QUANTUM DETERMINANTS IN POLYNOMIAL TIME

IGOR PAK AND DANIEL SOSKIN

Abstract. We give an algebraic branching program of polynomial size which computes Cayley determinant of right quantum matrices. This is a rare example of an efficient computation of a noncommutative determinant, and the first such example for quantum groups. We extend the results to the q-Cayley determinant of q-right quantum matrices, as well as to their multiparameter generalization.

The proofs are entirely combinatorial, as we relate Cayley, Moore and Valiant determinants using bijections/involutions on words. We then employ the celebrated determinant construction of Mahajan and Vinay (SODA'97), to obtain the results.

1. Introduction

1.1. **Foreword.** The *determinant* is one of the most classical and well studied polynomials, ever since it was introduced by Leibnitz in 1693, see e.g. [Muir28]. The *noncommutative determinant*, first studied by Cayley [Cay45], plays a similar role (see §2.1). While computing commutative determinants is extremely well understood (see §2.3), computing noncommutative determinants is quite difficult, and remains a fundamental problem in its own right. In the complexity theoretic setting, this problem was first considered by Hyafil [Hya77], building on the work of Winograd [Win70] and Strassen [Str73], and has attracted considerable attention over the past few decades.

In a landmark paper [Nis91], Nisan established an $\Omega(2^n)$ lower bound for the size of the *algebraic branching program* (ABP) for computing the noncommutative $n \times n$ determinant over free algebra $\mathbb{C}\langle x_{11}, \ldots, x_{nn}\rangle$. This makes the noncommutative determinant similar to the noncommutative permanent, for which Nisan proved the same lower bound, and markedly different from the determinant over commutative rings, where there is a polynomial size ABP, see e.g. [Bür26].

Motivated in part by the *permanent vs. determinant paradigm* introduced by Valiant [Val79a], much effort has been made to build on Nisan's paper. Notably, Chien and Sinclair [CS07] showed that the exponential lower bound holds for matrices over the quaternion algebra, or over the algebra of 2×2 matrices with entries in a field of characteristic zero (among other examples). They asked for which algebras does the determinant requires large ABP size.

Since Nisan's ABP model is somewhat constrained, one can ask if there is a polynomial size *arithmetic circuit* for the noncommutative determinant. In a surprising discovery, Arvind and Srinivasan [AS18] showed that computing the noncommutative determinant is at least as hard as the permanent (up to polynomial factors), and thus at least as hard as the commutative permanent. Soon after, Chien et al. [CHSS11] and Bläser [Blä15] emulated Valiant's proof that 0/1 permanent is #P-complete [Val79b], to resolve the problem over which associative algebras $\mathcal{A}$ the determinant is hard. They showed that if $\mathcal{A}/\mathrm{rad}\mathcal{A}$ is noncommutative, then computing the determinant is #P-hard; it is in FP otherwise.[1] See also Gentry [Gen14], for a simpler approach via Barrington's theorem [Bar86].

Another motivation for studying noncommutative determinants lies in their ability to approximate the 0/1 permanent. In a commutative setting, this approach was introduced by Godsil and Gutman [GG81] over the reals, and further developed by Karmarkar et al. [K+93] and by Barvinok [Bar99], over the complex numbers and quaternions, respectively. The noncommutative version was further studied in [Bar00, CRS03, MR12].

In a positive direction, computing noncommutative determinants clearly hit a wall, with the progress stifled by the powerful negative results described above. This is somewhat surprising, since

[1] Here we are assuming that the algebra $\mathcal{A}$ is defined over $\mathbb{Q}$.

there are extremely well-known noncommutative associative algebras[2] used across representation theory and algebraic combinatorics, which are not of the matrices-over-algebras type.

1.2. **Our results.** In this paper, we prove that (noncommutative) determinants of quantum matrices have a polynomial size ABP.[3] By a determinant we mean (the usual) *Cayley determinant* of a matrix $A = (a_{ij})$ with noncommutative entries:

$$\operatorname{Cdet}(A) \ := \ \sum_{\sigma \in S_n} (-1)^{\operatorname{inv}(\sigma)} \, a_{1\sigma_1} \cdots a_{n\sigma_n} \, . \tag{1.1}$$

We define *quantum matrices* step-by-step, as they comes in flavors of varying difficulty, and the most general results combine the flavors.

First, we consider *Cartier–Foata (CF) matrices* [CF69, Foa65], defined to have commuting elements in different rows (see a discussion in §3.1 for our matrix notation):

$$a_{\ell j} \, a_{ki} \ = \ a_{ki} \, a_{\ell j} \quad \text{for} \quad i \neq j \, . \tag{1.2}$$

Even though the products in Cayley determinants of CF matrices have commuting entries, the usual linear algebra technology no longer applies.

The celebrated approach by Mahajan and Vinay [MV97, MV99], gives an ABP to compute the determinant as a commutative polynomial in matrix entries. With some modifications (see Remark 7.5), we show that this approach can be extended to compute the Cayley determinant of CF matrices. This is the simplest of our results, see below.

Second, for a complex parameter $q \neq 0$, we consider *q-CF matrices* :

$$\begin{cases} a_{\ell j} \, a_{ki} \ = \ a_{ki} \, a_{\ell j} & \text{for} \ \ i < j, \, k < \ell, \\ a_{\ell j} \, a_{ki} \ = \ q^2 \, a_{ki} \, a_{\ell j} & \text{for} \ \ i < j, \, k > \ell, \\ a_{kj} \, a_{ki} \ = \ q \, a_{ki} \, a_{kj} & \text{for} \ \ i < j \, . \end{cases} \tag{1.3}$$

When $q = 1$ we obtain the usual CF matrices. In this case, we compute the *q-Cayley determinant*, also called the *quantum determinant*:

$$\operatorname{Cdet}_q(A) \ := \ \sum_{\sigma \in S_n} (-q)^{-\operatorname{inv}(\sigma)} \, a_{1\sigma_1} \cdots a_{n\sigma_n} \tag{1.4}$$

When $q = -1$ this is *Cayley's permanent*. See [HK23] for the hardness of computing Cdet_q at roots of unity in the commutative case.

Third, for an array $\boldsymbol{q} = (q_{ij})_{1 \leq i < j \leq n}$ of nonzero complex parameters, we define the ***q*-CF matrices** and the ***q*-Cayley determinant**, giving a multiparameter quantum deformations of (1.3) and (1.4), respectively. See §3.2, §3.3 for the definitions.

Fourth and most important, we consider *right-quantum (RQ) matrices* :

$$\begin{cases} a_{kj} \, a_{ki} \ = \ a_{ki} \, a_{kj} & \text{for} \ \ i < j, \\ a_{ki} \, a_{\ell j} \ - \ a_{kj} \, a_{\ell i} \ = \ a_{\ell j} \, a_{ki} \ - \ a_{\ell i} \, a_{kj} & \text{for} \ \ i < j, \, k < \ell. \end{cases} \tag{1.5}$$

Note that the RQ matrices generalize the CF matrices by introducing new noncommutative relations on 2×2 minors. In this case, we give an ABP construction to compute the Cayley determinant.

Next, we consider *q-RQ matrices* defined as a q-deformation of relations (1.5), and that also generalize the q-CF matrices:

$$\begin{cases} a_{kj} \, a_{ki} \ = \ q \, a_{ki} \, a_{kj} \, , & \text{for} \quad i < j, \\ a_{kj} \, a_{\ell i} \ - \ q \, a_{ki} \, a_{\ell j} \ = \ q^2 a_{\ell i} \, a_{kj} \ - \ q \, a_{\ell j} \, a_{ki} & \text{for} \quad i < j, \, k < \ell. \end{cases} \tag{1.6}$$

This algebra was introduced by Faddeev, Reshetikhin and Takhtajan [FRT88] in connection with quantum groups, and motivated by representation theory and mathematical physics (see below). When $q = -1$, we obtain *antisymmetric right-quantum (ARQ) matrices*. Our general algorithm in

[2] All algebras in this paper are associative, so we will not specify this in the future.

[3] Over $\mathbb{Q}$, this implies that the determinants can be computed in polynomial time.

this case gives unusual examples, where the noncommutative permanent (but not necessarily the determinant!) can be computed efficiently.

Finally, we consider the *$\boldsymbol{q}$-RQ matrices*, which give a multiparameter generalization of relations (1.6), and which generalize the $\boldsymbol{q}$-CF matrices. See (3.2) for the definition. These matrices were introduced by Manin [Man88, Man89], motivated additionally by the ideas from noncommutative geometry, in an effort to construct a noncommutative linear algebra.

Theorem 1.1 (Main theorem)**.** *For all nonzero $\boldsymbol{q} = (q_{ij})$ as above, there is an ABP of polynomial size, which computes the $\boldsymbol{q}$-Cayley determinant of $\boldsymbol{q}$-RQ matrices. In particular, it computes the (usual) Cayley determinant of RQ matrices and the Cayley permanent of ARQ matrices.*

This is a rare positive result for noncommutative determinants, the first complexity result on quantum matrices in either direction, and an unusual polynomial time result for the exact computation of the permanent. An extensive background overview is given in the next section.

In the RQ case, the proof is based on two arguments in bijective combinatorics, see Theorem 4.1 for the main nonalgorithmic result of the paper. First, we modify the Konvalinka–Pak bijection in [KP07], to show that the Cayley determinant is equal to the Moore determinant, where the latter corresponds to factorizations of permutations into cycles. Second, we build on the Mahajan–Vinay involution to show that Moore's determinant is equal to Valiant's determinant, which is defined in terms of *clows* (closed walks), combined with algebraic properties of RQ matrices. In the $\boldsymbol{q}$-RQ case, a great deal of careful bookkeeping is also involved.

Remark 1.2. Some special cases of the theorem are easy to state and give striking conclusions. For example, one can take certain explicit 2×2 matrices a_{ij} which do not commute, but satisfy relations (1.5). See e.g. [HL07b], where these matrices are given in connection with the Jones polynomial in knot theory. Our Main Theorem 1.1 in this case gives a polynomial size ABP to compute the Cayley determinant of such matrices. This is in sharp contrast with the above mentioned result by Chien and Sinclair [CS07], that to compute the Cayley determinant of *all* such matrices one needs an exponential size ABP.

2. Background

2.1. **Noncommutative determinants.** In contrast with the commutative case, there are multiple notions of noncommutative determinants of varying degree of generality and applicability. The earliest definition (1.1) indeed goes back to Cayley [Cay45], and was motivated by Hamilton's work on quaternions. To put it in historical perspective, the *Cayley–Hamilton theorem* was obtained only in the 1850s. We refer to [Abe11] for the historical introduction.

Next, we have *Moore's determinant* defined by E. H. Moore [Moo22], in the context of Hermitian matrices over quaternions. It is a summation over all permutations given as an ordered product of cycles. Moore's determinant will play a crucial role throughout the paper, see §3.5 for the definition. An extensive discussion of Moore's determinant in a historical context is given in [Asl96].

Other notable examples include the *Dieudonné determinant* [Die43] for matrices over skew fields, see e.g. [Dra83, §20], and *Berezinian* for superalgebras[4] introduced by Berezin in [Ber87]. More recently, Barvinok [Bar00] introduced the *symmetrized determinant* with the goal of giving an approximation of the permanent, see also [ABG26] for complexity aspects.

The richest and most insightful are *quasideterminants* by Gelfand and Retakh [GR92, GGRW05]. This is not a single object but a family of rational functions in the division ring of free variables, evaluated at noncommutative matrix entries, and related to each other by certain "homological relations". Although this definition unites many other and underlies the philosophy of this paper, we will not need them for our results, see however §10.2

[4]These are $\mathbb{Z}_2$-graded algebras describing supersymmetries, originally studied in theoretical physics.

2.2. **Quantum matrices.** The CF matrices are sometimes called *row-commutative* and *partially commutative*. They were introduced by Cartier and Foata [Foa65, CF69] in an attempt to understand, generalize and give a combinatorial proof of the *MacMahon's Master Theorem* (MMT), a classical result in enumerative combinatorics, see e.g. [GJ83, §1.2.11].

