Enumeration of spanning trees of certain graphs

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In this note we give an algorithm which enables us to encode and enumerate all the spanning trees of a multipartite graph (see below). This algorithm may turn out to be useful for the enumeration of spanning trees satisfying certain conditions.

The number of spanning trees of a given graph Γ without loops and without multiple edges will be denoted by $t(\Gamma)$. We shall consider the graphs $\Gamma = \Gamma(G;\ G_1, ..., G_k)$, where G is a graph with vertices $\bar{1}, \bar{2}, ..., \bar{k}$, and Γ is obtained from it by replacing the vertex $\bar{i}$ by G_i, where, for vertices $a \in G_i$, $b \in G_j$ $(i \neq j)$, the edge $(a, b) \in \Gamma$ if and only if $(\bar{i}, \bar{j}) \in G$.

Theorem.

$$(1) \qquad t(\Gamma(G; G_1, \ldots, G_k)) = \prod_{l=1}^{k} \left(\sum_{i=1}^{n_l} f_l(i)\, d(l)^{i-1} \right) \sum_{\gamma} \prod_{r=1}^{k} n_r^{\rho_\gamma(\bar{r})-1},$$

where $n_i := |G_i|$, $d_i = \sum_{j=1}^{k} m_{ij} n_j$, *$(m_{ij})$ is the adjacency matrix of the vertices of the graph G, $f_l(i)$ is the number of spanning rooted forests of G_l with i connected components* (a spanning rooted forest of a graph is a forest containing all the vertices of the graph in which a vertex called the *root* has been selected in each connected component); *the second summation is taken over all spanning trees γ of G, and $\rho_\gamma(\bar{r})$ denotes the degree of the vertex $\bar{r}$ in the graph γ.*

We shall describe here a method of encoding the spanning trees of the graph Γ. We label the vertices of the graph G_1 by the numbers $1, 2, ..., n_1$, those of the graph G_2 by $n_1+1, ..., n_1+n_2, ...,$ and those of the graph G_k by $N-n_k+1, ..., N$, so that $N = \sum_{i=1}^{k} n_i$. We shall encode each spanning tree α of Γ by the set of sequences $P_1, P_2, ..., P_k, R$, of vertices of Γ of length $n_1-1, n_2-1, ..., n_k-1, k-2$ respectively.

We shall first describe a method of encoding trees due to Prüfer (see for example [1], [2]).

Let T be a tree with vertices labelled by distinct natural numbers. Consider the sequence of edges (b_i, a_i) of T constructed as follows: b_1 is the terminal vertex in T labelled by the smallest number (a_1 is then uniquely determined); similarly b_2 is the terminal vertex with the smallest number in the tree $T \setminus (a_1, b_1)$, and so on. We have thus constructed a sequence of length $|T|-2$. The sequence $a_1, a_2, ..., a_{|T|-2}$ will be called the *Prüfer code.*

We now orient a spanning tree α of Γ towards the root at the vertex labelled N. Let μ_i denote the directed rooted forest $\alpha \cap G_i$ (all its trees are directed towards their roots). From each μ_i we shall form a tree $\tilde{\mu}_i$. To do this we join each root of μ_i to a formal vertex $\tilde{i}$. Let $G_i+\{\tilde{i}\}$ be the graph containing the vertex $\tilde{i}$ joined to all the vertices of G_i. Then $\tilde{\mu}_i$ is a spanning tree of $G+\{\tilde{i}\}$. We shall assume that the vertex $\tilde{i}$ has a maximal label, and we find P_i', the Prüfer code of the tree $\tilde{\mu}_i$, which we write down in the sequence P_i.

Consider the tree α' obtained from α by contraction of the rooted forests μ_i to their roots; we find the Prüfer sequence of edges (b_i, a_i) for the tree α'. If $b_1 \in G_j$, then we replace the first occurrence of $\tilde{j}$ in P_j by the vertex a_1' such that the edge $(b_1, a_1') \in \alpha$ (in the given orientation). We deal similarly with the edge (b_2, a_2) and so on. If at the rth step $b_r \in G_i$, but $\tilde{i}$ does not occur in P_i, then we write a_r' in the first free place in the sequence R. By repeating one of these operations we arrive at the final code: $P_1, P_2, ..., P_k, R$.

