SIGNED PUZZLES FOR SCHUBERT COEFFICIENTS

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Abstract. We give a signed puzzle rule to compute Schubert coefficients. The rule is based on a careful analysis of Knutson's recurrence [Knu03]. We use the rule to prove polynomiality of the sums of Schubert coefficients with bounded number of inversions.

1. Introduction

Schubert coefficients are extremely well studied yet deeply mysterious numbers which play a central role in Schubert calculus. A major open problem asks for a combinatorial interpretation of the coefficients [Sta00, Problem 11]. Puzzles are finite tilings with edge-labeled equilateral triangles which enumerate the desired numbers. Such puzzle rules were discovered for many special cases and for closely related problems; we refer to [Knu23] for an extensive overview.

While a manifestly positive combinatorial interpretation remains elusive, signed combinatorial interpretations are also of interest for various applications, see a discussion in [Pak24]. In [PR24a], the authors present signed combinatorial interpretations for a wide range of structure constants in algebraic combinatorics, including Schubert coefficients. For Schubert coefficients and their generalizations, several such formulas are known in the literature, see a discussion in §8.2. Unfortunately, neither of these signed combinatorial interpretations can be extended to a signed puzzle rule.

In this paper we present a signed puzzle rule for Schubert coefficients. Similar (signed) puzzle rules already exist in a few special cases, see [KZ21, KZ23]. Our result is the first signed puzzle rule which holds in full generality.

Theorem 1.1. *For every integer n, let $\mathcal{T}_n$ be a set of $O(n^9)$ puzzle pieces defined in Section 4. Let $u, v, w \in S_n$ be permutations with $\operatorname{inv}(u) + \operatorname{inv}(v) = \operatorname{inv}(w)$, and denote $\ell = \binom{n}{2} - \operatorname{inv}(u)$. Let $\Gamma = \Gamma(u, v, w)$ be an $n \times \ell$ parallelogram region with indicators and labels defined in Section 4. Then the number of signed puzzles of Γ with $\mathcal{T}_n$ is the Schubert coefficient $c^w_{u,v}$.*

The starting point of our construction is a special case of *Knutson's recurrence* given in [Knu03], see Section 3 below. Knutson's recurrence is an advanced extension of an earlier paper [Knu01]. Note that we do not use the equivariant variables and consider the results only in type A. See also Yong's implementation of Knutson's recurrence [Yong06].

The proof of Theorem 1.1 is completely combinatorial, and uses only basic notions from Schubert calculus. The construction of the puzzles is given in Section 4; it is somewhat involved but richly illustrated. The proof of the theorem is then given in Section 5, and a large example is given in Section 6. We also give the following unusual application of the signed puzzle rule:

Theorem 1.2. *Fix k, and let*

$$\gamma_k(n) := \sum_{u,v,w \in S_n \,:\, \operatorname{inv}(w)=k} c^w_{u,v}\,.$$

Then γ_k is a polynomial in n.

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Since we have $u, v \leqslant w$ in Bruhat order for the nonzero terms in the summation, this gives $\mathrm{inv}(u), \mathrm{inv}(v) \leq k$. Thus the total number of triples (u, v, w) in the summation above is *at most* $\binom{n}{2}^{3k} \leq n^{6k}$. To bound the Schubert coefficients in the summation, note that $|\mathrm{NF}(u) \cup \mathrm{NF}(v)| \leq 4k$, where $\mathrm{NF}(w) := \{i \in [n] : w(i) \neq i\}$ denotes the set of non-fixed points in w. Stanley's upper bound in [Sta17, §5] gives $c^w_{v,w} \leq 2^{(4k)^2}$. Therefore, $\gamma_k(n) = O_k(n^{6k})$. However, a priori there is no reason to believe that the sum γ_k is polynomial in n.

The proof of Theorem 1.2 is given in Section 7. It uses technical details of the puzzle construction in the proof of Theorem 1.1 and a geometric argument in Ehrhart theory. We conclude with final remarks in Section 8, where we give further comments on the nature of signed puzzle rules.

2. Standard definitions and notation

We refer to [Mac91, Man01] for the background on Schubert calculus, to [Ful97, Sta99] for definitions and standard results in algebraic combinatorics, and to [GS87, §14] and [vEB97] for basic results on Wang tilings. See [BR15a] for the introduction to Ehrhart theory of rational polyhedra, and [Bar97] for a concise survey on the subject.

For two functions $f, g : \mathbb{N} \to \mathbb{N}$, we use $f = O(g)$ if there is a universal constant $C > 0$ such that $f(n) \leq Cg(n)$ for all $n \in \mathbb{N}$. For two functions $f, g : \mathbb{N}^2 \to \mathbb{N}$, we use $f = O_k(g)$ if for all $k \in \mathbb{N}$ there is a constant $C(k) > 0$ such that $f(n, k) \leq C(k)\, g(n, k)$ for all $n \in \mathbb{N}$.

Let $[n] := \{1, \ldots, n\}$ and $\langle n \rangle := [n] \cup \{-\}$, where we view "$-$" as the *blank element*. Our notation for permutations will simplify if the context is clear, e.g. we write 4123 to mean a permutation $(4, 1, 2, 3)$ in S_4. We think of multiplication on the right as the action on positions, e.g. $4123 \cdot 2134 = 1423$. Let $\mathrm{inv}(w) := \{(i, j) : i < j,\, w(i) > w(j)\}$ denote the *number of inversions* in w.

For a permutation $w \in S_n$ we say that $i \in [n-1]$ is an *ascent* if $w(i) < w(i+1)$. Otherwise, i is a *descent*. Let $\mathrm{Des}(w)$ denote the set of descents in w. Let $\boldsymbol{w}_\circ = (n, n-1, \ldots, 1)$ denotes the *long permutation*, so $\mathrm{Des}(\boldsymbol{w}_\circ) = [n-1]$. We write $\mathbf{1}$ for the identity permutation $(1, 2, \ldots, n)$. We use $t_{ij} := (i, j)$ to denote a transposition in S_n, and let $s_i := (i, i+1)$ denote an adjacent transposition. By definition, if i a descent in w, then i is an ascent in ws_i and vice versa.

A *puzzle piece* τ is a region (tile) in a triangular grid $\mathbb{T}$ with certain labels/indicators on the boundary. In this paper all puzzle pieces will be unit triangles. For a collection $\mathcal{T}$ of puzzle pieces and a region Γ in $\mathbb{T}$, a *puzzle T of Γ with $\mathcal{T}$* is a tiling of Γ with copies of puzzle pieces $\tau \in \mathcal{T}$ (up to parallel translation), such that the labels/indicators match along all common edges between the puzzle pieces and along the boundary of Γ. A *signed puzzle* is a puzzle T with a sign function $s(T) \in \{\pm 1\}$. The *number of signed puzzles* of Γ with $\mathcal{T}$ is the sum of signs $s(T)$ over all puzzles T of Γ with $\mathcal{T}$.

3. Knutson's recurrence

It is well known that $c^w_{u,v} = 0$ unless the *dimension equation* holds:

$$(\oplus) \qquad \mathrm{inv}(u) + \mathrm{inv}(v) = \mathrm{inv}(w).$$

Thus we only consider coefficients satisfying $(\oplus)$ in Theorem 1.1. Additionally, $c^w_{u,v} = 0$ unless $u \leqslant w$ in Bruhat order. The following result is a special case of Knutson's recurrence in type A, adapted in the notation above.