Similarly, the q- and $\boldsymbol{q}$-CF matrices were introduced by Foata–Han [FH07b] and by Konvalinka–Pak [KP07], respectively, as special cases of the q-RQ and $\boldsymbol{q}$-RQ matrices. The motivation was to give a combinatorial proof of the q-MMT obtained earlier by Garoufalidis, Lê and Zeilberger [GLZ06]. Consequently, the $\boldsymbol{q}$-MMT generalization and a bijective proof was given in [KP07]. Another proof based on Koszul duality was given in [HL07a]. See also [FH07a] for the first combinatorial proof of the q-MMT.

The right-quantum (RQ) matrices form a Hopf algebra $\mathcal{O}_q(G)$ isomorphic to the coordinate ring of the algebraic group $G = \mathrm{GL}(n, \mathbb{C})$. As we mentioned above, q-RQ matrices were introduced in [FRT88], as a way to deform this Hopf algebra. It is Hopf dual to the quantum group construction by Drinfeld–Jimbo, which is a q-deformation of the universal enveloping algebra $U_q(\mathfrak{gl})$. This approach to define quantum matrices starting with the R-matrix from the *Yang–Baxter equation*, is now known as the *FRT construction*.

The quantum determinant (1.4) spans the center of the $\mathcal{O}_q(G)$, so $\mathcal{O}_q(G)/(\mathrm{Cdet}_q = 1)$ gives a q-deformation of the coordinate ring of $\mathrm{SL}(n, \mathbb{C})$, ibid. We refer to [BG02, Kas95] for standard introductions to the area, including the FRT construction. See also [DF93] for a *Gauss decomposition* in this setting.

Let us mention that both q-RQ matrices and the q-Cayley determinant also appear in connection to the *Geometric Complexity Theory* (GCT), as shown by Blasiak, Mulmuley and Sohoni [BMS15]. They employed these tools in an effort to advance the *Kronecker coefficients problem* that is central in GCT, see e.g. an extensive discussion in [Aar16, §6.6].

The first connection between quantum determinants and quasideterminants was given by Krob and Leclerc [KL95]. We refer to [GR91, ER99] and [KP07], for explicit formulas relating the q- and $\boldsymbol{q}$-Cayley determinants with quasideterminants of q- and $\boldsymbol{q}$-RQ matrices, respectively. See also [ESS00, Rem. 1.3] for an advanced generalization.

Computing q-Cayley determinants of q-RQ matrices has also important applications in knot theory, as they determine the *colored Jones polynomial* originally introduced in [Jon87]. Motivated by the q-MMT, it was proved by Huynh and Lê [HL07b], that this polynomial is the inverse of the q-Cayley determinant of an almost q-RQ matrix with entries given by certain 2×2 matrices over q-differential operators corresponding to crossing of the knot. Another substitution gives the Alexander polynomial (ibid.) See also [HL18] for an implementation of this formula for knots with few crossings.

As we mentioned in the introduction, the $\boldsymbol{q}$-RQ matrices were defined by Manin in [Man88, Man89]. They are also called *Manin* and *row-pseudo-commutative matrices*, and are extremely well-studied. The *$\boldsymbol{q}$-Cayley determinant* was defined in [Man89, §1.9] in terms of exterior powers, and expanded in terms of inversions in [Man88, §3.5], see (3.3). Manin also showed that it spans the center of the algebra of $\boldsymbol{q}$-RQ matrices. We refer to [CSS09, CFR09, CFRS14, JLZ24, Sil21] for numerous algebraic and combinatorial identities for these matrices, generalizing classical results in linear algebra.

2.3. **Algorithmic aspects.** Edmonds [Edm67] was the first to rigorously analyze a version of the Gaussian elimination algorithm over $\mathbb{Q}$, and show that it runs in polynomial time. Kalorkoti [Kal85] proved the lower bound $\Omega(n^3)$ on the size of the arithmetic formula for computing the $n \times n$ determinant.

In [Str73], Strassen showed that how any such algorithm can be made division-free, giving an algorithm for computing the determinant in $O(n^5)$ steps, see a discussion in [BGH82].[5] Berkowitz [Ber84] gave the first explicit division-free algorithm to compute the determinant. Building on the work of Chistov and Valiant [Chi85, Val92], Mahajan and Vinay [MV97] described a combinatorial

[5] After several modern improvements, the complexity of this algorithm drops to $O(n^{3.37})$, see e.g. [Urb10].

algorithm compute the determinant in time $O(n^4)$. We refer to [MV99, Rote01] for historical remarks and further references.

In [Urb10], Urbańska showed how the Mahajan–Vinay algorithm can be sped up to $O(n^{3.03})$, via a reduction to the matrix multiplication. An especially simple division-free algorithm also based on matrix multiplication, was given by Bird [Bird11]. Most recently, starting from purely linear algebraic considerations, Ikenmeyer [Ike25] constructed a new ABP for the determinant, and for the coefficients of the characteristic polynomial.

2.4. **Complexity aspects.** Let $\mathbb{k}$ be a field, let $X = \{x_1, \ldots, x_k\}$ be free variables, and let $f \in \mathbb{k}\langle X\rangle$ be a fixed polynomial of degree n. An *ABP* is a directed acyclic graph $G = (V, E)$ with one *source* S and one *sink* T. The vertices $v \in V$ are partitioned into levels $0, 1, \ldots, n$, with source a unique vertex at level 0, and sink a unique vertex at level n. Let edges $v \to v'$ be labelled with a homogeneous linear form in X. For a path $\gamma : S \to T$ in G, denote by p_γ the product of all linear forms over edges in γ. The *size* of the ABP is the number of vertices $|V|$.

We say that f is computed by the ABP, if $f = \sum_\gamma p_\gamma$, where the summation is over all paths $\gamma : S \to T$. More generally, for an algebra $\mathcal{A}$ over $\mathbb{k}$ generated by $A = \{a_1, \ldots, a_k\}$, an element $D \in \mathcal{A}$ is computed by the ABP as above, if $f(a_1, \ldots, a_k) = D$. See e.g. [Bür26] for an accessible introduction, and for the role of ABPs in computing determinants.

The (noncommutative) arithmetic circuit is defined similarly, with operations $(+)$ and $(\times)$ at each vertex, all of in-degree at most two, and without the level restriction. Obtaining (unconditional) superpolynomial circuit lower bounds for a Cayley determinant remains a major open problem, see e.g. [Wig19, §12.5]. We refer to Hrubeš, Wigderson and Yehudayoff [HWY11], for a remarkable connection of this problem to a (commutative) complexity of computing the sum-of-squares.

In [HW15], Hrubeš and Wigderson extend noncommutative circuits for computation of rational functions, including noncommutative matrix inversion closely related to quasideterminants. See also Garg et al. [GGOW20], for a *deterministic* polynomial time algorithm deciding whether a symbolic matrix in noncommutative variables is invertible over $\mathbb{Q}$. Note that this is not the same as deciding whether the Cayley determinant is nonzero, cf. §10.4.[6]

The algorithm of Mahajan and Vinay [MV97] discussed above, can be viewed as an ABP of size $O(n^3)$. From the Boolean complexity point of view, it shows that computing the determinant over $\mathbb{Z}$ is in $\mathsf{GapL} = \#\mathsf{L} - \#\mathsf{L}$, and thus GapL-complete. Famously, Valiant showed that the determinant is complete in VP up to quasipolynomial reductions [Val79a]. Slightly relaxing the syntaxis from formulas to *weakly-skew arithmetic circuits*, gives an algebraic complexity class VBP, where the determinant is complete up to polynomial reductions, see [Bür26, §2.5].

Finally, we note that the *permanent approximation problem* is one of the benchmarks in the area. After much effort, a FPRAS for the problem was given by Jerrum, Sinclair and Vigoda in [JSV04], via the MCMC methods. The simplicity of taking expectations of the norms of random noncommutative determinants gives an alternative approach that is yet to be proved effective.

3. Definitions and the setup

3.1. **Basic notation.** We use $[n] := \{1, \ldots, n\}$, and let $[n]^*$ be the set of finite sequences (words) in $[n]$. Similarly, we use $\binom{[n]}{2} = \big\{(i, j) : 1 \le i < j \le n\big\}$. We write all permutations σ in the symmetric group S_n using *one-line notation*, as the word $(\sigma_1, \ldots, \sigma_n)$.

In an important departure from standard conventions, we write matrix elements in a transposed form, for example:

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

[6] Over division rings, being invertible is equivalent to having nonzero Dieudonné determinant, see [Die43, Dra83].

The reason for this unfortunate choice is technical: it is to ensure that the correspondence between words and paths is natural. For example, we have the path $1 \to 3 \to 2 \to 4$ corresponds to the word $a_{13}\,a_{32}\,a_{24}$ instead of the word $a_{31}\,a_{23}\,a_{42}$. Helpfully, almost all of the paper can be read ignoring this change.[7]

3.2. **Quantum matrices.** Denote by $\mathscr{F}_n = \mathbb{C}\langle a_{ij} \,:\, 1 \le i,j \le n\rangle$ the *free associative algebra*, whose basis is the set of words in the alphabet $\{a_{ij}\}$, with concatenation as product.

Let I^q_{cf} be the two-sided ideals in $\mathscr{F}_n$ generated by the relations (1.3). The quotient $\mathcal{A}^q_n = \mathscr{F}_n/I^q_{\text{cf}}$ is called the *q-CF algebra*. We write that two words $u, v \in \mathscr{F}_n$ are *equal modulo* (1.3) if they are equal in $\mathcal{A}^q_n$.

Similarly, let I^q_{rq} be the two-sided ideals in $\mathscr{F}$ generated by the relations (1.6). The quotient $\mathcal{B}^q_n = \mathscr{F}_n/I^q_{\text{rq}}$ is called the *q-RQ algebra*. We write that two words $u, v \in \mathscr{F}_n$ are *equal modulo* (1.6) if they are equal in $\mathcal{B}^q_n$.

More generally, fix complex numbers $q_{ij} \in \mathbb{C}$, such that $q_{ij} \neq 0$ for all $1 \le i < j \le n$. Consider ***q*-CF matrices** defined as

$$\text{(3.1)} \qquad \begin{cases} q_{k\ell}\,a_{\ell j}\,a_{ki} \;=\; q_{ij}\,a_{ki}\,a_{\ell j} & \text{for } \; i<j,\; k<\ell, \\ a_{\ell j}\,a_{ki} \;=\; q_{ij}\,q_{\ell k}\,a_{ki}\,a_{\ell j} & \text{for } \; i<j,\; k>\ell, \\ a_{kj}\,a_{ki} \;=\; q_{ij}\,a_{ki}\,a_{kj}\,, & \text{for } \; i<j. \end{cases}$$

Similarly, consider ***q*-RQ matrices** defined as

$$\text{(3.2)} \qquad \begin{cases} a_{kj}\,a_{ki} \;=\; q_{ij}\,a_{ki}\,a_{kj}\,, & \text{for } \; i<j, \\ a_{kj}\,a_{\ell i} \,-\, q_{ij}\,a_{ki}\,a_{\ell j} \;=\; q_{k\ell}\,q_{ij}\,a_{\ell i}\,a_{kj} \,-\, q_{k\ell}\,a_{\ell j}\,a_{ki}\,, & \text{for } \; i<j,\; k<\ell. \end{cases}$$

When all $q_{ij} = q$ we obtain q-CF and q-RQ matrices, respectively. By analogy with the above, let $I^{\boldsymbol{q}}_{\text{cf}}$ and $I^{\boldsymbol{q}}_{\text{rq}}$ denote corresponding ideals in $\mathscr{F}_n$. Similarly, let $\mathcal{A}^{\boldsymbol{q}}_n = \mathscr{F}_n/I^{\boldsymbol{q}}_{\text{cf}}$ and $\mathcal{B}^{\boldsymbol{q}}_n = \mathscr{F}_n/I^{\boldsymbol{q}}_{\text{rq}}$ denote the corresponding ***q*-CF** and ***q*-RQ algebras**.

3.3. **Inversions and $\boldsymbol{q}$-Cayley determinant.** Let $a_{\lambda_1\mu_1}\,a_{\lambda_2\mu_2}\cdots$ be a word in $\mathscr{F}_n$. We abbreviate this product as $a_{\lambda,\mu}$, where $\lambda = \lambda_1\lambda_2\cdots$ and $\mu = \mu_1\mu_2\cdots$ are words in the alphabet $[n]$. For a word $\nu = \nu_1\nu_2\cdots \in [n]^*$, define the *set of inversions*

$$\operatorname{INV}(\nu) \;=\; \big\{(\nu_i,\nu_j) \,:\, i<j,\; \nu_i > \nu_j\big\},$$

and let $\operatorname{inv}(\nu) := |\operatorname{INV}(\nu)|$ be the *number of inversions*. When ν is a permutation in S_n, this is the usual number of inversions.

