Lattice Path Conference
21-25 June 2021
Presentation times for this poster:

Tuesday	6-7 pm
Thursday	6-7 pm

Log-concavity in posets and random walks

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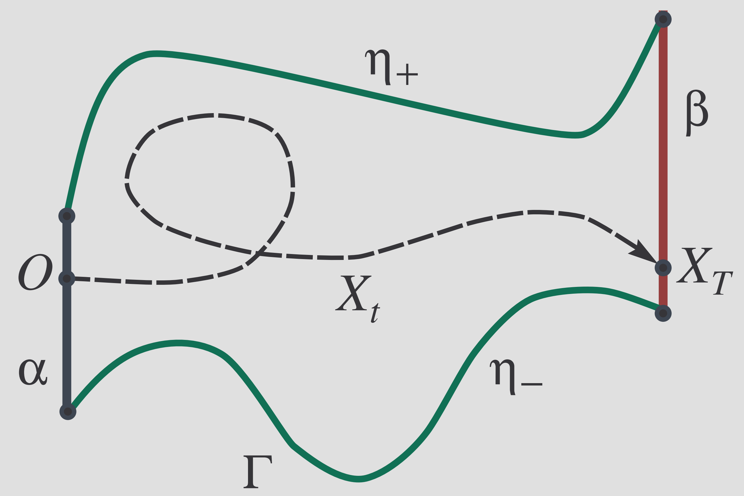
Simple random walk on $\mathbb{Z}^2$

- **Step set:** $\mathcal{S} = \{(0,1),(1,0),(0,-1),(-1,0)\} \subset \mathbb{Z}^2$
- ***n*-step lattice path:** sequence of steps $(v_1,\ldots,v_n) \in \mathcal{S}^n$
- **Probabilistic weights**: $\{p_{0,1}, p_{1,0}, p_{0,-1}, p_{-1,0}\}$, $p_s \in [0,1]$ s.t. $p_s \in [0,1]$ s.t. $\sum_{s\in\mathcal{S}} p_s = 1$.

Random walk constrained in a region Γ

- Vertical line α acts as: **Left-boundary** $p_{-1,0} = 0$.
- x-monotone curve η_+ acts as: **Upper-boundary** $p_{0,1} = 0$.
- x-monotone curve η_- acts as: **Lower-boundary** $p_{0,-1} = 0$.
- Vertical line β acts as: **Absorbing boundary** $p_{0,0} = 1$.

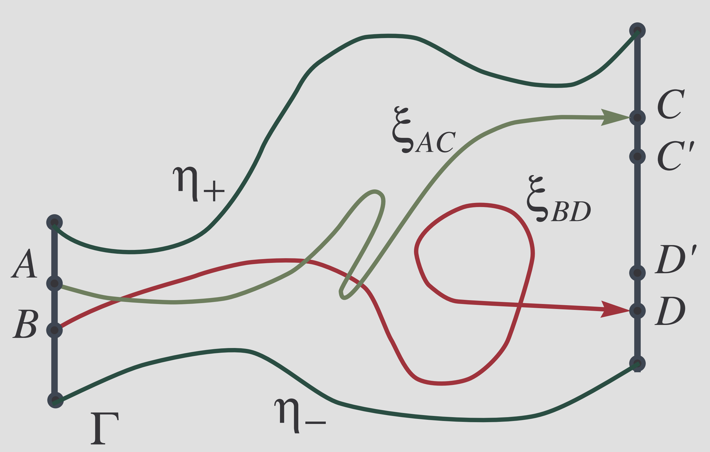


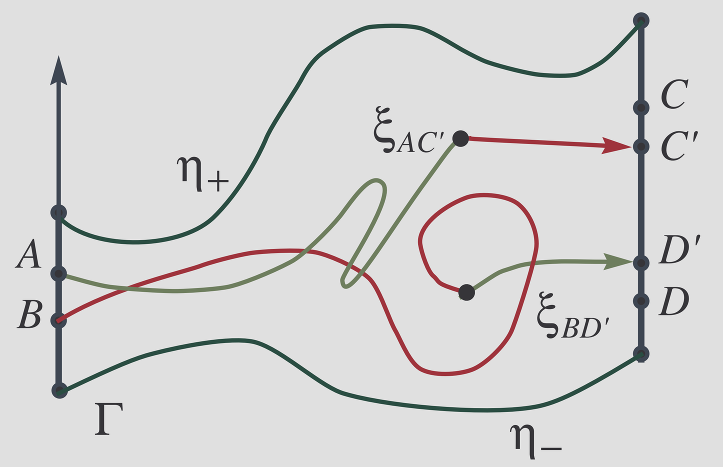
Log-concavity for the hitting probability

Let $P(k)$ be probability that the final altitude of the random walk is k.

Then $P(k)^2 \geq P(k-1)\,P(k+1)$ for every integer k.