Lemma 3.1 (*Knutson's recurrence* [Knu03]). *Let $u, v, w \in S_n$ and suppose $i \notin \mathrm{Des}(u)$. There are four cases:*

(0) *If $i \notin \mathrm{Des}(v)$ and $i \in \mathrm{Des}(w)$, then $c^w_{u,v} = 0$.*

(1) *If $i \notin \mathrm{Des}(v)$ and $i \notin \mathrm{Des}(w)$, then $c^w_{u,v} = c^{ws_i}_{us_i,v}$.*

(2) *If $i \in \mathrm{Des}(v)$ and $i \in \mathrm{Des}(w)$, then $c^w_{u,v} = c^w_{us_i,vs_i}$.*

(3) *If* $i \in \mathrm{Des}(v)$ *and* $i \notin \mathrm{Des}(w)$, *then*

$$c^w_{u,v} \, = \, c^{ws_i}_{us_i,v} \, + \, c^w_{us_i,vs_i} \, + \, \varepsilon(i,j,k) \sum_{(j,k)} c^w_{ut_{jk},vs_i}.$$

Here the summation is over all $1 \le j < k \le n$ *such that* $u(j) < u(k)$ *and* $|\{j,k\} \cap \{i,i+1\}| = 1$, *and we set*

$$\varepsilon(i,j,k) \, = \, \begin{cases} \;\;1 & \text{if } \; j = i \;\; \text{or} \;\; k = i+1 \\ -1 & \text{if } \; k = i \;\; \text{or} \;\; j = i+1 \\ \;\;0 & \text{otherwise.} \end{cases}$$

It is important to note that in all four cases of Knutson's recurrence where $(u,v,w) \to (u',v',w')$ in the lemma, we have $\mathrm{inv}(u') \ge \mathrm{inv}(u)+1$. Thus, after iterating the recurrence, it can stop only at $u = \boldsymbol{w}_\circ$ or at zero terms. In the former case we must also have $w = \boldsymbol{w}_\circ$, and dimension equation ($\oplus$) gives $v = \mathbf{1}$. Of course, we then have $c^{\boldsymbol{w}_\circ}_{\boldsymbol{w}_\circ,\mathbf{1}} = 1$.

It is also worth noting that when $\mathrm{inv}(u') > \mathrm{inv}(u)+1$, we have either $c^w_{u,v} = 0$ or $c^{w'}_{u',v'} = 0$ by the equation ($\oplus$). Therefore, $\mathrm{inv}(u') = \mathrm{inv}(u)+1$ is the only nontrivial possibility, and the total number of steps in the iteration of the recurrence is exactly $\mathrm{inv}(\boldsymbol{w}_\circ) - \mathrm{inv}(u) = \binom{n}{2} - \mathrm{inv}(u)$. In other words, after $\binom{n}{2} - \mathrm{inv}(u)$ iteration steps, we obtain a signed summation of Schubert coefficients $c^{\boldsymbol{w}_\circ}_{\boldsymbol{w}_\circ,\mathbf{1}}$, as all other terms in the summation are equal to zero.

In summary, every Schubert coefficient $c^w_{u,v}$ with u,v,w satisfying the dimension equation ($\oplus$) is equal to the number of positive minus the number of negative terms in the summation obtained by iterating Knutson's recurrence for $\binom{n}{2} - \mathrm{inv}(u)$ steps. Our signed puzzle rule is a reworking of a signed combinatorial interpretation given implicitly by this algorithm.

4. The construction

4.1. **The region.** Consider a parallelogram shaped region Γ of size $n \times \ell$, where $\ell = \binom{n}{2} - \mathrm{inv}(u)$ as in the introduction. We bicolor the region into equilateral triangles as in Figure 4.1 below.

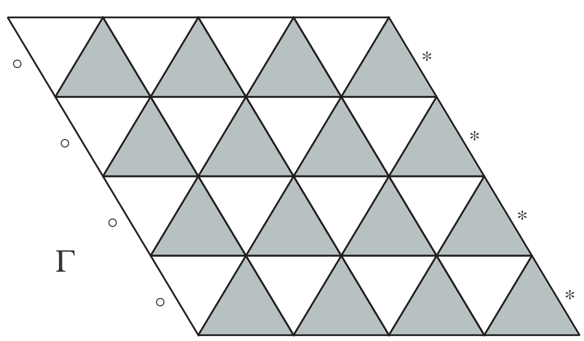

Figure 4.1. Region Γ.

We label the boundary of Γ as follows. Label all horizontal edges with triples $(a,b,c) \in [n]^3$. The top edges are labeled $(u(1),v(1),w(1)), \; \ldots \; ,(u(n),v(n),w(n))$. The bottom edges are labeled $(n,1,n), \; (n-1,2,n-1), \; \ldots \; ,(1,n,1)$. Left and right edges on the boundary of Γ are marked $\circ$ and $*$, respectively. We use the term *position* to mean a particular triangle in Γ.

4.2. **Puzzle pieces.** All puzzle pieces are equilateral triangles of three types: *white*, *shaded* and *dark*. We use *triangle* to mean puzzle piece, since all pieces will be triangular. In some cases we use the more specific terminology of *triangle tile* to refer to a triangle, to avoid confusion.

White triangle tiles are placed on white positions in Γ, while shaded and dark triangle tiles are placed on shaded positions in Γ. One should think of dark triangle tiles as "strongly shaded"; they encode a position where Knutson's recursion is applied. Dark triangles will come in three colors: dark yellow, dark blue and dark red. The red and blue dark triangles correspond to positive and negative contributions, respectively. *The sign of the puzzle will be the parity of the number of red triangles in the puzzle.* To simplify definitions we will illustrate the dark triangles as (uncolored) dark triangles, see Figure 4.2.

FIGURE 4.2. White, shaded and dark triangles. Three colors of dark triangles: dark yellow, dark blue and dark red.

No rotations or reflections of the pieces are allowed, only parallel translations. The sides of the triangles will have labels and indicators, described in Section 4.3. Triangles are allowed to share a side in the puzzle if corresponding side labels and indicators are identical.

Finally, in addition to color, all dark triangles have a *docket number*, which ranges from 1 to 4 for yellow triangles, and is either 1 or 2 for blue and red triangles, see Figure 4.3. Docket numbers of yellow triangles correspond to cases (1), (2), and first two terms in (3) of Lemma 3.1. Docket numbers of blue and red triangles correspond to cases of positive and negative terms in the summation in (3) of Lemma 3.1.

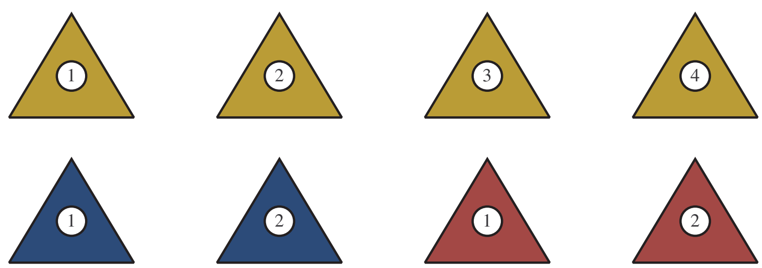

FIGURE 4.3. Possible docket numbers of dark triangles.

4.3. **Labels and indicators.** Here, labels are numbers and indicators are symbols. The labels and indicators on the triangles will be somewhat involved and defined in stages.

Level 0. We place an indicator $\circ$ or $*$ on the left and right edges of all triangles as follows. For both white and shaded triangles, the indicators on the left and right edges must be equal. For dark triangles, we mark the left edge with $\circ$ and right edge with $*$, see Figure 4.4. Since the left side of Γ is marked $\circ$ and the right is marked $*$, these indicators ensure that there is exactly one dark triangle in each row of Γ.

Level 1. All triangles have *permutation labels* on each edge. These are triples (a, b, c), where $a, b, c \in [n]$. White triangles have three identical permutation labels. Shaded and dark triangles can have distinct permutation labels on the left and right edges: $a \neq p$, $b \neq q$, and $c \neq r$, see Figure 4.5. In the figures we often include arrows to demonstrate the flow of permutation labels, i.e. how they shift through Γ. These are not part of the labels and have only explanatory meaning.