For all $J \subseteq \binom{n}{2}$, denote

$$\boldsymbol{q}^J \;:=\; \prod_{(i,j)\in J} q_{ji} \quad \text{and} \quad \boldsymbol{q}^{-J} \;:=\; \big(\boldsymbol{q}^J\big)^{-1}.$$

Define the ***q*-Cayley determinant** by

$$\text{(3.3)} \qquad \operatorname{Cdet}_{\boldsymbol{q}}(A) \;:=\; \sum_{\sigma\in S_n} (-1)^{\operatorname{inv}(\sigma)}\,\boldsymbol{q}^{-\operatorname{INV}(\sigma)}\,a_{1\sigma_1}\cdots a_{n\sigma_n}\,.$$

Here the factors of each term are listed in *column order*, the column index running $1,\ldots,n$ from left to right. Thus, $\operatorname{Cdet}_{\boldsymbol{q}}$ is a also called *column determinant*, where the order of the factors is essential since variable a_{ij} do not commute. Note also that $|\operatorname{INV}(\sigma)| = \operatorname{inv}(\sigma)$ and $\operatorname{Cdet}_{\boldsymbol{q}} = \operatorname{Cdet}_q$ when all $q_{ij} \leftarrow q$.

We will need the following algebraic property of the $\boldsymbol{q}$-Cayley determinant, generalizing the standard property for commutative determinants.

[7] The reader who cannot stomach this notation can pretend the matrix is given as usual, and switch the meaning of rows/columns, with *right-quantum* replaced by *left-quantum*. In fact, Cayley in [Cay45] did just that in a similar context, leading to all sorts of confusion [Abe11]. We decided to avoid this approach, so to be able to quote unchanged the results from earlier papers.

Lemma 3.1. *Let $A = (a_{ij}) \in \mathcal{B}_n$ be a RQ matrix with two equal columns. Then* $\operatorname{Cdet}(A) = 0$. *More generally, let $A = (a_{ij}) \in \mathcal{B}_n^{\boldsymbol{q}}$ be a $\boldsymbol{q}$-RQ matrix with two equal columns. Then* $\operatorname{Cdet}(A) = 0$.

Versions of this results are given across the literature. For q-RQ matrices, see e.g. [CFR09, Cor. 1(1)], [CFRS14, Prop. 3.3(3)], [FH07a, §2], and [GLZ06, Lemma 2.5]. For general $\boldsymbol{q}$-RQ matrices, see [KP07, Lemma 12.2(2) and 12.4] and [Sil21, Cor. 3.8]. The proofs are straightforward.

3.4. **Cycle decompositions.** We write every permutation $\sigma \in S_n$ as a product of disjoint cycles. We obtain a canonical form of a cycle decomposition $\boldsymbol{C} = \boldsymbol{C}(\sigma)$ by letting each cycle start at its smallest element, which is called the head of the cycle, and by rearranging the cycles in the increasing order of their heads.

Formally, a *cycle* $C = (c_1, c_2, \ldots, c_i)$ of length i is a sequence of $i \geq 1$ distinct elements in $[n]$, where the *head* $c_1 = \operatorname{head}(C)$ is the smallest element in C. The *weight* of C is the product

$$\operatorname{weight}(C) := a_{c_1 c_2}\, a_{c_2 c_3} \cdots a_{c_i c_1}\,, \tag{3.4}$$

which we view as the product of the weights of *arcs* (directed edges) in the cycle. Note that the ordering is crucial here, as the variables are noncommutative.

A *cycle decomposition* $\boldsymbol{C} = \boldsymbol{C}(\sigma)$ of a permutation $\sigma \in S_n$ is a sequence $\boldsymbol{C} = (C_1, \ldots, C_k)$ of disjoint cycles with increasing heads:

$$\operatorname{head}(C_1) < \ldots < \operatorname{head}(C_k). \tag{3.5}$$

Let $k(\boldsymbol{C})$ denote the number of cycles in $\boldsymbol{C}$. The *weight* and *sign* of $\boldsymbol{C}$ are defined as

$$\operatorname{weight}(\boldsymbol{C}) := \operatorname{weight}(C_1) \cdots \operatorname{weight}(C_k)\,, \qquad \operatorname{sign}(\boldsymbol{C}) := \operatorname{sign}(\sigma) = (-1)^{n-k}. \tag{3.6}$$

Denote by $\mathcal{S}_n = \{\boldsymbol{C}(\sigma) : \sigma \in S_n\}$ the set of cycle decompositions of length n.

For a cycle decomposition $\boldsymbol{C} \in \mathcal{S}_n$, denote by $\lambda^{\boldsymbol{C}}$ and $\mu^{\boldsymbol{C}}$ the corresponding words in $[n]^n$. We will need the following simple observation.

Lemma 3.2. $\operatorname{inv}\big(\mu^{\boldsymbol{C}}\big) - \operatorname{inv}\big(\lambda^{\boldsymbol{C}}\big) = n - k(\boldsymbol{C})$.

The proof is by induction. We omit the details.

Example 3.3. For a permutation $\sigma = (2,5,1,8,3,6,4,7) \in S_8$, the corresponding cycle decomposition $\boldsymbol{C} = (C_1, C_2, C_3)$, where $C_1 = (1,2,5,3)$, $C_2 = (4,8,7)$, and $C_3 = (6)$. In this case, we have: $\operatorname{weight}(\boldsymbol{C}) = a_{12}\, a_{25}\, a_{53}\, a_{31}\, a_{48}\, a_{87}\, a_{74}\, a_{66}$ and $\operatorname{sign}(\boldsymbol{C}) = -1$. In the notation above, we have $\lambda^{\boldsymbol{C}} = 12534876$ and $\mu^{\boldsymbol{C}} = 25318746$. Then $\operatorname{inv}(\lambda^{\boldsymbol{C}}) = 5$ and $\operatorname{inv}(\mu^{\boldsymbol{C}}) = 10$. Since $n = 8$ and $k(\boldsymbol{C}) = 3$, lemma states in this case that $8 - 3 = 10 - 5$.

3.5. **$\boldsymbol{q}$-weights and Moore's determinants.** For an $n \times n$ matrix $A = (a_{ij})$, define *Moore's determinant* as

$$\operatorname{Mdet}(A) := \sum_{\boldsymbol{C} \in \mathcal{S}_n} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}\big(\boldsymbol{C}\big). \tag{3.7}$$

Note that for $A \in \mathcal{A}_{\mathrm{cf}}$, we trivially have $\operatorname{Mdet}(A) = \operatorname{Cdet}(A)$.

More generally, define the *q-weight* as

$$\begin{aligned} \operatorname{weight}_q(\boldsymbol{C}) \,&:=\, q^{\operatorname{inv}(\lambda^{\boldsymbol{C}}) - \operatorname{inv}(\mu^{\boldsymbol{C}})}\, \operatorname{weight}(\boldsymbol{C}) \\ &=\, q^{k(\boldsymbol{C}) - n}\, \operatorname{weight}(\boldsymbol{C}), \end{aligned} \tag{3.8}$$

where the second equality holds by Lemma 3.2. Now define the *q-Moore determinant* as

$$\operatorname{Mdet}_q(A) := \sum_{\boldsymbol{C} \in \mathcal{S}_n} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}_q(\boldsymbol{C}). \tag{3.9}$$

Similarly, define the *$\boldsymbol{q}$-weight* as

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \,:=\, \boldsymbol{q}^{\operatorname{INV}(\lambda^{\boldsymbol{C}})}\, \boldsymbol{q}^{-\operatorname{INV}(\mu^{\boldsymbol{C}})}\, \operatorname{weight}(\boldsymbol{C}), \tag{3.10}$$

and note that $\boldsymbol{q}$-weight is equal to the q-weight when all $q_{ij} = q$. Finally, define the *$\boldsymbol{q}$-Moore determinant* as

$$\operatorname{Mdet}_{\boldsymbol{q}}(A) := \sum_{\boldsymbol{C}\in\mathcal{S}_n} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}_{\boldsymbol{q}}\big(\boldsymbol{C}\big). \tag{3.11}$$

3.6. **Clow sequences.** If we relax the requirement that all elements of a cycle are distinct, we arrive at the concept of a *closed ordered walk*, or *clow* for short, in the terminology of Mahajan and Vinay [MV97].[8] Formally, a *clow* is a sequence $C = (c_1, c_2, \ldots, c_i)$ of length $\operatorname{length}(C) = i \geq 1$, such that the head c_1 of C, denoted $\operatorname{head}(C) = c_1$, is the unique smallest element: $c_1 < \min\{c_2, \ldots, c_i\}$. The weight of C is again defined by (3.4).

A *clow sequence* $\boldsymbol{C} = (C_1, \ldots, C_k)$ is a sequence of clows with strictly increasing sequence of heads, as in (3.5). Let $k(\boldsymbol{C}) = k$ denote the number of clows in $\boldsymbol{C}$. The *weight* and the *sign* of $\boldsymbol{C}$ are defined by (3.6). Similarly, *length* of $\boldsymbol{C}$ is defined as the sum of lengths of the clows:

$$\operatorname{length}(\boldsymbol{C}) := \operatorname{length}(C_1) + \ldots + \operatorname{length}(C_k).$$

Denote by $\mathcal{C}_n$ the set of clow sequences of length n. By definition, every cycle decomposition as above is a clow sequence, i.e. $\mathcal{S}_n \subseteq \mathcal{C}_n$. Denote by $\mathcal{C}_n^{\circlearrowleft} := \mathcal{C}_n \smallsetminus \mathcal{S}_n$ the set of clow sequences which are not cycle decompositions. For a clow $\boldsymbol{C} \in \mathcal{C}_n$, denote by $\lambda^{\boldsymbol{C}}$ and $\mu^{\boldsymbol{C}}$ the corresponding words in $[n]^n$. Let *q-weight* and *$\boldsymbol{q}$-weight* be defined by (3.8) and (3.10), respectively.

3.7. **Valiant's determinant.** Let $A = (a_{ij})$ be an $n \times n$ matrix in $\mathscr{F}_n$. By analogy with Moore's determinant, define *Valiant's determinant* as

$$\operatorname{Vdet}(A) := \sum_{\boldsymbol{C}\in\mathcal{C}_n} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}\big(\boldsymbol{C}\big), \tag{3.12}$$

where the only difference with (3.7) is the summation over a larger set of clow sequences.

Similarly, define the *q-Valiant determinant* as

$$\operatorname{Vdet}_q(A) := \sum_{\boldsymbol{C}\in\mathcal{C}_n} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}_q(\boldsymbol{C}), \tag{3.13}$$

and the *$\boldsymbol{q}$-Valiant determinant* as

$$\operatorname{Vdet}_{\boldsymbol{q}}(A) := \sum_{\boldsymbol{C}\in\mathcal{C}_n} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}_{\boldsymbol{q}}\big(\boldsymbol{C}\big). \tag{3.14}$$

By design, except for the summation running over a larger set of clow sequences, these coincide with deformations of Moore determinants given in (3.9) and (3.11), respectively.

3.8. **Noncommutative permanents.** Let $A = (a_{ij})$ be an $n \times n$ matrix in $\mathscr{F}_n$. The *ARQ matrices* mentioned in the introduction are defined as

$$\begin{cases} a_{kj}\, a_{ki} &= -\, a_{ki}\, a_{kj}\,, & \text{for} \quad i < j\,, \\ a_{kj}\, a_{\ell i} + a_{ki}\, a_{\ell j} &= a_{\ell i}\, a_{kj} + a_{\ell j}\, a_{ki} & \text{for} \quad i < j,\, k < \ell\,. \end{cases} \tag{3.15}$$

We use $\mathcal{B}_n^-$ denote the algebra of these matrices, and note that $\mathcal{B}_n^- = \mathcal{B}_n^q$ for $q = -1$. This algebra goes back to Manin [Man89], and represents a typical example of quantum algebras whose properties change at roots of unity, see e.g. [Lus90, GK93].

