics plays a central role in *Geometric Complexity Theory* (GCT), as discussed at length in [Mul09, MNS12]. Notably, an important part of GCT was motivated by an observation that the saturation theorem implies that vanishing of LR–coefficients is in P, ibid. We refer to [Aar16, §6.6.3] for a high level overview of this connection. Let us mention that in [Mul09, §3.7], Mulmuley singled out Schubert coefficients as "one of the fundamental structural constants in representation theory and algebraic geometry," whose vanishing needs to be understood.

5.2. As we mentioned in §1.3, in type A the vanishing of LR coefficients (1.5) can be decided by linear programming via the saturation property. An alternative approach was given by Bürgisser and Ikenmeyer in [BI13]. Their algorithm uses flows in hive graphs and is specifically designed for the LR vanishing. While we make no effort to optimize our algorithm, below we include a brief comparison of the time complexity of these algorithms, and the algorithm we obtain in Corollary 1.3.

We assume that the three partitions λ, μ, ν are given in binary as vectors in $\mathbb{N}^\ell$, so $|\lambda| = |\mu| + |\nu|$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. Denote by $a := \log_2 \lambda_1$ the bit-size of the maximal part of the input. According to [BI13, p. 1640], the *ellipsoid method* applied to the hive polytopes given in [KT99], takes $O(\ell^{10} a \vartheta)$, where ϑ denotes the cost of arithmetic operations. They do not compute the cost of the *interior point method*, but observe that it is at least $\ell^9(a + \log \ell)\vartheta$. By contrast, the Bürgisser–Ikenmeyer (BI) algorithm takes $O(\ell^3 a \vartheta)$ [BI13, Thm 5.4].

Note that our probabilistic algorithm uses unary input, which makes the complexity not directly comparable. This is because for the standard embedding of the LR vanishing into the Schubert vanishing, we have $n = (\ell + \lambda_1)$. For comparison sake, assume that $\lambda_1 = \Theta(\ell)$. In this case, the LP methods take $O(\ell^{10+o(1)}\vartheta)$ and $O(\ell^{9+o(1)}\vartheta)$, respectively, while the BI algorithm takes $O(\ell^{3+o(1)}\vartheta)$ in this case.

Now, our algorithm in Corollary 1.3 takes $O(\ell^{8.75}\vartheta)$, i.e. slightly faster than the LP methods, but *much* slower than the BI algorithm. Note, however, that our analysis in §4.4 is not especially sharp since the matrices we are multiplying/inverting are very sparse (see Appendix A). It would be interesting to improve our analysis, especially in the Grassmannian case.

5.3. It is natural to ask if the saturation property (1.4) can be extended beyond LR coefficients. The exuberance which followed Knutson–Tao's proof led to a plethora of potential generalizations, see e.g. a large compendium in [Kir04]. Following the logic of [DM06, MNS12], such results could potentially give *deterministic* poly-time algorithms for the vanishing problems. With few notable exceptions, almost none of these potential generalizations are proved, and many have been refuted. We refer to our forthcoming paper [PR25+] for a disproof of Kirillov's conjectural saturation property for Schubert coefficients and further references.

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Appendix A. Examples

We illustrate the construction in §3.3 for two problems in $\mathsf{SL}_4(\mathbb{C})$ and for one problem in $\mathsf{SO}_7(\mathbb{C})$. In the first of the two $\mathsf{SL}_4(\mathbb{C})$ examples, the outcome determines the coefficient vanishes, and in the second, the outcome determines the coefficient is positive.