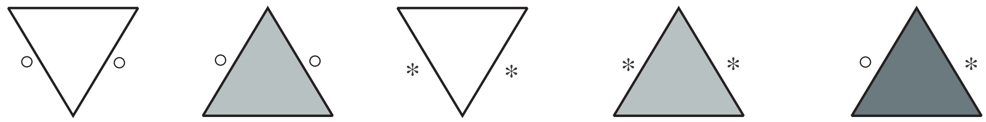

FIGURE 4.4. Indicators $\circ$ and $*$ on the left and right edges of three types of triangles.

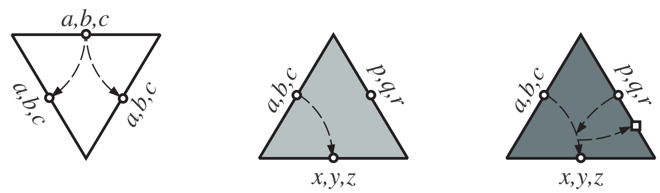

FIGURE 4.5. Permutation labels on three types of triangles.

Level 2. Some triangles have additional triples of *feedback labels* on the edges. These will be of the form (d, e, f), where $d, e, f \in \langle n\rangle$ and at least one of these is blank. All dark triangles have feedback labels, which will appear only on their right edge. Shaded triangles may have feedback labels, which will appear only on their left edge. For shaded triangles *with no* feedback nor transmuter labels (see Level 3), the permutation labels on their left edge and bottom edge will be equal. Finally, white triangles may have equal feedback labels on both left and right edges, see Figure 4.6.

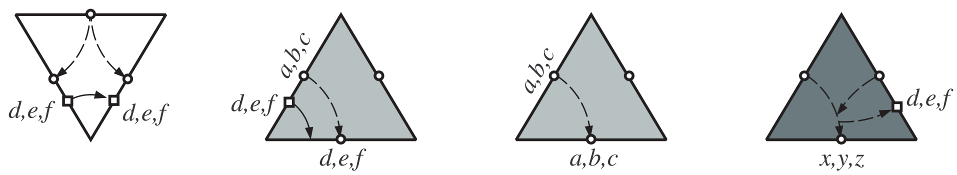

FIGURE 4.6. Feedback labels on three types of triangles, and a shaded triangle without feedback labels.

Level 3. Finally, triangles may have additional *transmuter labels* on the left and right edges of the form (g, h), where $g, h \in \langle n\rangle$. In white triangles transmuter labels on the left and right edges must be equal. Shaded and dark triangles can have either equal transmuter labels on the left and right edges, or have transmuter labels on only one edge. These transmuter labels can be combined with permutation and feedback labels as in Figure 4.7. For non-blank transmuter labels (g, h), we always have the inequality $g < h$.

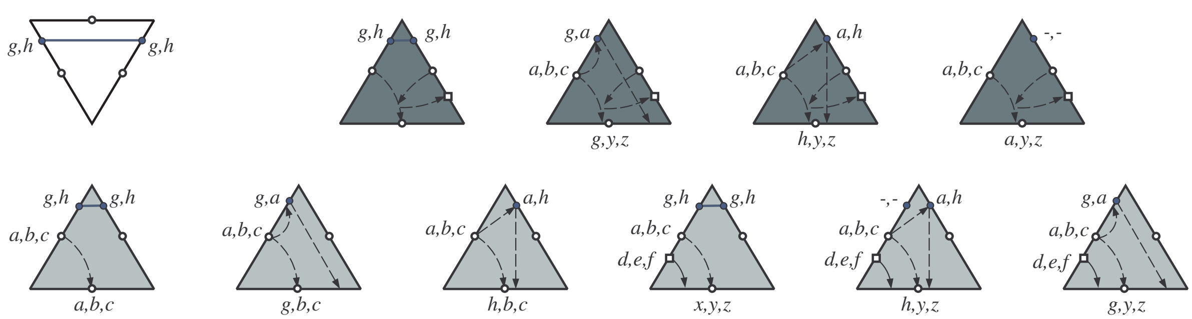

FIGURE 4.7. Transmuter labels on three types of triangles.

Note: For clarity, we distinguish the labels in the figures by marking permutation labels with $\circ$, the feedback labels with $\square$, and the transmuter labels with $\bullet$. To see these markings, the reader might want to zoom in.

4.4. **Constraints.** We now describe constraints on the labels and indicators. Roughly, there are very few additional constraints on white and shaded triangles other than those described above. Thus many types of labels can arise for white and shaded triangles. On the other hand, the dark triangles are heavily constrained, such that the docket number and color will uniquely determine the constraints on the edges.

White triangles: There are five types of labelings of white triangles depending on whether they have feedback labels, transmuter labels, or both, see Figure 4.8. In the fourth and fifth triangles (from the left), two of the feedback labels are blank. Additionally, in the fifth triangle both transmuter labels are blank. For technical reasons, to align with the left boundary of Γ, we allow the left permutation label to be empty. There are two choices of the $\circ/*$ indicators of the first two triangles, as described in Figure 4.4. We force that the last three triangles have $*$ on both edges. This gives in total $O(n^6)$ white triangles.

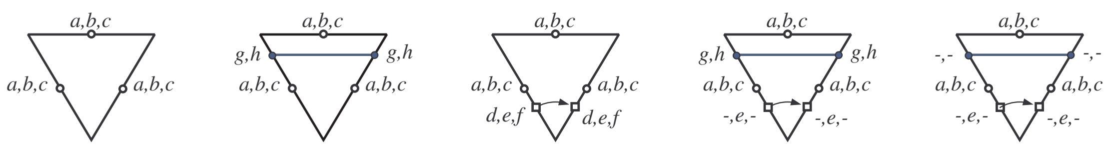

FIGURE 4.8. Five types of white triangles.

Shaded triangles: There are ten types of labelings for shaded triangles depending on whether they have feedback labels, transmuter labels, or both, see Figure 4.9. For technical reasons, to align with the right boundary of Γ, we allow the right permutation label to be empty. The third triangle in the second row has 9 labels which can all be distinct; other triangles have eight of fewer distinct labels. This gives in total $O(n^9)$ shaded triangles.

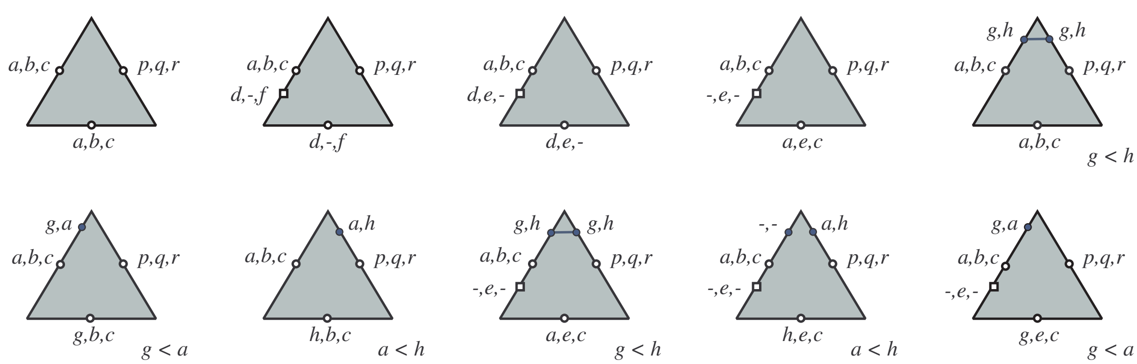

FIGURE 4.9. Ten types of shaded triangles.

For shaded triangles, there is an additional condition relating indicators and permutation labels: a shaded triangle where the first entries in the permutation labels form an ascent must have $*$ indicators, see Figure 4.10. We call this the *ascent condition*. When these labels form a descent, there is no additional constraint on the indicators.