For a matrix $A = (a_{ij}) \in \mathscr{F}_n$, we can now define a (noncommutative) *Cayley permanent*:

$$\operatorname{Cper}(A) := \sum_{\sigma\in S_n} a_{1\sigma_1} \cdots a_{n\sigma_n}\,, \tag{3.16}$$

Similarly, define (noncommutative) *Moore* and *Valiant permanents*:

$$\operatorname{Mper}(A) := \sum_{\boldsymbol{C}\in\mathcal{S}_n} \operatorname{weight}(\boldsymbol{C}), \qquad \operatorname{Vper}(A) := \sum_{\boldsymbol{C}\in\mathcal{C}_n} \operatorname{weight}(\boldsymbol{C}). \tag{3.17}$$

[8] Valiant [Val92], who introduced this concept, called this a *c_1-loop*.

Since $\operatorname{sign}(\boldsymbol{C}) = (-1)^{\operatorname{inv}(\sigma)} = (-1)^{n-k(\boldsymbol{C})}$ for $\boldsymbol{C} = \boldsymbol{C}(\sigma)$, substituting $q = -1$ in definitions (1.1), (1.4) and (3.3), gives:

$$\operatorname{Cper}(A) \,=\, \operatorname{Cdet}_{-1}(A), \qquad \operatorname{Mper}(A) \,=\, \operatorname{Mdet}_{-1}(A), \qquad \operatorname{Vper}(A) \,=\, \operatorname{Vdet}_{-1}(A). \tag{3.18}$$

4. Proof outline

The elaborate setup we presented allows us to reduce Main Theorem 1.1 to two results: one combinatorial theorem and one algorithmic lemma.

4.1. **Equality of determinants.** Given the different nature of three families of noncommutative determinants we consider, the following result is quite unexpected.

Theorem 4.1 (*Determinantal equality*)**.** *Let $A = (a_{ij}) \in \mathcal{B}_n^{\boldsymbol{q}}$ be a $\boldsymbol{q}$-RQ matrix. Then we have:*

$$\operatorname{Cdet}_{\boldsymbol{q}}(A) \,=\, \operatorname{Mdet}_{\boldsymbol{q}}(A) \,=\, \operatorname{Vdet}_{\boldsymbol{q}}(A). \tag{4.1}$$

In particular, for $A \in \mathcal{B}_n$ we have:

$$\operatorname{Cdet}(A) \,=\, \operatorname{Mdet}(A) \,=\, \operatorname{Vdet}(A). \tag{4.2}$$

Similarly, for $A \in \mathcal{B}_n^-$ we have:

$$\operatorname{Cper}(A) \,=\, \operatorname{Mper}(A) \,=\, \operatorname{Vper}(A). \tag{4.3}$$

Note that in (4.1), both equalities are special for $\boldsymbol{q}$-right-quantum matrices and do not hold in greater generality. Indeed, the Cayley and Moore determinants are summations with equal number of product terms, but the terms are different, so the equality fails in $\mathscr{F}_n$ for $n \geq 2$. Worse, the summations in the Moore and Valiant determinants have different number of terms, so the equality fails in $\mathscr{F}_n$ again. As before, (4.2) and (4.3) follow from (4.1) and (3.18), by setting all $q_{ij} = 1$ and $q_{ij} = -1$, respectively.

4.2. **Two combinatorial lemmas.** We prove Theorem 4.1 in two separate lemmas, each by a combinatorial argument.

Lemma 4.2 (=Theorem 9.1)**.** $\operatorname{Cdet}_{\boldsymbol{q}}(A) = \operatorname{Mdet}_{\boldsymbol{q}}(A)$ *for all* $A \in \mathcal{B}_n^{\boldsymbol{q}}$.

We prove the lemma by an explicit many-to-many combinatorial argument generalizing bijections. Our argument is somewhat similar and inspired by the Konvalinka–Pak proof of the $\boldsymbol{q}$-MMT mentioned in §2.2.

The idea of the proof is as follows. It is easier to consider RQ case first. Both determinants are sums of products over permutations $\sigma \in S_n$, one in natural order, and one in cyclic decomposition order. Start rearranging products in natural order to products in cycle decomposition order by swapping adjacent letters, one at a time.

During the swaps, by the commutation relations (1.5) products will either remain (when the letters commute), or change (in case of the second relation). Despite the change, we can group products in pairs, and establish a bijection at every swap. Iterating the procedure gives the desired many-to-many map proving the lemma.

In the $\boldsymbol{q}$-RQ case, additional products of constants q_{ij} will emerge under every swap under relations in (3.2). Fortunately, products of these constants give the desired $\boldsymbol{q}$-weights as in (3.10), regardless of the order of swaps.

Lemma 4.3 (=Theorem 9.2)**.** $\operatorname{Mdet}_{\boldsymbol{q}}(A) = \operatorname{Vdet}_{\boldsymbol{q}}(A)$ *for all* $A \in \mathcal{B}_n^{\boldsymbol{q}}$.

We prove the lemma by an explicit involution which cancels the terms on the right, corresponding to clows which are *not* cycle decompositions, thus leaving only the terms on the left. Our argument emulates the explicit involution introduced by Mahajan–Vinay [MV97], but with many complications added by the noncommutative setting.

The idea of the Mahajan–Vinay involution is roughly as follows. For a clow of length n that is not a cycle decomposition, find the "first violation", which in this case is a repetition of a label in the same clow or different clows. Setting aside the precise meaning of the "first violation", which are different in [MV97] and this paper anyway, let's say the repetition occurs in the same clow. Extract the portion of the clow between repeated labels, and let it form a new clow with the head being the repeated label. When the repetition occurs in the different clows, employ the opposite procedure. Since the parity of the number of clows changes, the corresponding signs in (3.12) are the opposite, allowing the cancellation.

Now, in the RQ case, the process of "extracting" a portion of the clow is complicated by the fact that it has to be moved across several other clows before it can be placed into its place in the clow sequence whose heads are increasing as in (3.5). This requires a large number of swaps which need to be made according to relations (1.5), so a naive approach immediately falls apart.

We make several major changes in this approach, starting with the notions of the "first violation", and modifying the two-for-two swapping discussed above into a delicate process involving cancellations of terms. At the end, we reduce the problem to computing Moore's determinant or a matrix with two equal columns. By Lemma 4.2, this reduces to Cayley's determinant of a matrix with equal columns, which is known to be zero by Lemma 3.1.

In the $\boldsymbol{q}$-RQ case, the whole process is further complicated by the need to involve bookkeeping of the products of constants q_{ij}, which we ultimately do by mildly modifying the case of RQ matrices.

4.3. **Proof of Main Theorem 1.1.** The rest of the proof, at least for $\boldsymbol{q} = 1$, is very close to that in Mahajan–Vinay argument by design.

Lemma 4.4 (*Dynamic programming*, see [MV97])**.** *For all nonzero $\boldsymbol{q} = (q_{ij})$ as above, there is an ABP of size $O(n^3)$, which computes the $\boldsymbol{q}$-Valiant determinant in free associative algebra $\mathscr{F}_n = \mathbb{C}\langle a_{ij} \,:\, 1 \le i,j \le n\rangle$. In particular, it computes the $\boldsymbol{q}$-Valiant determinant of $\boldsymbol{q}$-RQ matrices, the (usual) Valiant determinant of RQ matrices, and the Valiant permanent of ARQ matrices.*

The proof follows Mahajan and Vinay [MV97], with delicate modifications to account for the constants q_{ij} computed along the way. The proof given in Section 6, crucially relies on the Factorization Lemma 5.2 for the $\boldsymbol{q}$-weights of clow sequences. While the proof of the lemma is very short, this technical result allows us to use the dynamic programming approach as in the original $\boldsymbol{q} = 1$ case.

In summary, by Theorem 4.1, we obtain an ABP of polynomial size, which computes the $\boldsymbol{q}$-Cayley determinant of $\boldsymbol{q}$-RQ matrices. Consequently, after specializations $q_{ij} = 1$ and $q_{ij} = -1$, it computes the Cayley determinant and the Cayley permanent, of right-quantum and antisymmetric right-quantum matrices, respectively. This completes the proof of Theorem 1.1. □

4.4. **Structure of the paper.** We begin with Factorization Lemma in Section 5. Then, in Section 6, we give a $\boldsymbol{q}$-version of the Mahajan–Vinay argument in the proof of Lemma 4.4. This is the only argument which applies to the whole free associative algebra $\mathscr{F}_n$. The reader familiar with the [MV97] and interested only in the right-quantum ($\boldsymbol{q} = 1$) case, can safely skip this section.

To get the reader familiar with the setting of words in quantum algebras, we have Sections 7 and 8, where we proof Theorem 4.1 for the $\boldsymbol{q}$-CF and RQ matrices, respectively. The proof of the $\boldsymbol{q}$-RQ case combines the details of both previous sections, and is given in Section 9. We conclude with final remarks and open problems in Section 10.

5. Factorization lemma

In this section we proof that the $\boldsymbol{q}$-weight of any clow sequence factors into $\boldsymbol{q}$-weights of its clows. This factorization lemma justifies the adjustment of the Mahajan and Vinay [MV97] dynamic programming by the constants q_{ij}. We start with computation of the $\boldsymbol{q}$-weight of a single clow.

Lemma 5.1. *For any clow $C = (c_1, c_2, \ldots, c_\ell)$, we have*

$$\operatorname{weight}_{\boldsymbol{q}}(C) \;=\; a_{c_1c_2}\, a_{c_2c_3} \,\cdots\, a_{c_\ell c_1} \prod_{i=2}^{\ell} q_{c_1\,c_i}^{-1}\,.$$

Proof. For a single cycle $C = (c_1, c_2, \ldots, c_\ell)$ with head c_1, we have: $\lambda^C = (c_1, \ldots, c_\ell)$ and $\mu^C = (c_2, \ldots, c_\ell, c_1)$, so μ^C is a cyclic shift of λ^C by one position. Since $c_1 < c_i$ any pair $(c_i, c_1) \in \mathrm{INV}(\mu)$ and any pair $(c_1, c_i) \notin \mathrm{INV}(\lambda)$ for all $2 \le i \le \ell$. For all other pairs (c_i, c_j) with $2 \le i < j \le \ell$ we have that $(c_i, c_j) \in \mathrm{INV}(\lambda)$ if and only if $(c_i, c_j) \in \mathrm{INV}(\mu)$. Then, by the definition (3.10), we have

$$\operatorname{weight}_{\boldsymbol{q}}(C) \;=\; \boldsymbol{q}^{\mathrm{INV}(\lambda^C)}\, \boldsymbol{q}^{-\mathrm{INV}(\mu^C)}\, \operatorname{weight}(C) \;=\; a_{c_1c_2}\, a_{c_2c_3} \,\cdots\, a_{c_\ell c_1} \prod_{i=2}^{\ell} q_{c_1\,c_i}^{-1}\,,$$

as desired. □

Lemma 5.2 (Factorization Lemma**).** *For a clow sequence $\boldsymbol{C} = (C_1, \ldots, C_k) \in \mathcal{C}_n$, we have:*

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \;=\; \operatorname{weight}_{\boldsymbol{q}}(C_1) \,\cdots\, \operatorname{weight}_{\boldsymbol{q}}(C_k)\,.$$

Proof. Since the clows occupy contiguous blocks of positions, we have

$$\operatorname{weight}(\boldsymbol{C}) \;=\; \operatorname{weight}(C_1) \,\cdots\, \operatorname{weight}(C_k)\,.$$

For the product, write $\lambda = \lambda^{C_1} \cdots \lambda^{C_k}$ and $\mu = \mu^{C_1} \cdots \mu^{C_k}$. An inversion straddling two distinct clows' blocks pairs the same two letters in λ as in μ, since each clow word carries the same multiset of letters; its contributions to the $\mathrm{INV}(\lambda)$ and $\mathrm{INV}(\mu)$ are therefore equal and cancel. Thus only within-clow inversions survive, which asserts the claim of the lemma. □

Example 5.3. Consider the clow sequence $\boldsymbol{C} = (C_1, C_2)$, where $C_1 = (1424)$ and $C_2 = (354)$. Here

$$\operatorname{weight}(\boldsymbol{C}) \;=\; a_{14}\, a_{42}\, a_{24}\, a_{41} \;\cdot\; a_{35}\, a_{54}\, a_{43}\,,$$

so that $\lambda^{\boldsymbol{C}} = 1424354$ and $\mu^{\boldsymbol{C}} = 4241543$. By Lemma 5.1 applied to each clow, we have:

$$\operatorname{weight}_{\boldsymbol{q}}(C_1) \;=\; a_{14}\, a_{42}\, a_{24}\, a_{41}\; q_{14}^{-1}\, q_{12}^{-1}\, q_{14}^{-1} \qquad \text{and} \qquad \operatorname{weight}_{\boldsymbol{q}}(C_2) \;=\; a_{35}\, a_{54}\, a_{43}\; q_{35}^{-1}\, q_{34}^{-1}\,.$$

By the Factorization Lemma 5.2, we conclude:

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \;=\; q_{12}^{-1}\, q_{14}^{-2}\, q_{34}^{-1}\, q_{35}^{-1}\;\, \operatorname{weight}(\boldsymbol{C})\,.$$

Alternatively, computing directly from the definition (3.10), we have $\mathrm{inv}(\lambda^{\boldsymbol{C}}) = 4$ and $\mathrm{inv}(\mu^{\boldsymbol{C}}) = 9$, with

$$\boldsymbol{q}^{\mathrm{INV}(\lambda^{\boldsymbol{C}})} \;=\; q_{24}\, q_{34}^2\, q_{45} \qquad \text{and} \qquad \boldsymbol{q}^{-\mathrm{INV}(\mu^{\boldsymbol{C}})} \;=\; q_{12}^{-1}\, q_{14}^{-2}\, q_{24}^{-1}\, q_{34}^{-3}\, q_{35}^{-1}\, q_{45}^{-1}\,,$$

giving the same result.