A.1. **Vanishing SL_4 example.** Take $u = 3214$, $v = 1423$, and $w = 4312$. This gives $w_\circ w = 1243$. We have:

$$\begin{aligned} \Phi_+(u) &= \{e_1 - e_2, e_1 - e_3, e_2 - e_3\}, \\ \Phi_+(v) &= \{e_2 - e_3, e_2 - e_4\}, \\ \Phi_+(w_\circ w) &= \{e_3 - e_4\}. \end{aligned}$$

Using these roots, we construct the bases S_u, S_v, and $S_{w_\circ w}$:

$$x_1\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_2\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_3\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_4\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_5\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_6\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We then build matrices ρ_i as follows:

$$\rho_1 = \begin{pmatrix} 1 & a_0 & a_1 & a_2 \\ 0 & 1 & a_3 & a_4 \\ 0 & 0 & 1 & a_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} 1 & b_0 & b_1 & b_2 \\ 0 & 1 & b_3 & b_4 \\ 0 & 0 & 1 & b_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \rho_3 = \begin{pmatrix} 1 & c_0 & c_1 & c_2 \\ 0 & 1 & c_3 & c_4 \\ 0 & 0 & 1 & c_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After computing $\rho_1 S_u {\rho_1}^{-1}$, $\rho_2 S_v {\rho_2}^{-1}$, and $\rho_3 S_{w_\circ w} {\rho_3}^{-1}$, we restrict to the strictly upper diagonal entries to build column vectors π_i. We illustrate this for the first basis element in $\rho_1 S_u {\rho_1}^{-1}$:

$$\begin{aligned} &\rho_1 x_1\,(\mathrm{E}_{e_2-e_3}){\rho_1}^{-1} = \\ &= \begin{pmatrix} 1 & a_0 & a_1 & a_2 \\ 0 & 1 & a_3 & a_4 \\ 0 & 0 & 1 & a_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a_0 & a_0a_3 - a_1 & -a_2 + a_1a_5 + a_0(a_4 - a_3a_5) \\ 0 & 1 & -a_3 & a_3a_5 - a_4 \\ 0 & 0 & 1 & -a_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_1 & -a_3x_1 & (a_3a_5 - a_4)x_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\tau} [x_1,\ -a_3x_1,\ (a_3a_5 - a_4)x_1,\ 0,\ 0,\ 0]^T \end{aligned}$$

Repeating this process for each basis element, we obtain $\{\pi_i\}$ to build the following matrix:

$$M = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 \\ -a_3x_1 & 0 & a_0x_3 & b_0x_4 & 0 & 0 \\ (a_3a_5 - a_4)x_1 & x_2 & -a_0a_5x_3 & -b_0b_5x_4 & b_0x_5 & c_1x_6 \\ 0 & -a_5x_2 & x_3 & x_4 & 0 & 0 \\ 0 & 0 & -a_5x_3 & -b_5x_4 & x_5 & c_3x_6 \\ 0 & 0 & 0 & 0 & 0 & x_6 \end{pmatrix}.$$

One can check that $\det(M) \equiv 0$ in this case, so $c^w_{u,v} = 0$.

Rather than compute $\det(M) \equiv 0$ directly, in the algorithm we instead randomly evaluate the variables above in the interval [121] to produce an evaluated matrix $\overrightarrow{M}$. Of course, we will always have $\det(\overrightarrow{M}) = 0$ in this case.

A.2. **Nonvanishing SL_4 example.** Now we take $v = 1342$ to consider the triple $u = 3214$, $v = 1342$, and $w = 4312$. Again $w_\circ w = 1243$. This time we have $\Phi_+(v) = \{e_2 - e_4, e_3 - e_4\}$. Take ρ_i as in Example A.1.

By the same process as Example A.1, we obtain $\{\pi_i\}$ to build the following matrix:

$$M = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 & 0 \\ -a_3x_1 & 0 & a_0x_3 & 0 & 0 & 0 \\ (a_3a_5 - a_4)x_1 & x_2 & -a_0a_5x_3 & b_0x_4 & b_1x_5 & c_1x_6 \\ 0 & -a_5x_2 & x_3 & 0 & 0 & 0 \\ 0 & 0 & -a_5x_3 & x_4 & b_3x_5 & c_3x_6 \\ 0 & 0 & 0 & 0 & x_5 & x_6 \end{pmatrix}.$$

We find $\det(M) \not\equiv 0$, so $c^w_{u,v} > 0$.