***Lemma* 1.** *A set of sequences $P_1, P_2, ..., P_k, R$ of lengths $n_1-1, n_2-1, ..., n_k-1, k-2$ respectively is the code of some tree α if and only if the following conditions are satisfied:*

1) *for each i, $a \in P_i \Rightarrow a \in G_i \cup D_i$, where $D_i := \bigcup_{m_{ij}>0} G_j$;*

2) *let P_i' be the sequence formed from P_i by replacing every b in P_i that is not a vertex of G_i by $\tilde{i}$; then for each i the sequence P_i' is the Prüfer code of a spanning tree of the graph $G_i+\{i\}$;*

3) *let the sequence R' be formed from R by replacing every $a \in G_i$ by $\bar{i}$. Then R' is the Prüfer code of some spanning tree of G.*

***Lemma* 2.** *This encoding sets up a bijection between the spanning trees of the graph* Γ *and the sequences satisfying the conditions of Lemma* 1.

***Lemma* 3.** *For each set of spanning rooted forests* $\mu_1, \mu_2, \dots, \mu_k$ *of graphs* $G_1, G_2, \dots, G_k$ *respectively and spanning tree* β *of the graph* G, *the number of spanning trees of the graph* Γ *corresponding to* $\mu = (\mu_1, \dots, \mu_n)$ *and* β *(see Lemma* 1, parts 2), 3)) *is equal to*

$$t(\Gamma, \mu, \beta) = \prod_{l=1}^{k} \left(d(l)^{\delta_l - 1} n_l^{\rho_\beta(l)-1}\right), \tag{2}$$

where δ_l *is the number of connected components of the forest* μ_l.

It is not difficult to find the method of decoding inverse to the encoding algorithm above, which, given a sequence satisfying the conditions of Lemma 1, constructs a spanning tree of the graph Γ. We have thus obtained a method of running through all spanning trees. Lemmas 1 and 2 follow from this, and the proofs of Lemma 3 and the Theorem now follow easily.

***Corollary* 1.** *Let* $n_1 = n_2 = \dots = n_k = 1$, $G = \Gamma = K_k$ *the complete graph with* k *vertices. Then*

$$t(\Gamma) = k^{k-2}. \tag{3}$$

This is Cayley's well-known formula [3]. The idea of encoding trees to compute $t(\Gamma)$ is due to Prüfer [2]. In this case our code just consists of the sequence R coinciding with the Prüfer code.

***Corollary* 2.** *Let* $n_1 = p$, $n_2 = q$, $k = 2$, $\Gamma = K_{p,q}$ *the bipartite graph with* p *vertices in one part and* q *in the other. Then*

$$t(\Gamma) = p^{q-1} q^{p-1}. \tag{4}$$

Formula (4) was obtained by Scoins [4] and proved by means of the Rényi encoding [5].

***Corollary* 3.** *Let* $G_i = O_{n_i}$ *(where* O_m *is the empty graph with* m *vertices),* $G = K_k$. *Then* $\Gamma = K_{n_1, n_2, \dots, n_k}$ *and*

$$t(\Gamma) = (N - n_1)^{n_1 - 1} (N - n_2)^{n_2 - 1} \dots (N - n_k)^{n_k - 1} N^{k-2}. \tag{5}$$

Formula (5) is a generalization of (3) and (4). A proof by Austin may be found in [6], and an encoding in a paper of Oláh [7]; in this case his code is the same as ours.

***Corollary* 4.** *Let* $G_i = O_{n_i}$ $(i = 1, 2, \dots, k)$; *then* $t(\Gamma) = \prod_{i=1}^{k} d(i)^{n_i - 1} \sum_{\gamma} \prod_{j=1}^{k} n_j^{\rho_\gamma(j)-1}$.

The authors are grateful to A.V. Zelevinskii for the interest he has shown in this paper.

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Received by the Baord of Governors 16 January 1990