6. Proof of Dynamic Programming Lemma 4.4

The proof follows the approach by Mahajan and Vinay [MV97] and the exposition of Rote [Rote01], with adjustments for the constants q_{ij}.

Proof of Lemma 4.4. We build up each clow sequence one arc at a time. Formally, a *partial clow sequence* to be a prefix of the weight (3.6) with heads satisfying (3.5). Here we have several (finished) clows trailed by one *open* (incomplete) clow. For a partial clow sequence $\boldsymbol{C} = C_1 \cdots C_s\,(c_0\, c_1\, c_2 \cdots c_t$, define its *partial* $\boldsymbol{q}$*-weight* as

$$\operatorname{pweight}_{\boldsymbol{q}}(\boldsymbol{C}) \,:=\, \operatorname{weight}_{\boldsymbol{q}}(C_1) \cdots \operatorname{weight}_{\boldsymbol{q}}(C_s) \,\cdot\, a_{c_0 c_1}\, a_{c_1 c_2} \cdots a_{c_{t-1}\, c_t} \,\cdot\, \prod_{j=1}^{t} q_{c_0\, c_j}^{-1}\,.$$

Here we take the product of the $\boldsymbol{q}$-weights of its completed clows, the weight (not the $\boldsymbol{q}$-weight) of the open clow, and the scalar correction of the open clow. The remainder of the computation depends on is the head and the present endpoint of the open clow, plus the sign $(-1)^k$ recording the parity of the count k of finished clows.

The desired ABP is given by the recursion, where the level of vertices will be denoted by ℓ. The recursion computes a polynomial $F_v(a_{ij})$ for every vertex $v = [\ell, c, c_0, s]$, where $1 \le \ell \le n$, $1 \le c_0 < c \le n$, and $s \in \{\pm 1\}$. The rule is:

> For $v = [\ell, c, c_0, s]$, let F_v be the sum of partial $\boldsymbol{q}$-weights over all partial clow sequences that have reached length ℓ, currently stand at vertex c inside a clow with head c_0, and have completed a number of clows of parity $s = \pm 1$.

Add an auxiliary source vertex S and set $F_S := 1$, and take edges to all vertices on level 1. For $v = [1, r, r, 1]$, let $F_v := a_{rr}$. From a given $[\ell, c, c_0, s]$, a partial clow sequence advances in exactly two ways, which we record as two families of outgoing edges:

- *extend* the current clow: edge to $[\ell + 1, c', c_0, s]$ of weight $a_{cc'} \cdot q_{c_0\, c'}^{-1}$, for all $c' > c_0$,
- *close* the current clow and open a new one: edges to $[\ell + 1, c_0', c_0', -s]$ of weight a_{cc_0}, for all $c_0' > c_0$.

Add a sink vertex $T = [n+1, n+1, n+1, \pm 1]$, with edges from all vertices at level n. By the construction and definition (3.14), the sum of weights over all $S \to T$ path, taken with the sign given by s, equals the $\boldsymbol{q}$-Valiant determinant. Finally, note that the total number of vertices is $O(n^3)$, which completes the proof of the first part.

For the $\boldsymbol{q}$-Valiant determinant and the (usual) Valiant determinant, the lemma follows from definition and by setting all $q_{ij} = 1$, respectively. Finally, for the Valiant permanent, the proof follows by the definition, from the relation (3.18), and by setting all $q_{ij} = -1$. □

Remark 6.1. In the proof above, we are using the fact that for a complete clow sequence $\boldsymbol{C} = (C_1, \ldots, C_k) \in \mathcal{C}_n$, the partial $\boldsymbol{q}$-weight coincides with the $\boldsymbol{q}$-weight:

$$\operatorname{pweight}_{\boldsymbol{q}}(\boldsymbol{C}) \,=\, \operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}).$$

This follows immediately from Lemma 5.1 and Factorization Lemma 5.2. However, for general partial clow sequences, the two are not necessarily equal, and can differ by a scalar (product of constants q_{ij}). For example, for a partial clow $(132$ we have:

$$\operatorname{weight}_{\boldsymbol{q}}\big(a_{13}\, a_{32}\big) \,=\, a_{13}\, a_{32}\; q_{23}^{-1} \qquad \text{while} \qquad \operatorname{pweight}_{\boldsymbol{q}}\big(a_{13}\, a_{32}\big) \,=\, a_{13}\, a_{32}\; q_{13}^{-1}\, q_{12}^{-1}\,,$$

while for a complete clow (132) we have:

$$\operatorname{pweight}_{\boldsymbol{q}}\big(a_{13}\, a_{32}\, a_{21}\big) \,=\, a_{13}\, a_{32}\, a_{21}\; q_{13}^{-1}\, q_{12}^{-1} \,=\, \operatorname{weight}_{\boldsymbol{q}}\big(a_{13}\, a_{32}\, a_{21}\big).$$

Note that the partial $\boldsymbol{q}$-weight can be computed recursively, as at each step the scalar is multiplied by $q_{c_0\, c'}^{-1}$, while the usual $\boldsymbol{q}$-weight might gain a few new inversions not involving the head c_0, and determining them requires a further search.

7. Cartier–Foata matrices

7.1. **Cayley–Moore determinantal equality.** In some sense, Cayley determinant (3.3) and Moore determinant (3.11) are the most economical formulas for the $\boldsymbol{q}$-determinant. In the $\boldsymbol{q}$-CF algebra, they are equal up to a constant.

We start with a general observation which will be used repeatedly. For a word $a_{\lambda,\mu} = a_{\lambda_1\mu_1}\, a_{\lambda_2\mu_2} \cdots$ in $\mathscr{F}_n$, define its *$\boldsymbol{q}$-weight* by analogy with (3.10):

$$\operatorname{weight}_{\boldsymbol{q}}\left(a_{\lambda,\mu}\right) \,:=\, \boldsymbol{q}^{\operatorname{INV}(\lambda)}\, \boldsymbol{q}^{-\operatorname{INV}(\mu)}\, a_{\lambda,\mu}, \tag{7.1}$$

so that $\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) = \operatorname{weight}_{\boldsymbol{q}}\left(a_{\lambda c,\mu c}\right)$ for all clow sequences $\boldsymbol{C} \in \mathcal{C}_n$.

Lemma 7.1. *Let $a_{\lambda',\mu'}$ be obtained from $a_{\lambda,\mu}$ by interchanging two adjacent factors $a_{\ell j}\, a_{ki}$ with $i \neq j$. Then:*

$$\operatorname{weight}_{\boldsymbol{q}}\left(a_{\lambda',\mu'}\right) \,=\, \operatorname{weight}_{\boldsymbol{q}}\left(a_{\lambda,\mu}\right) \quad \textit{in the } \boldsymbol{q}\textit{-CF algebra } \mathcal{A}_n^{\boldsymbol{q}}.$$

Proof. Without loss of generality, let $i < j$. The interchange changes the inversion sets of λ and μ only in the slot these factors occupy: it deletes the inversion of μ there, so the $\boldsymbol{q}$-weight loses the factor q_{ij}^{-1}; in λ it deletes the inversion contributing $q_{k\ell}$ when $k < \ell$, adds the inversion contributing $q_{\ell k}$ when $k > \ell$, and changes nothing when $k = \ell$. Hence the interchange multiplies the $\boldsymbol{q}$-weight by

$$c \,=\, q_{ij}\, q_{k\ell}^{-1} \quad (k < \ell), \qquad c \,=\, q_{ij}\, q_{\ell k} \quad (k > \ell), \qquad c \,=\, q_{ij} \quad (k = \ell).$$

On the other hand, in each of these three cases relations (3.1) read $a_{\ell j}\, a_{ki} \,=\, c a_{ki}\, a_{\ell j}$, so the interchange multiplies the product of entries by c^{-1}. The two factors c and c^{-1} cancel, and the $\boldsymbol{q}$-weight is unchanged. □

Now, comparing (3.3) and (3.11), we need to show that

Lemma 7.2. *For all $\sigma \in S_n$ and the corresponding cycle decomposition $\boldsymbol{C} = \boldsymbol{C}(\sigma)$, we have*

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \,=\, \boldsymbol{q}^{-\operatorname{INV}(\sigma)}\, a_{1\sigma_1} \cdots a_{n\sigma_n}\,.$$

Proof. Both $\operatorname{weight}(\mathbf{C})$ and the column word $a_{1\sigma_1} \cdots a_{n\sigma_n}$ are products of the same n factors, with distinct row indices and distinct column indices, so one is obtained from the other by interchanging adjacent factors $a_{\ell j}\, a_{ki}$ with $i \neq j$. By Lemma 7.1, we have $\operatorname{weight}_{\boldsymbol{q}}(\mathbf{C}) = \operatorname{weight}_{\boldsymbol{q}}\left(a_{1\sigma_1} \cdots a_{n\sigma_n}\right)$ in $\mathcal{A}_n^{\boldsymbol{q}}$. For the column word we have $\lambda = (1, \ldots, n)$ and $\mu = \sigma$, so $\boldsymbol{q}^{\operatorname{INV}(\lambda)} = 1$ and $\boldsymbol{q}^{-\operatorname{INV}(\mu)} = \boldsymbol{q}^{-\operatorname{INV}(\sigma)}$, and the result follows from (7.1). □

Corollary 7.3. *Let $A = (a_{ij}) \in \mathcal{A}_n^{\boldsymbol{q}}$ be a $\boldsymbol{q}$-CF matrix. Then we have:*

$$\operatorname{Cdet}_{\boldsymbol{q}}(A) \,=\, \operatorname{Mdet}_{\boldsymbol{q}}(A).$$

Proof. The result follows by comparing the definitions (3.3) and (3.11) term by term, via Lemma 7.2. □

7.2. **Moore–Valiant determinantal equality.** Let us prove Lemma 4.3 for $\boldsymbol{q}$-CF matrices.

Lemma 7.4. *Let $A = (a_{ij}) \in \mathcal{A}_n^{\mathrm{cf}}$ be a $\boldsymbol{q}$-CF matrix. Then we have:*

$$\operatorname{Mdet}_{\boldsymbol{q}}(A) \,=\, \operatorname{Vdet}_{\boldsymbol{q}}(A).$$

Proof. Comparing the definitions (3.11) and (3.14), it suffices to show that the signed $\boldsymbol{q}$-weights of the clow sequences in $\mathcal{C}_n^{\circlearrowleft} = \mathcal{C}_n \smallsetminus \mathcal{S}_n$ cancel each other. We pair off all such clow sequences, as follows.