Dark triangles: Recall all dark triangles have indicators $\circ$ on the left edge and $*$ on the right. Additionally, they all have an ascent in the first entries of the permutation labels: $a < p$. Below are dark triangles arranged by color and docket number as in Figure 4.3. Note that the additional

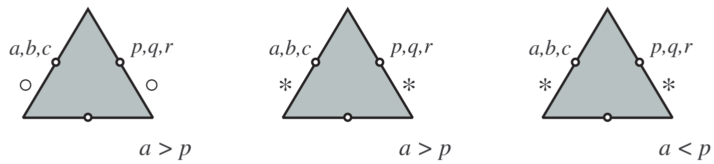

Figure 4.10. Indicators of shaded triangles that are allowed under the ascent condition.

inequalities for permutation labels correspond precisely to the cases in Knutson's recursion. This gives $O(n^8)$ dark triangles in total.

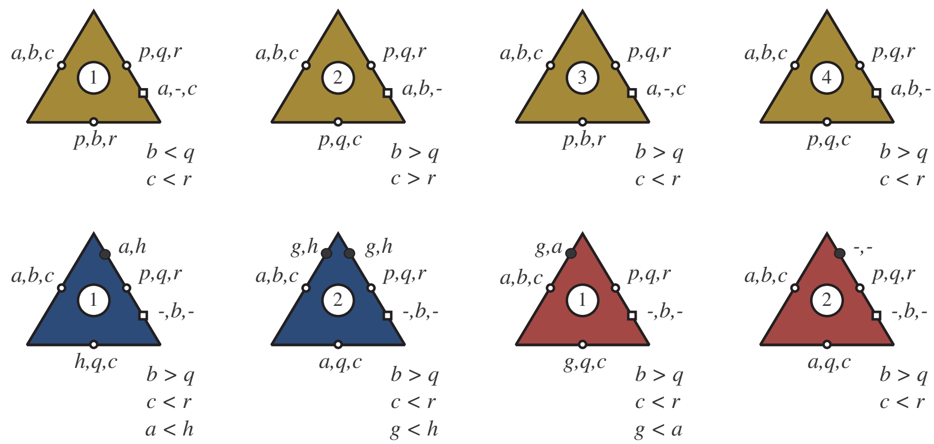

Figure 4.11. Label constraints on dark triangles by docket number.

4.5. **Summary.** We gave a construction of $O(n^9)$ puzzle pieces. All pieces are triangles with three types of labels (permutation, feedback and transmuter), two types of indicators ($\circ$ and $*$), five colors (white, shaded, dark yellow, dark blue, and dark red), and docket numbers to further distinguish dark colors. We denote this set of triangle tiles by $\mathcal{T}_n$.

Given three permutations $u, v, w \in S_n$ which satisfy the dimension equation $(\oplus)$, we constructed a parallelogram shaped region $\Gamma = \Gamma(u, v, w)$ on a triangular grid with particular indicators and labels on the boundary. For each puzzle T of Γ, the sign $s(T)$ is $(-1)^p$, where p is the number of red triangles in T. Denote by $t_+(u, v, w)$ and $t_-(u, v, w)$ the number of puzzles T with signs $s(T) = 1$ and $s(T) = -1$, respectively. Theorem 1.1 states that $c^w_{u,v} = t_+(u, v, w) - t_-(u, v, w)$. We prove this in the next section.

5. Proof of Theorem 1.1

We prove the result by induction on the number ℓ of rows of Γ. Thus, it suffices to show that the first row of triangles in the puzzle is given by Knutson's recurrence as in Lemma 3.1. Formally, we show that one step of the recurrence $(u, v, w) \to (u', v', w')$ corresponds to top and bottom labels of triangles in the top row of a puzzle.

Recall there is exactly one dark triangle in each row of Γ, and that the dark triangles correspond to index i in Lemma 3.1. Then this dark triangle tile is placed the i-th shaded position in its row of the region Γ. By the constraints on dark triangles we must have $i \notin \mathrm{Des}(u)$. As mentioned before, the indicators $\circ/*$ on the boundary of Γ and on the triangles constrain the indicators which may appear. Thus by the ascent condition on shaded triangle tiles, the index i must be the first ascent in the permutation u.

We think of white triangles as splitters, which input the information, i.e. permutation labels $(u(i), v(i), w(i))$, and transmit it to triangles in shaded positions to its right and left. When there are feedback or transmuter labels, they transmit the signal with no changes from the left edge to the right or vice versa, see Figure 4.8.

Shaded triangles often play a similar role. In particular, if no feedback or transmuter labels appear, these triangles transmit the permutation labels from the left to the bottom edge. In these cases the permutation labels $(u(i), v(i), w(i))$ are transmitted unchanged from i-th top edge to i-th bottom edge of the first row of Γ, see Figure 5.1.

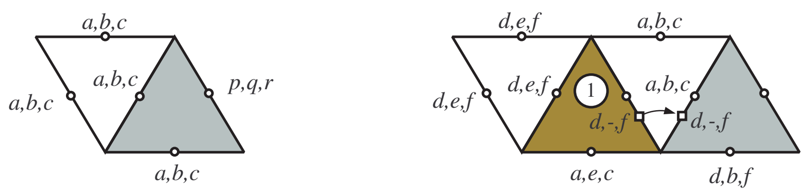

FIGURE 5.1. Permutation labels transmitted from top to bottom. Feedback signal dominated the permutation signal resulting in two transpositions: $d \leftrightarrow a$ and $f \leftrightarrow c$.

The feedback labels in Figure 4.6 model transpositions on the permutation labels. While the transpositions are initiated in dark triangles, since shaded triangles are not adjacent to them, the feedback labels transmit relevant permutation labels from left to right. Thus when shaded triangles have feedback labels, these labels are on their left edge to receive a signal sent by the dark triangle. This signal is transmitted by a neighboring white triangle immediately to the left of the shaded triangle. The feedback labels dominate the permutation labels in determining the permutation label on the bottom edge of the shaded triangle; only blanks are substituted with permutation labels, see Figure 5.1.

Dark triangles are distinguished by their color and docket numbers, each corresponding to different cases of Knutson's recurrence. The assumptions in these cases are directly translated into constraints on the dark triangles given in Figure 4.11. For the dark yellow triangles, the transpositions are local and the feedback labels enforce them. However, for both the dark blue and dark red triangles, this enactment of transpositions t_{jk} can involve distant triangles. Such transpositions are implemented with transmuter labels.

Transmuter labels are placed near the top of the triangle to be "above the fray", see Figure 4.7. For some triangles (either dark or shaded), they may affect permutation labels. However, in most cases, they introduce no constraints. For transmuter labels (g, h), we always have label g moving to the right, while h to the left. Similarly to feedback labels, transmuter labels dominate the permutation labels in determining the permutation label on the bottom edge of the triangle.

By the dimension condition $(\oplus)$, the recurrence (4) in Lemma 3.1 always involves transpositions which increase the number of inversions in u by one:

$$(\diamond) \qquad \operatorname{inv}(ut_{jk}) \,=\, \operatorname{inv}(u) \,+\, 1.$$

This gives a restriction $g < h$ for all transmuter labels. This alone is not a sufficient condition to ensure $(\diamond)$. However, since the total number of rows in Γ is $\ell = \binom{n}{2} - \operatorname{inv}(u)$, and since there is at least one inversion in *every* row, condition $(\diamond)$ holds automatically for all puzzles.

From this point, all conditions on dark blue and dark red triangles are immediate translations of the last term of the summation (3) in Lemma 3.1. Transmuter labels can initiate at either dark or shaded triangles. The former possibility corresponds to having $j = i$ and dark blue docket number 1, or having $k = i$ and dark red docket number 1. The latter corresponds to having $j = i+1$ and dark blue docket number 2, or having $k = i + 1$ and the dark red docket number 2.