Now, the algorithm tests $\{\det(M) \stackrel{?}{\equiv} 0\}$. We randomly evaluate the variables above in the interval [121] to produce an evaluated matrix $\overrightarrow{M}$. With probability $> 2/3$, our random choices will result in $\overrightarrow{M}$ which will output $\det(\overrightarrow{M}) \neq 0$. Since the algorithm has one–sided error, if ***any*** evaluation $\overrightarrow{M}$ produces $\det(\overrightarrow{M}) \neq 0$, then we have $c^w_{u,v} > 0$ with certainty.

For example, take $(x_1, x_2, x_3, x_4, x_5, x_6) \leftarrow (6, 5, 4, 3, 2, 1)$ and

$$\overrightarrow{\rho_1} \leftarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \overrightarrow{\rho_2} \leftarrow \begin{pmatrix} 1 & 7 & 8 & 9 \\ 0 & 1 & 10 & 11 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \overrightarrow{\rho_3} \leftarrow \begin{pmatrix} 1 & 13 & 14 & 15 \\ 0 & 1 & 16 & 17 \\ 0 & 0 & 1 & 18 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This substitution gives $\overrightarrow{M}$ with $\det(\overrightarrow{M}) = 181440 \neq 0$, so we conclude that $c^w_{u,v} > 0$.

A.3. **An SO_7 example.** Here we illustrate the construction in §3.3 for a problems in SO_7. After the setup below, the remainder of the algorithm follows precisely like the $\mathsf{SL}_4(\mathbb{C})$ cases above.

In this example, we consider an example in the maximal isotropic Grassmannian of type B_3. Take $u = \overline{2}13$, $v = \overline{21}3$, and $w = \overline{32}1$. This gives $w_\circ w = 32\overline{1}$ since $w_\circ = \overline{321}$. We have:

$$\begin{aligned} \Phi_+(u) &= \{e_1, e_1 + e_2\}, \\ \Phi_+(v) &= \{e_1, e_2, e_1 + e_2\}, \\ \Phi_+(w_\circ w) &= \{e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}. \end{aligned}$$

Using these roots, we construct the bases S_u, S_v, and $S_{w_\circ w}$:

$$x_1\,\mathrm{E}_{e_1} = \begin{pmatrix} 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_2\,\mathrm{E}_{e_1+e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_3\,\mathrm{E}_{e_1} = \begin{pmatrix} 0 & 0 & 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad x_4\,\mathrm{E}_{e_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_5\,\mathrm{E}_{e_1+e_2} = \begin{pmatrix} 0&0&0&0&0&x_5&0\\ 0&0&0&0&0&0&-x_5\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix} \qquad x_6\,\mathrm{E}_{e_3} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&x_9&0&0&0\\ 0&0&0&0&-x_9&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$x_7\,\mathrm{E}_{e_1-e_2} = \begin{pmatrix} 0&x_7&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-x_7\\ 0&0&0&0&0&0&0 \end{pmatrix} \qquad x_8\,\mathrm{E}_{e_1-e_3} = \begin{pmatrix} 0&0&x_8&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-x_8\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

$$x_9\,\mathrm{E}_{e_2-e_3} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&x_9&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-x_9&0\\ 0&0&0&0&0&0&0 \end{pmatrix}$$

We then build matrices ρ_i as follows:

$$\kappa_1 = \begin{pmatrix} 1&a_0&a_1&a_2&a_3&a_4&0\\ 0&1&a_5&a_6&a_7&0&-a_4\\ 0&0&1&a_8&0&-a_7&-a_3\\ 0&0&0&1&-a_8&-a_6&-a_2\\ 0&0&0&0&1&-a_5&-a_1\\ 0&0&0&0&0&1&-a_0\\ 0&0&0&0&0&0&1 \end{pmatrix}$$

Here we have highlighted the matrix entries above with positions in $N(\mathsf{SO}_7)$ in blue. We now set $\rho_1 := (\mathrm{I}_7+\kappa_1)^{-1}(\mathrm{I}_7-\kappa_1)$. We then construct matrices ρ_2 and ρ_3 similarly. The remainder of the algorithm proceeds analogously as in Examples A.1 and A.2. As matrices become rather large, we omit the details.