Let $\boldsymbol{C} = (C_1, \ldots, C_k) \in \mathcal{C}_n^{\circlearrowleft}$. We successively add the clows $C_k, C_{k-1}, \ldots$ until a repetition occurs: assume that $C_{p+1}, \ldots, C_k$ is a set of disjoint cycles, but $C_p, C_{p+1}, \ldots, C_k$ contains a repeated element, where $1 \le p \le k$. Let $C_p = (c_1, \ldots, c_i)$, and let c_j be the first element of this clow, moving

from the right end to the left towards the clow head, which is either (A) equal to an element c_ℓ in the tail of the clow ($j < \ell \le i$), or (B) equal to an element of one of the cycles $C_{p+1}, \ldots, C_k$.

Precisely one of these two cases occurs: we call $\boldsymbol{C}$ *merged* in case (A), and *split* in case (B). In case (A), we remove the cycle $(c_j = c_\ell, c_{j+1}, \ldots, c_{\ell-1})$ from the clow C_p and turn it into a separate cycle, which we insert into $\boldsymbol{C}$ in the position prescribed by (3.5), after cyclically shifting its head to the front. In case (B), we insert the corresponding cycle into the clow C_p just before the element c_j. These two operations are inverse to each other, and hence they pair off all clow sequences in $\mathcal{C}_n^{\circledcirc}$ into *orbits* $\{\boldsymbol{C}, \boldsymbol{C}'\}$ of a merged and the corresponding split clow sequence, with $k(\boldsymbol{C}') = k(\boldsymbol{C}) + 1$; see Example 7.6 below.

By (3.6), we have $\operatorname{sign}(\boldsymbol{C}') = -\operatorname{sign}(\boldsymbol{C})$. On the other hand, one word is obtained from the other by repeatedly interchanging two adjacent factors $a_{\ell j}\, a_{ki}$ with $i \neq j$. By Lemma 7.1, we conclude:

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \,=\, \operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}') \quad \text{in the algebra} \quad \mathcal{A}_n^{\boldsymbol{q}}\,.$$

Therefore, the two terms of every orbit in (3.14) cancel each other, and only the terms corresponding to cycle decompositions $\boldsymbol{C} \in \mathcal{S}_n$ survive, giving (3.11), as desired. □

Remark 7.5. Our pairing between merged and split clow sequences differs from the one of Mahajan and Vinay [MV97]. The key difference is that we read each clow from its right end towards its head on the left, whereas in [MV97] each clow is read from its head towards the right. This change is crucial: in the non-commutative classes of matrices we consider, entries in the same row do not commute at all, and since the heads of the split clow sequence must increase from left to right, the given involution assures that we do not encounter factors from the same row as we perform swaps to reach its split sequence. This was not the case in the original Mahajan–Vinay involution.

Example 7.6. Let $\boldsymbol{C} = (C_1, C_2, C_3)$ be the merged clow sequence, where $C_1 = (172553857)$, $C_2 = (2)$ and $C_3 = (469)$. Extracting the cycle (385) from the clow C_1 gives the corresponding split clow sequence

$$\boldsymbol{C}' \,=\, (172557)(2)(385)(469),$$

so that $\{\boldsymbol{C}, \boldsymbol{C}'\}$ is an orbit. In this case, we have:

$$\operatorname{weight}(\boldsymbol{C}) \,=\, a_{17}\, a_{72}\, a_{25}\, a_{55}\, \underline{a_{53}}\, \underline{a_{38}}\, \underline{a_{85}}\, a_{57}\, a_{71} \,\cdot\, a_{22} \,\cdot\, a_{46}\, a_{69}\, a_{94}$$

and

$$\operatorname{weight}(\boldsymbol{C}') \,=\, a_{17}\, a_{72}\, a_{25}\, a_{55}\, a_{57}\, a_{71} \,\cdot\, a_{22} \,\cdot\, \underline{a_{38}}\, \underline{a_{85}}\, \underline{a_{53}} \,\cdot\, a_{46}\, a_{69}\, a_{94}\,,$$

where the letters of the relocated cycle are underlined. Note that $k(\boldsymbol{C}) = 3$ and $k(\boldsymbol{C}') = 4$, so that $\operatorname{sign}(\boldsymbol{C}') = -\operatorname{sign}(\boldsymbol{C})$, in agreement with (3.6). By Lemma 5.1 and the Factorization Lemma 5.2, we also have:

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \,=\, q_{12}^{-1}\, q_{13}^{-1}\, q_{15}^{-3}\, q_{17}^{-2}\, q_{18}^{-1}\, q_{46}^{-1}\, q_{49}^{-1} \;\; \operatorname{weight}(\boldsymbol{C})$$

and

$$\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}') \,=\, q_{12}^{-1}\, q_{15}^{-2}\, q_{17}^{-2}\, q_{35}^{-1}\, q_{38}^{-1}\, q_{46}^{-1}\, q_{49}^{-1} \;\; \operatorname{weight}(\boldsymbol{C}'),$$

which are equal in the algebra $\mathcal{A}_n^{\boldsymbol{q}}$, as in the proof of Lemma 7.4.

8. Right-quantum matrices

Unlike in the Cartier–Foata case $\mathcal{A}_n^{\boldsymbol{q}}$, showing that $\operatorname{Cdet}_{\boldsymbol{q}}$, $\operatorname{Mdet}_{\boldsymbol{q}}$ and $\operatorname{Vdet}_{\boldsymbol{q}}$ are equal in $\mathcal{B}_n^{\boldsymbol{q}}$ is significantly harder. We postpone the proof until the next section. In this section, we start with the RQ case, which captures the essence of the proof for the right-quantum matrices.

8.1. **Cayley–Moore determinantal equality.** In the RQ case, we have the following special case of Lemma 4.2:

Lemma 8.1. *Let $A = (a_{ij}) \in \mathcal{B}_n$ be an RQ matrix. Then we have:*

$$\mathrm{Cdet}(A) \;=\; \mathrm{Mdet}(A).$$

We prove Lemma 8.1 following the bijective approach of Konvalinka and the first author [KP07]. Consider lattice steps of the form $(x, i) \to (x+1, j)$ for $x, i, j \in \mathbb{Z}$, $1 \le i, j \le n$. We think of x as drawn along the x-axis, increasing from left to right, and refer to i and j as the *starting* and *ending height*, respectively. In keeping with our weight convention, where the arc from i to j carries the entry a_{ij}, we represent the step $(x, i) \to (x+1, j)$ by the variable a_{ij}, and a finite sequence of steps by a word in the alphabet $\{a_{ij}\}$.

We will consider sequences consisting of n letters a_{ij} whose sets of starting points and of ending points each have no repetitions, and are exactly $\{1, \dots, n\}$. We call such a sequence *balanced*. An *o-sequence* is a balanced sequence whose columns are arranged in increasing order.

For a balanced sequence

$$(0, i_1) \to (1, j_1),\ (1, i_2) \to (2, j_2),\ \dots,\ (n-1, i_n) \to (n, j_n),$$

define the *rank* r as

$$r \;:=\; \big|\{(s,t) \;:\; i_s > i_t,\ 1 \le s < t \le n\}\big|.$$

Clearly, o-sequences are exactly the balanced sequences of rank 0.

Note that the words in the Cayley determinant (1.1) are precisely the o-sequences. Indeed, the term of σ lists its letters by increasing column, so its starting points run $1, 2, \dots, n$ in order, while its ending points are the distinct values $\sigma_1, \dots, \sigma_n$.

A balanced sequence is called a *p-sequence*, if it is the word weight$(\boldsymbol{C})$ of the canonical form of a cycle decomposition $\boldsymbol{C} \in \mathcal{S}_n$, that is, the clow-ordered weight obtained by reading each cycle from its head and listing the cycles by increasing heads. In this language, the summation in Moore's determinant, is taken over all p-sequences.

A balanced sequence is called a *q-sequence*, if it starts with a prefix of a p-sequence and is then continued with a suffix of an o-sequence, with at most one exceptional step which might have a starting point larger than those of the following steps to the right.

Example 8.2. For $\boldsymbol{C} = (1527)(4)(386)$, the three types of sequence are

$$\begin{aligned} p:&\quad a_{15}\,a_{52}\,a_{27}\,a_{71}\,a_{44}\,a_{38}\,a_{86}\,a_{63},\\ o:&\quad a_{15}\,a_{27}\,a_{38}\,a_{44}\,a_{52}\,a_{63}\,a_{71}\,a_{86},\\ q:&\quad a_{15}\,a_{52}\,a_{27}\,a_{38}\,a_{44}\,a_{71}\,a_{63}\,a_{86}, \end{aligned}$$

the q-sequence having p-prefix $a_{15}\,a_{52}\,a_{27}$ and a single exceptional step a_{71}. Drawing each letter a_{ij} as a step from height i to height j, the three paths are as follows.

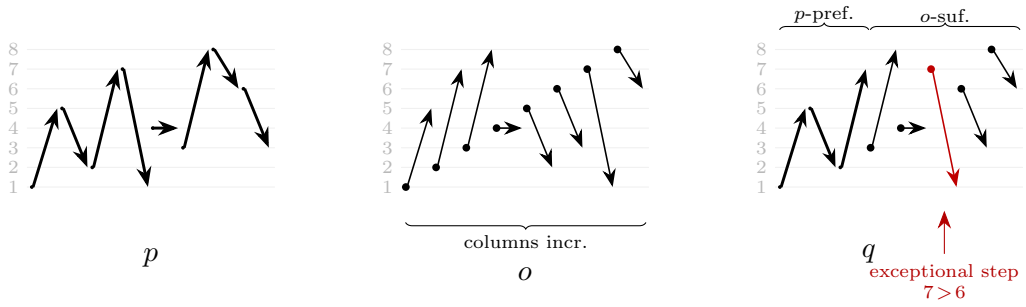

Figure 8.1 An example of p-, o- and q-sequences.

Proof of Lemma 8.1. We define a map Ψ on q-sequences, following the approach by Konvalinka and the first author [KP07]. Given a q-sequence, let its longest prefix that already obeys the rules of a p-sequence consist of x steps, and let i be the height at the end of step x. The map Ψ swaps the unique step starting at height i with its immediate left neighbor. This single swap raises the rank by 1.

A swap of two steps with distinct ends is not by itself a relation, so we carry it out together with a second q-sequence. This partner has the same rank and agrees with the first everywhere except that the ends of the two swapped steps are interchanged. If the first sequence corresponds to a permutation σ, the partner corresponds to $\sigma' = \sigma \cdot \tau$, where τ is the transposition of the two interchanged ending points; hence the two carry opposite signs, $\operatorname{sign}\sigma' = -\operatorname{sign}\sigma$. The second relation in (1.5) now performs the swap in both sequences at once, and because their signs are opposite the relation applies with the signs intact: $\mp\alpha \pm \beta = \mp\Psi\alpha \pm \Psi\beta$. Example 8.3 illustrates such a pair.

When a sequence is already a p-sequence, Ψ leaves it unchanged. Iterating the map therefore rewrites the Cayley determinant in $\mathcal{B}_n$ as the Moore determinant, proving Lemma 8.1. □

Example 8.3. Consider the two q-sequences α and β that differ only in the ends of the two red steps. Here α corresponds to $\sigma = (5,7,8,4,2,3,1,6)$ with $\operatorname{sign}\sigma = -1$, while interchanging the ends of the two red steps merges the fixed point 4 into the long cycle, giving $\sigma' = (5,7,8,1,2,3,4,6)$ with $\operatorname{sign}\sigma' = +1$; the signs shown are these permutation signs. Applying Ψ swaps the two red steps in each row and leaves the signs unchanged, see Figure 8.2.