The only delicate point is the blank transmuter label when using the dark red docket number 2 triangle. This blank label signals that the shaded triangle to the right should initiate the transpositions, see Figure 5.2. The details are straightforward.

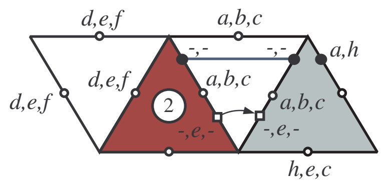

FIGURE 5.2. For dark red triangle docket number 2, the blank transmuter labels signal initiates transposition t_{jk} with $j = i+1$ at the following shaded triangle. Here $d = u(i)$, $a = u(i+1)$ and $h = u(k)$.

Finally, we note that some constraints of the recurrence follow from the setup and are never used. Notably, we do not check that permutation labels $(u(1), \dots, u(n))$, $(v(1), \dots, v(n))$ and $(w(1), \dots, w(n))$, do indeed form permutations. This is assumed in the input, and for every row of horizontal edges this holds by induction. Additionally, we never use the (0) case of the lemma. This is because if we have an ascent in both u and v, it remains so by induction, and the desired puzzle does not exist by the labeling of bottom edges of Γ. This completes the proof. $\square$

Remark 5.1. One can modify the construction to avoid the argument following equation ($\diamond$) above. Note for ($\diamond$) to hold, we must have $u(m)$ outside of the interval $[u(j), u(k)]$, for all $j < m < k$. This is a non-local constraint which can be implemented by adding further constraints on the transmuter labels. Formally, we can add the constraint that $a \notin [g, h]$ on white triangles of the second and fourth type in Figure 4.8. From above, this ensures that ($\diamond$) holds automatically, without referencing the number of rows ℓ of Γ. In the construction above, we opted to avoid this modification for simplicity, but we will need this version in the proof of Theorem 1.2 in Section 7.

6. EXAMPLE

Let $n = 7$. Take three permutations $u = 3251467$, $v = 4126537$, $w = 6271534$ in S_7. The following is an example of Knutson's recursion steps with cases identified:

$$\begin{bmatrix}3251647\\4126537\\6271534\end{bmatrix} \xrightarrow{i=2,(1)} \begin{bmatrix}3521647\\4126537\\6721534\end{bmatrix} \xrightarrow{i=1,(3b)} \begin{bmatrix}5321647\\1426537\\6721534\end{bmatrix} \xrightarrow[\varepsilon(4,5,7)=-1]{i=4,(3c)} \begin{bmatrix}5321746\\1425637\\6721534\end{bmatrix} \xrightarrow{i=4,(1)}$$

$$\begin{bmatrix}5327146\\1425637\\6725134\end{bmatrix} \xrightarrow{i=3,(1)} \begin{bmatrix}5372146\\1425637\\6752134\end{bmatrix} \xrightarrow{i=2,(2)} \begin{bmatrix}5732146\\1245637\\6752134\end{bmatrix} \xrightarrow{i=1,(1)} \begin{bmatrix}7532146\\1245637\\7652134\end{bmatrix} \xrightarrow[\varepsilon(5,4,7)=-1]{i=5,(3c)}$$

$$\begin{bmatrix}7532164\\1245367\\7652134\end{bmatrix} \xrightarrow{i=5,(1)} \begin{bmatrix}7532614\\1245367\\7652314\end{bmatrix} \xrightarrow[\varepsilon(4,2,5)=1]{i=4,(3c)} \begin{bmatrix}7632514\\1243567\\7652314\end{bmatrix} \xrightarrow{i=4,(1)} \begin{bmatrix}7635214\\1243567\\7653214\end{bmatrix} \xrightarrow{i=3,(2)}$$

$$\begin{bmatrix}7653214\\1234567\\7653214\end{bmatrix} \xrightarrow{i=6,(1)} \begin{bmatrix}7653241\\1234567\\7653241\end{bmatrix} \xrightarrow{i=5,(1)} \begin{bmatrix}7653421\\1234567\\7653421\end{bmatrix} \xrightarrow{i=4,(1)} \begin{bmatrix}7654321\\1234567\\7654321\end{bmatrix} = \begin{bmatrix}\boldsymbol{w}_\circ\\ \mathbf{1}\\ \boldsymbol{w}_\circ\end{bmatrix}$$

Here (3b) indicates the second summand in (3) in Lemma 3.1, and so on. The whole puzzle is quite large, so Figure 6.1 gives just the rows corresponding to the third line of the calculation above, i.e.

a quarter portion of the actual puzzle. To avoid cluttering we also omit some labels which are clear from the example.

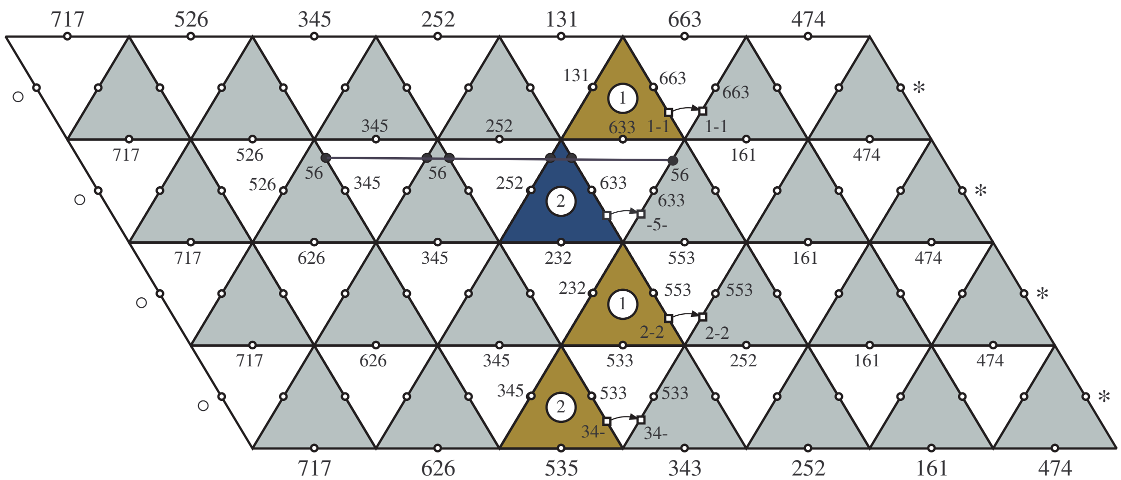

Figure 6.1. An example of a puzzle.

7. Proof of Theorem 1.2

7.1. The setup. First, note that $c^w_{u,v} = c^{\overline{u}}_{\overline{w},v}$, where $\overline{w} = w \cdot \boldsymbol{w}_\circ$. Thus we can rewrite

$$\gamma_k(n) \ = \sum_{u,v,w\in S_n \,:\, \mathrm{inv}(u)=\binom{n}{2}-k} c^w_{u,v}\,.$$

Now consider *all* puzzles of the $n \times k$ parallelogram region Γ as in Figure 4.1, where we remove the constraints on the top boundary of Γ. Such puzzles contribute to *some* triples of permutations (u, v, w) as above.

Through the proof we will work with a modified set $\mathcal{T}'_n$ of puzzle pieces given in Remark 5.1. Note that $\mathcal{T}'_n \subset \mathcal{T}_n$ by construction. This is needed to ensure $\mathrm{inv}(u) = k$ in every puzzle. Then we have:

$$\gamma_k(n) \ = \sum_{\text{puzzle } T \text{ of } \Gamma \text{ with } \mathcal{T}_n} s(T)\,.$$

From this point on, to simplify the counting we will work with a rectangular region obtained by an affine transformation of Γ as in the figure below. Here each position (of equilateral triangle shape) is turned into the right isosceles triangle, so two such triangles (one white and one shaded) form a unit square. We still refer to the resulting region as Γ. We now modify puzzle pieces, place them on the new region accordingly, and refer to them in the same way as in the proof above.