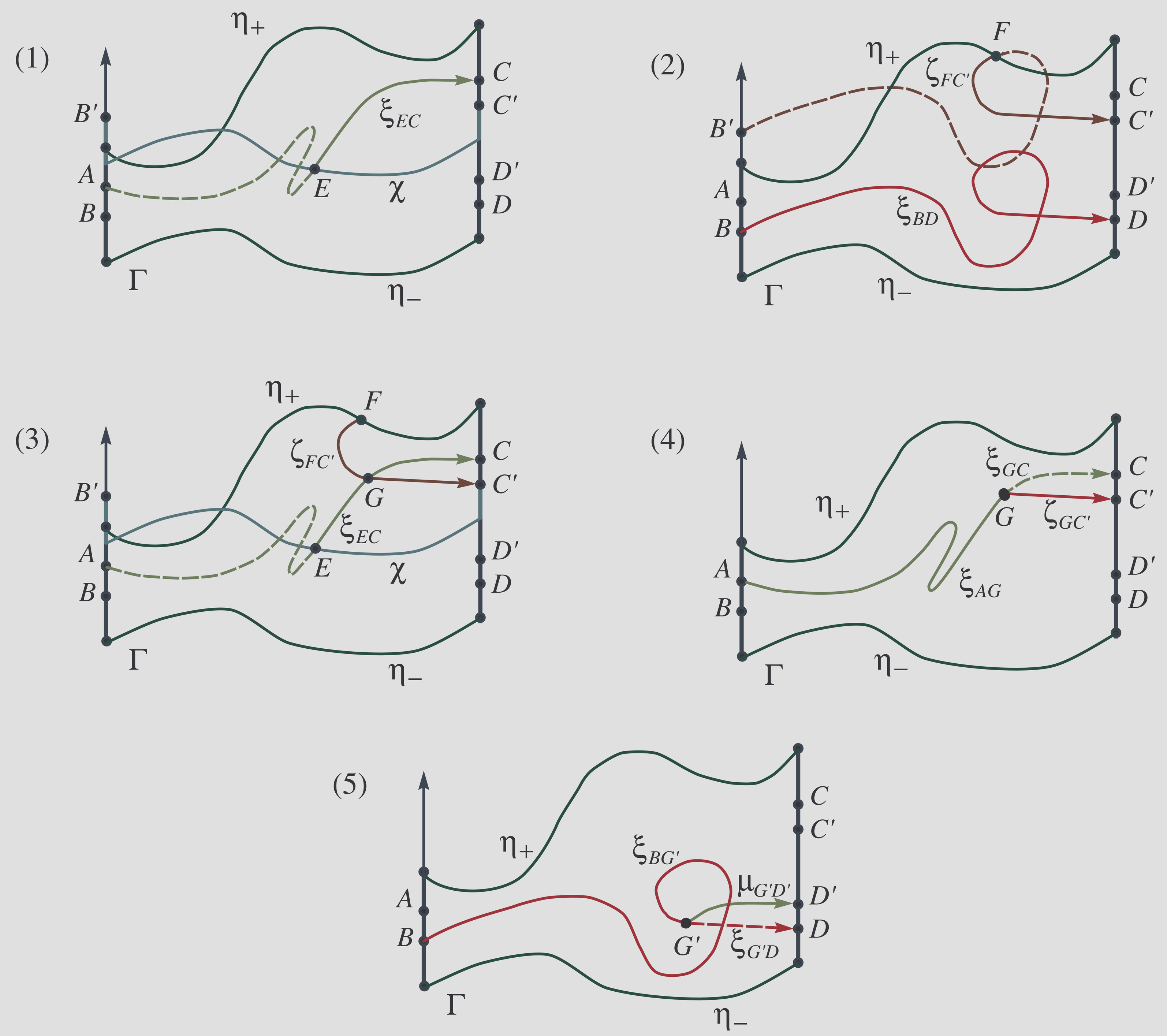
The injection that proves the log-concavity theorem





Paths start at:	A and B, with A below B.
Paths end at:	C, C', D, D', with C highest and D lowest.
Condition:	$\lvert AB\rvert \leq \lvert C'D\rvert$ and $\lvert CC'\rvert = \lvert DD'\rvert$.
Input:	Path ξ_{AC} from A to C, and path ξ_{BD} from B to D.
Output:	Path $\xi_{AC'}$ from A to C', and path $\xi_{BD'}$ from B to D'.