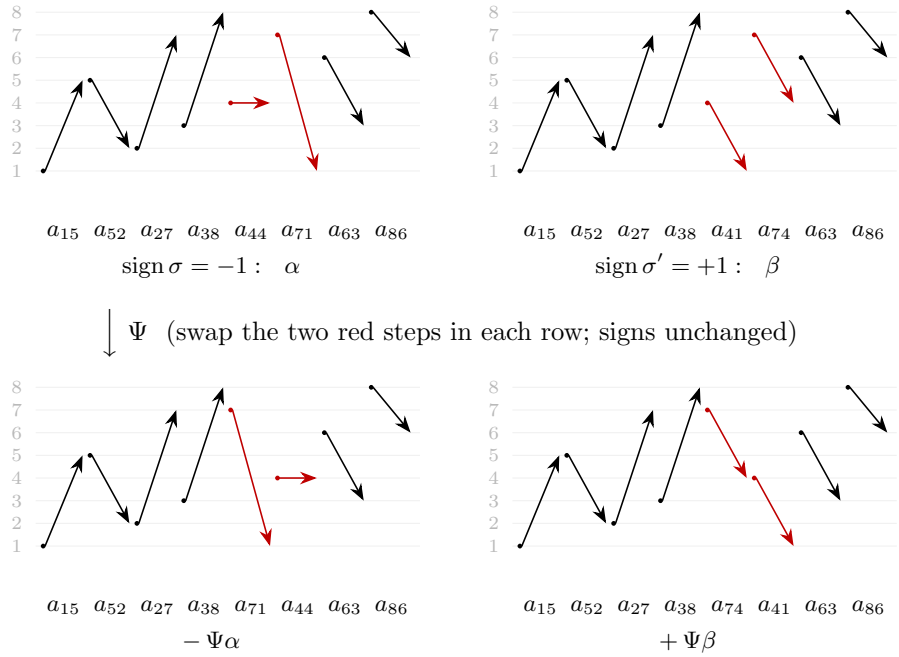

Figure 8.2 Map Ψ in Example 8.3.

8.2. **Moore–Valiant determinantal equality.** In the proof of the Cayley–Moore equality given above, pairs of q-sequences remain pairs of q-sequences after a swap. On the other hand, for pairs of clow sequences, a sequence of swaps can end up with a pair of balanced words with no clear description. Thus, we prove the Moore–Valiant equality using a different method, namely induction.

Lemma 8.4. *Let $A = (a_{ij}) \in \mathcal{B}_n$ be an RQ matrix. Then we have:*

$$\operatorname{Mdet}(A) \;=\; \operatorname{Vdet}(A).$$

Example 8.5. Take $n = 5$, and consider four clow sequences in $\mathcal{C}_5$:

$$\boldsymbol{C}_1 = (1223)(5), \qquad \boldsymbol{C}'_1 = (123)(2)(5), \qquad \boldsymbol{C}_2 = (1232)(5), \qquad \boldsymbol{C}'_2 = (12)(23)(5),$$

forming two orbits $\{\boldsymbol{C}_1, \boldsymbol{C}'_1\}$ and $\{\boldsymbol{C}_2, \boldsymbol{C}'_2\}$. In this case, we have:

$$\begin{aligned} \mathrm{weight}(\boldsymbol{C}_1) &= a_{12}a_{22}a_{23}a_{31}a_{55}\,, & \mathrm{weight}(\boldsymbol{C}'_1) &= a_{12}a_{23}a_{31}a_{22}a_{55}\,, \\ \mathrm{weight}(\boldsymbol{C}_2) &= a_{12}a_{23}a_{32}a_{21}a_{55}\,, & \mathrm{weight}(\boldsymbol{C}'_2) &= a_{12}a_{21}a_{23}a_{32}a_{55}\,. \end{aligned}$$

Here $\mathrm{sign}(\boldsymbol{C}_1) = \mathrm{sign}(\boldsymbol{C}_2) = -1$ and $\mathrm{sign}(\boldsymbol{C}'_1) = \mathrm{sign}(\boldsymbol{C}'_2) = +1$, and the column relation in (1.5) reduces the two signed orbits to

$$\begin{aligned} \mathrm{weight}(\boldsymbol{C}'_1) - \mathrm{weight}(\boldsymbol{C}_1) &= a_{12}a_{23}\big(a_{31}a_{22} - a_{22}a_{31}\big)a_{55}\,, \\ \mathrm{weight}(\boldsymbol{C}'_2) - \mathrm{weight}(\boldsymbol{C}_2) &= a_{12}a_{23}\big(a_{21}a_{32} - a_{32}a_{21}\big)a_{55}\,. \end{aligned}$$

Neither is zero in $\mathcal{B}_5$, since the relations (1.5) force neither a_{31} to commute with a_{22} nor a_{21} with a_{32}. Their sum, however, vanishes: the crossing relation in (1.5) with $i = 1$, $j = 2$, $k = 2$, $\ell = 3$ makes the inner factor $\big(a_{31}a_{22} - a_{22}a_{31}\big) + \big(a_{21}a_{32} - a_{32}a_{21}\big)$ identically zero, so

$$[\mathrm{weight}(\boldsymbol{C}'_1) - \mathrm{weight}(\boldsymbol{C}_1)] + [\mathrm{weight}(\boldsymbol{C}'_2) - \mathrm{weight}(\boldsymbol{C}_2)] = 0 \quad \text{in} \quad \mathcal{B}_5\,,$$

even though neither orbit cancels separately.

8.3. **Clow types.** In the CF case of Section 7, the proof of Lemma 7.4 shows that $\mathrm{weight}_{\boldsymbol{q}}(\boldsymbol{C}) = \mathrm{weight}_{\boldsymbol{q}}(\boldsymbol{C}')$ in $\mathcal{A}_n^{\boldsymbol{q}}$ for each orbit $\{\boldsymbol{C}, \boldsymbol{C}'\}$ separately. Here this fails: in Example 8.5, neither $\mathrm{weight}(\boldsymbol{C}'_1) - \mathrm{weight}(\boldsymbol{C}_1)$ nor $\mathrm{weight}(\boldsymbol{C}'_2) - \mathrm{weight}(\boldsymbol{C}_2)$ is zero in $\mathcal{B}_5$, and yet their sum is zero.

Since the orbit pairing no longer suffices, we prove the collective cancellation of the non-permutation clow sequences by a coarser grouping: we partition them into several types and show that the signed weights within each type sum to zero in $\mathcal{B}_n$.

Read the clow sequence from right to left, taking the clows in reverse order and each clow from its right end toward its head. Let e' be the first element of this reading that repeats one already read, let e be the earlier occurrence it repeats, and let h be the head of the clow containing e'. We call that clow *defected*.

Passing to the corresponding split clow sequence separates e and e' into two distinct clows. We call the one carrying e, namely the clow with the larger head, the *right split*, and the one carrying e' the *left split*.

Example 8.6. The merged clow sequence

$$\boldsymbol{C} = (\underset{h}{\underline{1}}725\underset{e'}{\underline{\mathbf{5}}}38\underset{e}{\underline{\mathbf{5}}}7)(2)(469)$$

of Example 7.6 is read from right to left as $9, 6, 4, 2, 7, \mathbf{5}, 8, 3, \mathbf{5}, \ldots$, the repeated pair shown in bold: the second occurrence is the first repeat $e' = \mathbf{5}$, and the earlier one is $e = \mathbf{5}$. The clow (172553857) is the defected clow of $\boldsymbol{C}$, and the corresponding split clow sequence is

$$\boldsymbol{C}' = (\underset{h}{\underline{1}}725\underset{e'}{\underline{\mathbf{5}}}7)(2)(38\underset{e}{\underline{5}})(469),$$

with the same three elements underlined: the right split (385) carries e and the left split (172557) carries e'.

Definition 8.7. An orbit is of *type* 1 if, in the defected clow of its merged sequence, the head h is followed by an element other than e'.

Example 8.8. The merged clow sequence

$$\boldsymbol{C} = (\underset{h}{\underline{1}}725\underset{e'}{\underline{5}}38\underset{e}{\underline{5}}7)(2)(469)$$

of Example 8.6 has the block 725 between h and e', so the head h is followed by $7 \neq e'$. Hence this orbit is of type 1.

Definition 8.9. An orbit is of *type* 2 if, in the defected clow of its merged sequence, the head h is followed by e', and the sequence contains another clow whose head is smaller than h.

Example 8.10. In the merged clow sequence

$$\boldsymbol{C} \,=\, (132)(\underset{h}{3}\underset{e'}{6}4\underset{e}{6})(52),$$

the defected clow is (3646), whose head $h = 3$ is immediately followed by $e' = 6$. Moreover the clow (132) has head $1 < h$, so this orbit is of type 2. Its corresponding split clow sequence is

$$\boldsymbol{C}' \,=\, (132)(\underset{h}{3}\underset{e'}{6})(4\underset{e}{6})(52),$$

with the same three elements underlined. Here the left split (36) is not the leftmost clow of $\boldsymbol{C}'$: the smaller-head clow (132) stands to its left.

Definition 8.11. An orbit is of *type* 3 if it is not of type 1 or type 2.

Example 8.12. In the merged clow sequence

$$\boldsymbol{C} \,=\, (\underset{h}{1}\underset{e'}{5}3\underset{e}{5}2)(48)(67),$$

the head h is followed by e' and no clow of smaller head precedes the defected clow, so the orbit is neither of type 1 nor of type 2. Hence this orbit is of type 3.

8.4. **Collective cancellation.** We now use a double induction, to prove that the signed sums of non-permutation clow sequences vanish in $\mathcal{B}_n$, type by type.

Remark 8.13. Whenever we wish to prove that a signed sum of clow sequences supported on a fixed multiset of elements vanishes in $\mathcal{B}_n$, we may assume without loss of generality that the support is an interval $\{1, 2, \ldots, k\}$, possibly with repetitions. Indeed, if the support has a gap, we shift all larger elements down, preserving their order, so as to fill it.

Inductive Assumption 8.14. Fix $m \geq 1$, the number of elements in the support multiset. We assume that for any $n \leq m$:

(i) the signed sum of the clow sequences of type 1 supported on a fixed multiset with n elements that share a common prefix

$$\underline{C_1 C_2 \cdots (h t_1 t_2 \cdots t_k e' \cdots)} \cdots$$

vanishes in $\mathcal{B}_n$;

(ii) the signed sum of the clow sequences of type 2 supported on a fixed multiset with n elements that share a common prefix

$$\underline{C_1 C_2 \cdots (h e' \cdots)} \cdots$$

vanishes in $\mathcal{B}_n$;

(iii) the signed sum of the clow sequences of type 3 supported on a fixed multiset with n elements that share a common prefix

$$\underline{(h e' \cdots)} \cdots$$

vanishes in $\mathcal{B}_n$.

Lemma 8.15. *For any integer $n \geq 1$, the signed sum of the clow sequences of type* 1 *supported on a fixed multiset with n elements that share a common prefix*

$$C_1 C_2 \cdots (h t_1 t_2 \cdots t_k e' \cdots) \cdots$$

vanishes in $\mathcal{B}_n$.

Proof. We argue under Inductive Assumption 8.14. Deleting the common clows left of the defected clow and the block $t_1 \cdots t_k$ turns the sum into the signed sum of the type-3 sequences of order $n < m$, beginning with $(h e' \cdots) \cdots$. This sum is zero in $\mathcal{B}_n$ by induction.

Note that every term in the sum above begins $a_{he'}\, a_{e',*} \cdots$, so factoring out $a_{he'}$ shows that the signed sum of the suffixes S is zero in $\mathcal{B}_n$. In the original sum, every term shares the prefix

$$\underbrace{\cdots a_{ht_1}\, a_{t_1 t_2} \cdots a_{t_{k-1} t_k}\, a_{t_k e'}}_{\text{common prefix}}\, a_{e',*} \cdots ,$$

and factoring it out leaves the same suffixes S. Multiplying by this common prefix on the left, we conclude that the original sum vanishes in $\mathcal{B}_n$. □

Lemma 8.16. *For any integer $n \geq 1$, the signed sum of the clow sequences of type 2 supported on a fixed multiset with n elements that share a common prefix*

$$C_1 C_2 \cdots (h e' \cdots) \cdots$$

vanishes in $\mathcal{B}_n$.

Proof. Completely analogous to the proof of Lemma 8.15. □

Remark 8.17. A non-permutation clow sequence that is not of type 1 or type 2 has only one pair of repeated elements, namely e' and e; all of its remaining elements are distinct.

Lemma 8.18. *For any integer $n \geq 1$, the signed sum of the clow sequences of type 3 supported on a fixed multiset with n elements that share a common prefix*

$$(h e' \cdots) \cdots$$

vanishes in $\mathcal{B}_n$.

Proof. We argue under Inductive Assumption 8.14. By Remark 8.17, we may assume that the support is $\{1, 2, \ldots, k, k, \ldots, n\}$ with the single repeated pair $e' = e = k$, and head $h = 1$. Let

$$T_n \;=\; \sum_{\boldsymbol{C}} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}(\boldsymbol{C})$$

be the signed sum of all type-3 clow sequences supported on this multiset.