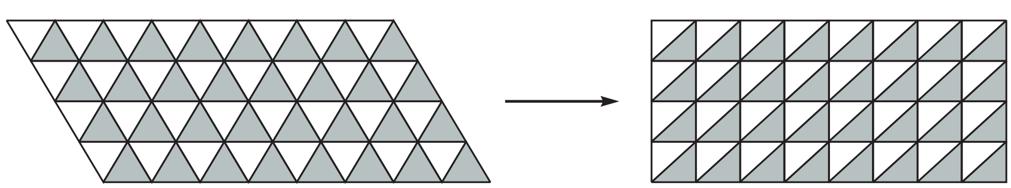

Figure 7.1. Turning parallelogram region Γ into a rectangle.

7.2. **Relative placements and labelings.** By construction, there is a finite number of types of dark triangles and shaded triangles with a transmuter label on one side. The positions and labelings of these triangles determines the puzzle. Recall that there are exactly k dark triangles and at most $2k$ such shaded triangles, where at most k are not immediately following the dark triangles. We call them *separated shaded triangles*, or *separated triangles* for short.

Since k is fixed, the number of relative placements π of these dark and separated shaded triangles is also finite and depends only on k but not on n. Here by a *relative placement* we mean how these triangles are arranged in Γ relative to each other (above, to the left, to the right, etc.), when one ignores the distances between them. Each relative placement corresponds to $f_\pi(n)$ actual placements, where f_π is polynomial in n. Here an *actual placement* refers to a choice of positions for the triangles that results in a puzzle.

One way to think of the actual placement is to think of the set I of columns of Γ which contains dark and separated triangles see Figure 7.2. Clearly, I determines the actual placement of these triangles, as their relative positions to others determines their positions within the columns.

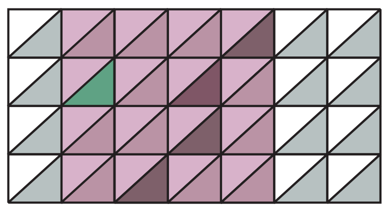

FIGURE 7.2. For the example in Figure 6.1, we have $I = \{2, 3, 4, 5\}$, and the columns are highlighted purple. Dark triangles are at $(1, 5)$, $(2, 4)$, $(3, 4)$ and $(4, 3)$. Separated triangle is at $(2, 2)$ and is highlighted green. In this case $I^* = \{1, 2, 3, 4, 5\}$.

We partition the set $I \subseteq [n]$ into blocks of consecutive integers: $I = I_1 \cup I_2 \cup \ldots$, where $I_1 = [i_1, i'_1]$, $I_2 = [i_2, i'_2]$, etc. Denote $I^* := I^*_1 \cup I^*_2 \cup \ldots$, where $I_m := [i_m - 1, i'_m]$ for all $m = 1, 2, \ldots$ Since $|I| \leq 2k$, we have $|I^*| \leq 4k$. From this point, we will work with I^*, see Figure 7.2.

Next, we need to take into account the number of possible permutation, feedback and transmuter labels of *all triangles* which lie in columns I^*. The number r of such labels satisfies $r \leq 9 \cdot k \cdot |I^*| = O(k^2)$. Given π, there is a large number of equalities and inequalities on these labels, resulting in polynomially many possible labelings. These labelings again can be characterized by the *relative labelings* determined by the at most $r^2 = O(k^4)$ inequalities on the labelings. We denote relative labelings by λ.

7.3. **Actual labelings.** Denote by $g_{\pi,\lambda}(n)$ the number of possible actual labelings, given the relative placement π and relative labeling λ. Although we will not need this fact, let us briefly show that $g_{\pi,\lambda}$ is polynomial in n. Indeed, $g_{\pi,\lambda}(n)$ is the number of integer points in the r-cube $[n]^r$ minus some half-spaces of the type $x < y$, restricted to hyperplanes of the type $x = z$, and outside of some hyperplanes of the type $y = z$. Resolving each $y \neq z$ as either $y > z$ or $y < z$, this shows that $g_{\pi,\lambda}$ the sum of *order polynomials* of a poset on the labels, which is a polynomial in n, see e.g. [Sta99, §3.15].[1] We will need a stronger argument of this type below.

From above, each such actual placement and actual labeling completely determines a puzzle, except for the permutation labels not given by λ. Since the remaining permutation labels are unchanged from row to row by the dark and separated triangles, the bottom boundary determines them as well. Indeed, these permutation labels in i-th column are given by $u_i = w_i = n - i + 1$ and $v_i = i$, for all $i \notin I^*$.

Notice that in each block $I^*_p = [i_p - 1, i'_p]$, the leftmost white triangles in the column $i_p - 1$ have equal permutation labels in all rows: $(n - i_p + 2, i_p - 1, n - i_p + 2)$. Similarly, the rightmost shaded

[1] More precisely, some poset inequalities become strict; this does not affect the argument.

triangles in column i'_p have permutation labels on the right edges $(n-i'_p, i'_p+1, n-i'_p)$. This implies that the actual labeling already contains the information about I, and thus actual placement of dark/separated triangles. In other words, relative placement π and actual labeling of triangles in columns $i \in I^*$ uniquely determines the whole puzzle.

Of course, the inequalities on the actual labelings as above can be inconsistent and the actual puzzle may not exist. In summary, for every π and actual labeling of triangles contained in columns of dark/separated triangles, there is either one or zero possible puzzles. Distinguishing between these possibilities is more difficult.

7.4. **Inequalities on actual labelings.** Fix π and λ as above. We use parameters $\alpha_1, \dots, \alpha_r \in [n]$ to denote the actual permutation, feedback and transmuter labelings of all triangles which lie in columns I^*. Whether the resulting puzzle exists or not introduces linear inequalities on these parameters with integer coefficients as in Figures 4.9 and 4.11, and the inequalities given by Remark 5.1.

First, as we mentioned above, the inequalities on the labelings coming from each triangle are of the form $\alpha_i < \alpha_j$, $\alpha_i = \alpha_j$ and $\alpha_i \neq \alpha_j$. These give $O(k^4)$ inequalities. Importantly, given π, λ and $\{\alpha\}$, not all permutation labels will form triples of permutations in each row. Translating this condition into permutation labels using relative orders π and λ, gives inequalities relating differences between the labels and distances between the columns of their positions. More precisely, we obtain inequalities of the form $\alpha_i - \alpha_j > m$, $\alpha_i - \alpha_j < m$ or $\alpha_i - \alpha_j = m$, which gives $O(r^2) = O(k^4)$ additional inequalities.

Finally, we need to include the inequalities coming from constrains on the transmuter labels, as they relate to other labels given by Remark 5.1. These inequalities are also of this type, but in logical combination. Indeed, for example, for the shaded triangle that is third in the second row on Figure 4.9, the inequalities are of the form the form $a, p \notin [g, h]$. This translates to

$$(\circledast) \qquad \big((a < g) \vee (a > h)\big) \wedge \big((p < g) \vee (p > h)\big),$$

where $a, g, h, p \in \{\alpha_1, \dots, \alpha_r\}$. Note that the number of such inequalities on the parameters is $O(k^2)$. In total, the number of inequalities on the labels is thus $O(k^4)$.

7.5. **Counting actual labelings.** We now proceed to the counting of the set of labelings $\{\alpha_i\}$ in $[n]^r$ which satisfy the inequalities as above. The exact inequalities or even their exact number will prove unimportant. We will use only their form and the upper bound $O(k^4)$ of their number.

Fix a pair (π, λ) of relative placements and labelings as before. For simplicity, relabel all parameters α_i according to the relative order λ, so we have $1 \leq \alpha_1 \leq \ldots \leq \alpha_r \leq n$. Denote by $J_{\pi,\lambda} \subset [n]^d$ the set of possible vectors $\alpha = (\alpha_1, \dots, \alpha_r)$ as above for which there exists a puzzle.