How the injection works



1. χ is the path η_- shifted up by $\overrightarrow{DC'}$.
   E is the last point in ξ_{AC} that intersects χ.
2. $\zeta_{B'C'}$ is the path ξ_{BC} shifted up by $\overrightarrow{DC'}$.
   F is the last point in $\zeta_{B'C'}$ that intersects η_+.
3. G is lexicographically smallest point in the intersection of ξ_{EC} and $\zeta_{FC'}$.
4. To construct $\xi_{AC'}$, first follow ξ_{AG}, then follow $\zeta_{GC'}$.
5. $\mu_{G'D'}$ is the path $\zeta_{GC'}$ shifted down by $\overrightarrow{C'D}$.
   To construct $\xi_{BD'}$, first follow $\xi_{BG'}$, then follow $\mu_{G'D'}$.

Partially ordered sets of width 2

- **Ground set** X is union of $C_1 = \{\alpha_1,\ldots,\alpha_a\}$ and $C_2 = \{\beta_1,\ldots,\beta_b\}$.
- **Partial order** $\prec$ satisfies $\alpha_1 \prec \cdots \prec \alpha_a$ and $\beta_1 \prec \cdots \prec \beta_b$.
  (The partial order $\prec$ can have more relations.)
- **Linear extension** is **order preserving function** from X to $[a+b]$.

Linear extensions are in bijection with lattice paths

- **Linear extension** L corresponds to **lattice path** $v_1,\ldots,v_{a+b}$ from $(0,0)$ to (a,b), where
  $$v_i = (1,0) \;\text{ if }\; L^{-1}(i) \in C_1, \quad\text{and}\quad v_i = (0,1) \;\text{ if }\; L^{-1}(i) \in C_2.$$
- The **boundaries** η_+ and η_- are lattice paths corresponding to C_1-maximal and C_1-minimal linear extensions, respectively.

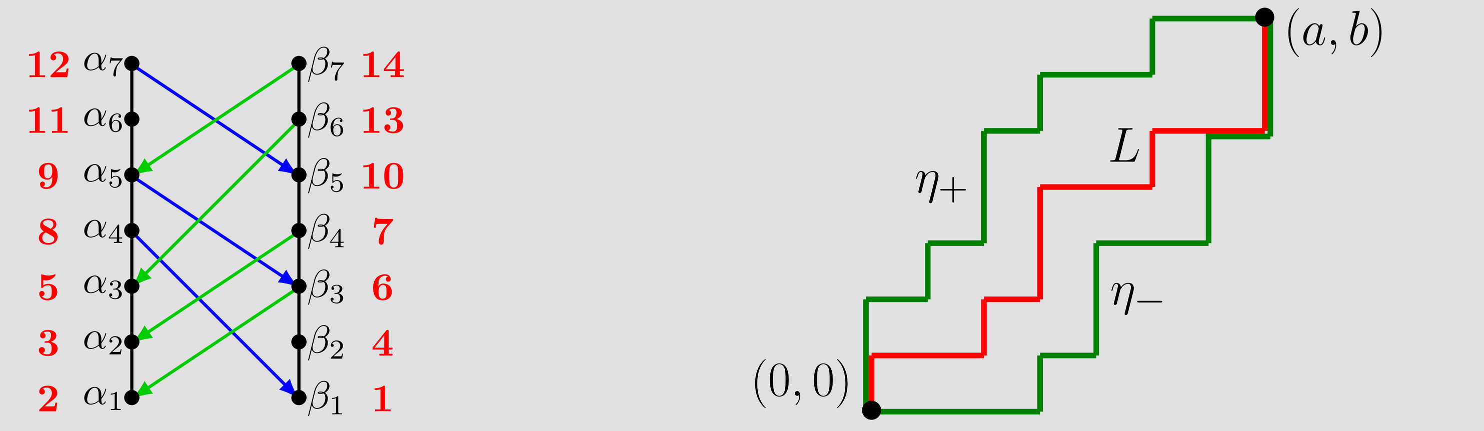


Figure: Left: Hasse diagram of a width 2 poset and a linear extension (red labels). Right: The associated lattice path (in red) with boundaries η_+, η_- (in green).

Application: Stanley inequality for width 2 posets

Fix $x \in X$. Let $N(k)$ counts linear extensions L with $L(x) = k$.

Then $N(k)^2 \geq N(k-1)\,N(k+1)$ for every integer k.

Application: Kahn–Saks inequality for width 2 posets

Fix $x, y \in X$. Let $F(k)$ counts linear extensions L with $L(y) - (x) = k$.

Then $F(k)^2 \geq F(k-1)\,F(k+1)$ for every integer k.

Other results

- Equality conditions for all these inequalities are attained.
- Extensions to multivariate versions of Stanley, Kahn–Saks inequalities.
- Methods can be generalized to prove cross-product inequalities and other correlation inequalities for posets of width 2.

References

[1] S. H. Chan, I. Pak, G. Panova, *Extensions of the Kahn–Saks inequality for posets of width two*, arXiv:2106.07133.

[2] S. H. Chan, I. Pak, G. Panova, *Log-concavity in planar random walks*, arXiv:2106.10640.