Richard Stanley: D-finiteness of certain series associated with group algebras

Let G be a group and $\mathbb{Z}G$ its integral group algebra. For $u \in \mathbb{Z}G$ let $f_u(n) = [1]u^n$, the coefficient of the identity element of G when u^n is expanded in terms of the basis G. Set $F_u(x) = \sum_{n\geq 1} f_u(n)x^n$. If $F = F_d$, the free group on d generators, then it is known that $F_u(x)$ is algebraic. This goes back to Chomsky and Schützenberger and seems first to have been explicitly stated by Haiman. If $G = \mathbb{Z}^d$ then it follows from standard facts about D-finite series that $F_u(x)$ is D-finite, though it need not be algebraic.

Maxim Kontsevich asked whether $F_u(x)$ is always D-finite when $G = \mathrm{GL}(d, \mathbb{Z})$. This remains open, though it is known that the question of whether $F_u(x) = 0$ is undecidable. More generally, we can ask for which groups G is $F_u(x)$ algebraic for all $u \in \mathbb{Z}G$, and for which groups is $F_u(x)$ D-finite for all $u \in \mathbb{Z}G$.

Nick Wormald: Reduction of degree in the coefficients of a generating function

Problem: Let $[y]_k$ denote $y(y-1)\cdots(y-k+1)$ and $A_t(y,z)$ the coefficient of x^t in

$$\log \sum_{k\geq 0} \frac{[y]_k [z]_k}{k!} x^k.$$

Clearly A_t has total degree at most $2t$. Show that A_t has total degree at most $t+1$ for $t \geq 1$.

Notes:

(1) A short solution was quickly found by Ira Gessel, Gilles Schaeffer and Richard Stanley, each independently.

(2) A similar question: show that for $t \geq 1$ the coefficient of x^t in

$$\log \sum_{k\geq 0} \frac{[y]_{2m}}{m!} x^m$$

has degree $t+1$. This was also solved by Ira Gessel, using the solution to the main problem.

(3) Ira Gessel has obtained the leading coefficients in both questions.

(4) The effect of 'reduction of degree' occurring in a similar context is explained in a paper of Valentin Féray, "Asymptotic behavior of some statistics in Ewens random permutations" (Electron. J. Probab. 18 (2013), no. 76, 32 pp).