Resolve all inequalities $(\circledast)$ into two pairs of strict inequalities. Similarly, resolve all inequalities $\alpha_i \neq \alpha_j$ as either $\alpha_i < \alpha_j$ or $\alpha_i > \alpha_j$. These define a partition of $J_{\pi,\lambda}$ as a disjoint union of subsets $J_{\pi,\lambda,\varkappa}$, where $1 \leq \varkappa \leq \zeta(k)$ and $\zeta(k) = 2^{O(k^4)}$.

Observe that $J_{\pi,\lambda} = nQ_{\pi,\lambda} \cap \mathbb{Z}^r$, where the set $Q_{\pi,\lambda} \subset [0,1]^r$ is a disjoint union of $\zeta(k)$ convex polyhedra $P_{\pi,\lambda,\varkappa} \subset [0,1]^r$ with rational vertices. Here each $P_{\pi,\lambda,\varkappa}$ is given by the integral inequalities as above, where strict inequalities of the form $a < b$ are converted into nonstrict inequalities $a \leq b-1$.

Consider the *Ehrhart quasi-polynomials* $h_{\pi,\lambda,\varkappa}(n) := |nP_{\pi,\lambda,\varkappa} \cap \mathbb{Z}^r|$, and observe that

$$h_{\pi,\lambda}(n) := |J_{\pi,\lambda}| = |nQ_{\pi,\lambda} \cap \mathbb{Z}^r| = \sum_{\varkappa=1}^{\zeta(k)} h_{\pi,\lambda,\varkappa}(n)$$

is also quasi-polynomial, see [Sta99, §4.6] for the definitions. Clearly, $h_{\pi,\lambda}(n) \leq f_\pi(n) \cdot g_{\pi,\lambda}(n)$.

Now, write each polyhedron $P = P_{\pi,\lambda,\varkappa}$ by the defining inequalities as $A\boldsymbol{x} \leqslant \boldsymbol{b}$. From above, every inequality can be rewritten in the form $\alpha_i - \alpha_j \leq b$. Thus, all maximal minors are determinants of $r \times r$ matrices with entries in $\{0, \pm 1\}$, with at most two ± 1's in every row of opposite sign.

Thus these minors are themselves in $\{0, \pm 1\}$ by the same argument as in the standard proof of the *matrix-tree theorem*, see e.g. [Sta99, §5.6]. Therefore, all polyhedra $P_{\pi,\lambda,\varkappa}$ are *unimodular*, and thus have integral vertices, see e.g. [Bar97]. This implies that the quasi-polynomial $h_{\pi,\lambda,\varkappa}(n)$ is in fact a polynomial in n, for all $(\pi, \lambda, \varkappa)$, ibid.

7.6. **Putting everything together.** Observe also that the sign of a puzzle $s(T) \in \{\pm 1\}$ is determined solely by π, so by a mild abuse of notation we can write $s(T) = s(\pi)$. Summing over all relative placements and labelings, we have:

$$\gamma_k(n) \;=\; \sum_{(\pi,\lambda)} s(\pi) \cdot h_{\pi,\lambda}(n) \;=\; \sum_{(\pi,\lambda)} \sum_{\varkappa=1}^{\zeta(k)} s(\pi) \cdot h_{\pi,\lambda,\varkappa}(n).$$

The double summation has a constant number of terms for a fixed k. From above, each $h_{\pi,\lambda,\varkappa}(n)$ is a polynomial in n. Thus, $\gamma_k(n)$ is also a polynomial in n, as desired. □

8. Final remarks

8.1. While discussing the background of puzzle rules in Schubert calculus, Knutson and Zinn-Justin make the following observation:

> "We take [from above] the oracular statement that *puzzles should be related to Schubert calculus.*" [KZ17, p. 2][a]
>
> [a]Original emphasis.

The signed puzzle rule in this paper is quite elaborate and uses a relatively large number $\Theta(n^9)$ of puzzle pieces which are not allowed to be rotated. It is worth comparing this with some of the earlier puzzle rules.

In the celebrated *Knutson–Tao puzzles* [KT03] for the Littlewood–Richardson coefficients, there are only three puzzle pieces and all 60° rotations are allowed. In the case of the equivariant K-theory structure constants, Pechenik and Yong [PY17, Cor. 1.3] modify and prove the previously conjectured *Knutson–Vakil puzzle rule*. Their new puzzle pieces can still be rotated, but now have complicated shapes (this can be corrected by introducing new edge labels).

In the 3-step case (for permutations with at most 3 descents), Knutson and Zinn-Justin gave several puzzle rules with the largest involving 3591 rhombi and some triangles, where now only 180° rotations are allowed [KZ17, KZ21]. In the 4-step case, the number of puzzle pieces is even larger and some of them have negative weight [KZ21]. Finally, in the *separated descents* case, Knutson and Zinn-Justin [KZ23] have $\Theta(n^2)$ puzzle pieces. We leave it to the reader to decide how our puzzles fit with these earlier puzzle designs, and whether this gives additional support to the quote above.

8.2. There are several *signed rules* for Schubert coefficients known in the literature, sometimes in disguise. They are also called *signed combinatorial interpretations*, *cancellative formulas* and GapP *formulas* in different contexts. Perhaps, the cleanest signed rule was given by Morales as a consequence of the Postnikov–Stanley formula [PS09, §17] and the pipe dream combinatorial interpretation of Kostka–Schubert numbers, see [Pak24, §10.2]. Our own signed rule in [PR24a, §5] is somewhat similar but less explicit and stated in a more general context. More involved (and much more general) signed rules are given by Duan [Duan05] and Berenstein–Richmond [BR15b].[2] Further generalizations of these rules are also known; we refer to [Knu23] for an overview. While all these rules have their own advantages, they seem incompatible with signed puzzle rules.

Additionally, there are several recursive formulas for computing Schubert coefficients which can in principle be converted to a formulas for Schubert coefficients. Notably, these include *Billey's formula* [Bil99, Eq. (5.5)], and the recent *Goldin–Knutson formula* [GK21]. Note that Billey's formula requires a division, which is a major obstacle to making it a signed rule. In the case of the Goldin–Knutson formula, the issue is the equivariant variables and their derivatives, which are unavoidable, even in cases in which the dimension equation (⊕) holds.

[2]The authors' insistence on using "Littlewood–Richardson coefficients" to refer to Schubert coefficients is somewhat unfortunate as it initially obscures the very general nature of their results, see [BR15b, Remark 1.2].

8.3. It would be interesting to find a conceptual proof of Theorem 1.2. Can one compute the polynomials γ_k explicitly? At the moment we do not know even the degrees of γ_k beyond small special cases. Curiously, the proof above only gives $\deg \gamma_k = O(k^2)$, since the total number of labels of puzzle pieces can be rather large. This is weaker than the elementary bound $\deg \gamma_k \leq 6k$ given in the introduction.

In a different direction, since Knutson's recurrence is originally stated in the generality of equivariant cohomology, we expect that Theorem 1.2 would also generalize in this direction. It would be interesting to see if the theorem generalizes to other cohomology theories mentioned in [Knu23], notably to K-theory and quantum cohomology.

8.4. To end on a philosophical note, it is worth pondering whether combinatorial interpretations (rules), and, specifically, *signed* combinatorial interpretations are worth studying. As one would expect, there are several schools of thought on the matter; see [PR25, App. B] for some background quotes.

In [Knu23, §1.4], Knutson lists three reasons why positive (unsigned) combinatorial rules are better than signed: the vanishing problem, computational efficiency and possibility of categorification. We find the computational efficiency reason to be unconvincing, at least from a theoretical point of view. Indeed, conjecturally the problem of computing Schubert coefficients is #P-hard [PR24b, Conj 1.2]. Even if Schubert coefficients had a #P formula, this formula might be quite hard to compute. Since subtraction can be done in linear time, it is possible and even likely that writing Schubert coefficients as the difference of two #P functions can lead to a faster algorithm. For example, famously, *fast integer multiplication* and *fast matrix multiplication* algorithms are fast *because* they allow subtractions. In other words, if computing is the goal, then constraining oneself to positive functions might not be a good strategy.

For the vanishing problem, we explain the state of art in our recent papers [PR24b, PR25]. There, we obtain the best known vanishing results in full generality, completely bypassing combinatorial arguments. Unfortunately, even for the best studied 2-step case, where a puzzle rule was proved in [BKPT16] (first conjectured by Knutson in 1999) the complexity of the vanishing problem remains wide open. Specifically, in the 2-step case, it would be very interesting to see if the vanishing problem is in P (this is known in the 1-step case). We are very far from resolving this problem despite having the puzzle rule.

In [PR24a], we adopt the opposite point of view, suggesting that signed combinatorial rules have an intrinsic value, apart from being a stepping stone towards a positive rule. In fact, a *really good* signed rule can be incredibly useful. For example, the celebrated *Murnaghan–Nakayama rule* for the S_n characters (see e.g. [Sta99, §7.17]) is omnipresent in the algebraic combinatorics literature, and has many powerful applications in other areas. This is despite the fact that its absolute value has no positive rule, unless the polynomial hierarchy collapses [IPP24].

This paper gives further evidence in favor of this point of view, as our Theorem 1.2 gives a *structural result* that was not easily attainable prior to the signed puzzle rule in Theorem 1.1. In fact, even the definition of $\gamma_k(n)$ is mysterious from algebraic point of view. However, fixing the height of the region is completely natural in the tiling literature, see e.g. [MSV00, Moo99]. There, the number of tilings is usually computed using generating functions and the *transfer-matrix method*, see e.g. [Sta99, §4.7]. While the technical details are quite different, the connections to Ehrhart theory have a similar flavor.

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References

[Bar97] Alexander Barvinok, Lattice points and lattice polytopes, in *Handbook of discrete and computational geometry*, CRC, Boca Raton, FL, 1997, 133–152.

[BR15a] Matthias Beck and Sinai Robins, *Computing the continuous discretely* (Second ed.), Springer, New York, 2015, 285 pp.

[BR15b] Arkady Berenstein and Edward Richmond, Littlewood–Richardson coefficients for reflection groups, *Adv. Math.* **284** (2015), 54–111.

[Bil99] Sara Billey, Kostant polynomials and the cohomology ring for G/B, *Duke Math. J.* **96** (1999), 205–224.

[BKPT16] Anders S. Buch, Andrew Kresch, Kevin Purbhoo and Harry Tamvakis, The puzzle conjecture for the cohomology of two-step flag manifolds, *J. Algebraic Combin.* **44** (2016), 973–1007.

[Duan05] Haibao Duan, Multiplicative rule of Schubert classes, *Invent. Math.* **159** (2005), 407–436.

[Ful97] William Fulton, *Young tableaux*, Cambridge Univ. Press, Cambridge, UK, 1997, 260 pp.

[GK21] Rebecca Goldin and Allen Knutson, Schubert structure operators and $K_T^*(G/B)$, *Pure Appl. Math. Q.* **17** (2021), 1345–1385.

[GS87] Branko Grünbaum and Geoffrey C. Shephard, *Tilings and patterns*, Freeman, New York, 1987, 700 pp.

[IPP24] Christian Ikenmeyer, Igor Pak and Greta Panova, Positivity of the symmetric group characters is as hard as the polynomial time hierarchy, *Int. Math. Res. Not.* (2024), no. 10, 8442–8458.

[Knu01] Allen Knutson, Descent-cycling in Schubert calculus, *Experiment. Math.* **10** (2001), 345–353.

[Knu03] Allen Knutson, A Schubert calculus recurrence from the noncomplex W-action on G/B, preprint (2003), 10 pp.; `arXiv:math/0306304`.

[Knu16] Allen Knutson, Schubert calculus and puzzles, in *Schubert calculus*, MSJ, Tokyo, 2016, 185–209.

[Knu23] Allen Knutson, Schubert calculus and quiver varieties, in *Proc. ICM*, Vol. VI, Sec. 12–14, EMS Press, Berlin, 2023, 4582–4605.

[KT03] Allen Knutson and Terence Tao, Puzzles and (equivariant) cohomology of Grassmannians, *Duke Math. J.* **119** (2003), 221–260.

[KZ17] Allen Knutson and Paul Zinn-Justin, Schubert puzzles and integrability I: invariant trilinear forms, preprint (2017), 51 pp.; `arXiv:1706.10019`.

[KZ21] Allen Knutson and Paul Zinn-Justin, Schubert puzzles and integrability II: multiplying motivic Segre classes, preprint (2021), 80 pp.; `arXiv:2102.00563`.

[KZ23] Allen Knutson and Paul Zinn-Justin, Schubert puzzles and integrability III: separated descents, preprint (2023), 42 pp.; `arXiv:2306.13855`.

[Mac91] Ian G. Macdonald, *Notes on Schubert polynomials*, Publ. LaCIM, UQAM, Montreal, 1991, 116 pp.

[Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS, Providence, RI, 2001, 167 pp.

[MSV00] Donatella Merlini, Renzo Sprugnoli and M. Cecilia Verri, Strip tiling and regular grammars, *Theoret. Comput. Sci.* **242** (2000), 109–124.

[Moo99] Cristopher Moore, Some polyomino tilings of the plane, preprint (1999), 13 pp.; `arXiv:math/9905012`.

[Pak24] Igor Pak, What is a combinatorial interpretation?, in *Open Problems in Algebraic Combinatorics*, AMS, Providence, RI, 2024, 191–260.

[PR24a] Igor Pak and Colleen Robichaux, Signed combinatorial interpretations in algebraic combinatorics, preprint (2024), 24 pp.; to appear in *Algebraic Combinatorics*; `arXiv:2406.13902`.

[PR24b] Igor Pak and Colleen Robichaux, Vanishing of Schubert coefficients (Appendix C joint with David E Speyer), preprint (2024), 30 pp. `arXiv:2412.02064`; extended abstract to appear in *Proc. 57th STOC* (2025), 12 pp.

[PR25] Igor Pak and Colleen Robichaux, Vanishing of Schubert coefficients is in $\mathsf{AM} \cap \mathsf{coAM}$ assuming the GRH, preprint (2024), 18 pp.

[PY17] Oliver Pechenik and Alexander Yong, Equivariant K-theory of Grassmannians II: the Knutson–Vakil conjecture, *Compos. Math.* **153** (2017), 667–677.

[PS09] Alexander Postnikov and Richard P. Stanley, Chains in the Bruhat order, *J. Algebraic Combin.* **29** (2009), 133–174.

[Sta99] Richard P. Stanley, *Enumerative Combinatorics*, vol. 1 (Second ed.) and vol. 2, Cambridge Univ. Press, 2012 and 1999, 626 pp. and 581 pp.

[Sta00] Richard P. Stanley, Positivity problems and conjectures in algebraic combinatorics, in *Mathematics: frontiers and perspectives*, AMS, Providence, RI, 2000, 295–319.

[Sta17] Richard P. Stanley, Some Schubert shenanigans, preprint (2017), 8 pp.; `arXiv:1704.00851`.

[vEB97] Peter van Emde Boas, The convenience of tilings, in *Complexity, logic, and recursion theory*, Dekker, New York, 1997, 331–363.

[Yong06] Alexander Yong, Maple 7 code to compute Schubert calculus in G/B, 2006; software available at ayong.web.illinois.edu/papers.html