Every such sequence has its defected clow of the form $(h e' \cdots)$, so its weight begins with the arc $h \to e'$, that is, with the letter $a_{he'}$. Writing $\operatorname{weight}(\boldsymbol{C}) = a_{he'}\, w(\boldsymbol{C})$, where $w(\boldsymbol{C})$ denotes the product of the remaining n letters of $\operatorname{weight}(\boldsymbol{C})$, we factor this common letter out:

$$T_n \;=\; a_{he'}\, R_n\,, \qquad R_n \;=\; \sum_{\boldsymbol{C}} \operatorname{sign}(\boldsymbol{C})\, w(\boldsymbol{C}).$$

We claim that

$$R_n \;=\; -\operatorname{Mdet}(Q),$$

where $Q = (Q_{cr})$ is the $n \times n$ matrix obtained from A by replacing its first column with a copy of its k-th column:

$$Q_{1r} \;=\; a_{kr}\,, \qquad Q_{cr} \;=\; a_{cr} \quad (2 \leq c \leq n).$$

Indeed, each entry of Q is one of the original entries a_{cr}, and the columns of Q form a sub-collection of the columns of A, so the entries of Q satisfy the column and crossing relations in (1.5) and $Q \in \mathcal{B}_n$ is again an RQ matrix. Now, deleting the distinguished repeated k that follows the head of the defected clow turns a type-3 clow sequence on $\{1, \ldots, k, k, \ldots, n\}$ into a cycle decomposition $\boldsymbol{C} \in \mathcal{S}_n$, the split arcs $h \to k \to (\cdot)$ collapsing to a single arc whose letter belongs to the first column of Q, and conversely. This bijection preserves the weights and reverses the signs, since the number of clows is unchanged while the length drops by one, which proves the claim.

By Lemma 8.1 applied to Q, we have $\operatorname{Mdet}(Q) = \operatorname{Cdet}(Q)$ in $\mathcal{B}_n$. The first and the k-th columns of Q are equal, both being the k-th column of A, and the Cayley determinant of an RQ matrix with two equal columns vanishes in $\mathcal{B}_n$ by Lemma 3.1. Hence

$$R_n \;=\; -\operatorname{Mdet}(Q) \;=\; -\operatorname{Cdet}(Q) \;=\; 0 \quad \text{in} \quad \mathcal{B}_n.$$

Multiplying by the common left factor $a_{he'}$, we conclude that $T_n = a_{he'}\, R_n$ vanishes in $\mathcal{B}_n$, as desired. □

Example 8.19. We illustrate the proof on the multiset $\{1,2,2,3\}$, so $k=2$ and $n=3$. There are six type-3 clow sequences, all with defected clow $(12\cdots)$, namely $(12)(2)(3)$, $(12)(23)$, $(122)(3)$, (1223), (1232) and $(123)(2)$, and T_3 is equal to

$$-a_{12}a_{21}a_{22}a_{33} + a_{12}a_{21}a_{23}a_{32} + a_{12}a_{22}a_{21}a_{33} - a_{12}a_{22}a_{23}a_{31} - a_{12}a_{23}a_{32}a_{21} + a_{12}a_{23}a_{31}a_{22}.$$

Factoring out a_{12} (the arc $1 \to 2$) leaves

$$R_3 \;=\; -a_{21}a_{22}a_{33} + a_{21}a_{23}a_{32} + a_{22}a_{21}a_{33} - a_{22}a_{23}a_{31} - a_{23}a_{32}a_{21} + a_{23}a_{31}a_{22}.$$

The bijection of the proof inserts the vertex 2 after the head 1. For example, the cycle decomposition $\boldsymbol{C} = (12)(3) \in \mathcal{S}_3$ has weight $Q_{12}Q_{21}Q_{33} = a_{22}\,a_{21}\,a_{33}$ and $\operatorname{sign}(\boldsymbol{C}) = -1$, and it maps to the clow sequence $(122)(3)$ of weight $a_{12}\cdot a_{22}\,a_{21}\,a_{33}$ and sign $+1$. Carrying out all six gives $R_3 = -\operatorname{Mdet}(Q)$, where

$$Q \;=\; \begin{pmatrix} a_{21} & a_{21} & a_{31} \\ a_{22} & a_{22} & a_{32} \\ a_{23} & a_{23} & a_{33} \end{pmatrix}$$

is obtained from A by setting the first column equal to the second. By Lemma 8.1, we have $\operatorname{Mdet}(Q) = \operatorname{Cdet}(Q)$ in $\mathcal{B}_3$. Finally, $\operatorname{Cdet}(Q) = 0$ in $\mathcal{B}_3$, since the first and second columns of Q coincide.

Proof of Lemma 8.4. Comparing the definitions (3.7) and (3.12), it suffices to show that the signed sum of the weights of the clow sequences in $\mathcal{C}_n^{\circledcirc}$ vanishes in $\mathcal{B}_n$. Observe that every clow sequence $\boldsymbol{C} \in \mathcal{C}_n^{\circledcirc}$ has a unique type and a unique prefix as above. Thus the set $\mathcal{C}_n^{\circledcirc}$ is a disjoint union, over the three types and over all possible common prefixes, of the families of clow sequences in Lemmas 8.15, 8.16 and 8.18. By these lemmas, the signed sum over each family vanishes in $\mathcal{B}_n$. Therefore, we have:

$$\operatorname{Vdet}(A) \;-\; \operatorname{Mdet}(A) \;=\; \sum_{\boldsymbol{C} \in \mathcal{C}_n^{\circledcirc}} \operatorname{sign}(\boldsymbol{C})\, \operatorname{weight}(\boldsymbol{C}) \;=\; 0 \quad \text{in} \quad \mathcal{B}_n,$$

as desired. □

9. $\boldsymbol{q}$-RIGHT-QUANTUM MATRICES

In this section we prove that $\operatorname{Cdet}_{\boldsymbol{q}}$, $\operatorname{Mdet}_{\boldsymbol{q}}$ and $\operatorname{Vdet}_{\boldsymbol{q}}$ are equal in $\mathcal{B}_n^{\boldsymbol{q}}$ (Theorem 4.1), the case of $\boldsymbol{q}$-RQ matrices, which is the most general case. The proof combines the proofs of the corresponding statements for the $\boldsymbol{q}$-CF and RQ cases, given in Sections 7 and 8, respectively.

9.1. Cayley–Moore determinant equality.

We are now ready to prove the most general result of this type:

Theorem 9.1 (=Lemma 4.2)**.** *Let $A = (a_{ij}) \in \mathcal{B}_n^{\boldsymbol{q}}$ be a $\boldsymbol{q}$-RQ matrix. Then we have:*

$$\operatorname{Cdet}_{\boldsymbol{q}}(A) \;=\; \operatorname{Mdet}_{\boldsymbol{q}}(A).$$

When all $q_{ij} = 1$, we have $\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) = \operatorname{weight}(\boldsymbol{C})$, and Theorem 9.1 becomes Lemma 8.1. As in §8.1, the summation is taken over all p-sequences.

Proof of Theorem 9.1. We repeat the bijective scheme of the proof of Lemma 8.1, now keeping track of the $\boldsymbol{q}$-weight. Recall the map Ψ on q-sequences defined as follows. Given a q-sequence whose longest p-sequence prefix consists of x steps, with i the height at the end of step x, the map Ψ swaps the unique step starting at height i with its immediate left neighbor, raising the rank by 1.

We carry out the swap together with a partner q-sequence, exactly as before: the partner has the same rank and agrees with the first everywhere except that the ends of the two swapped steps are

interchanged, so if the first corresponds to a permutation σ, the partner corresponds to $\sigma' = \sigma \cdot \tau$ and the two carry opposite signs, $\operatorname{sign}\sigma' = -\operatorname{sign}\sigma$. The two swapped steps are adjacent factors $a_{\ell j}\, a_{ki}$ with $i \neq j$; say $i < j$. By the computation in the proof of Lemma 7.1, interchanging $a_{\ell j}\, a_{ki}$ multiplies the $\boldsymbol{q}$-weight (7.1) of each sequence by

$$c \;=\; q_{ij}\, q_{k\ell}^{-1} \quad (k < \ell), \qquad c \;=\; q_{ij}\, q_{\ell k} \quad (k > \ell), \qquad c \;=\; q_{ij} \quad (k = \ell),$$

and in each of these three cases relations (3.2) give $a_{\ell j}\, a_{ki} = c\, a_{ki}\, a_{\ell j}$ in the two summed sequences at once. Because the two partners carry opposite signs, the relations apply with the signs intact, with the factor c cancelling against the $\boldsymbol{q}$-weight exactly as in Lemma 7.1:

$$\mp\, \mathrm{weight}_{\boldsymbol{q}}(\alpha) \,\pm\, \mathrm{weight}_{\boldsymbol{q}}(\beta) \;=\; \mp\, \mathrm{weight}_{\boldsymbol{q}}(\Psi\alpha) \,\pm\, \mathrm{weight}_{\boldsymbol{q}}(\Psi\beta).$$

When a sequence is already a p-sequence, Ψ leaves it unchanged. Iterating the map therefore rewrites the $\boldsymbol{q}$-Cayley determinant in $\mathcal{B}_n^{\boldsymbol{q}}$ as the $\boldsymbol{q}$-Moore determinant, proving Theorem 9.1. □

9.2. **Moore–Valiant determinantal equality.**

Theorem 9.2 (=Lemma 4.3)**.** *Let $A = (a_{ij}) \in \mathcal{B}_n^{\boldsymbol{q}}$ be a $\boldsymbol{q}$-RQ matrix. Then we have:*

$$\mathrm{Mdet}_{\boldsymbol{q}}(A) \;=\; \mathrm{Vdet}_{\boldsymbol{q}}(A).$$

The proof occupies the rest of this section. We carry over the double induction of §8.4 verbatim, replacing $\mathrm{weight}(\boldsymbol{C})$ by $\mathrm{weight}_{\boldsymbol{q}}(\boldsymbol{C})$ throughout. The reduction of the support to an interval $\{1, 2, \ldots, k\}$, the Inductive Assumption 8.14, and the type-1, type-2, type-3 grouping all apply unchanged.

Lemma 9.3. *For any integer $n \geq 1$, the signed $\boldsymbol{q}$-weighted sum of the clow sequences of type 1 supported on a fixed multiset with n elements that share a common prefix*

$$C_1 C_2 \cdots (h t_1 t_2 \cdots t_k e' \cdots) \cdots$$

vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$.

Proof. We use the analogue of Inductive Assumption 8.14 for the $\boldsymbol{q}$-weighted sums. Reading the arcs $h \to t_1 \to \cdots \to t_k \to e'$, every term factors at the word level as

$$\mathrm{weight}(\boldsymbol{C}) \;=\; \Pi\, P\, \sigma(\boldsymbol{C}), \qquad \Pi = \mathrm{weight}(C_1)\,\mathrm{weight}(C_2)\cdots, \qquad P = a_{ht_1}\, a_{t_1 t_2} \cdots a_{t_k e'},$$

with Π (the finished clows) and P (the path into e') common to all terms and $\sigma(\boldsymbol{C})$ the tail, the arc leaving e' onward. Deleting Π and the block $t_1 \cdots t_k$ turns $\boldsymbol{C}$ into a type-3 sequence $\overline{\boldsymbol{C}}$ of order $m < n$, with defected clow $(h e' \cdots)$ and $\mathrm{weight}(\overline{\boldsymbol{C}}) = a_{he'}\, \sigma(\boldsymbol{C})$.

By the Factorization Lemma 5.2, the $\boldsymbol{q}$-weight factors over clows and inversions between distinct clows cancel. Sorting the surviving inversions of $\mathrm{weight}_{\boldsymbol{q}}(\boldsymbol{C})$ by the blocks $\Pi, P, \sigma(\boldsymbol{C})$ gives:

$$\mathrm{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \;=\; [\mathrm{weight}_{\boldsymbol{q}}(C_1)\ \mathrm{weight}_{\boldsymbol{q}}(C_2)\cdots] \;\cdot\; P \;\cdot\; \gamma \;\cdot\; \mathrm{weight}_{\boldsymbol{q}}\big(\sigma(\boldsymbol{C})\big), \tag{9.1}$$

where $\mathrm{weight}_{\boldsymbol{q}}(C_j)$ and $\mathrm{weight}_{\boldsymbol{q}}(\sigma(\boldsymbol{C}))$ are the $\boldsymbol{q}$-weights of the finished clows and the tail, and γ is the *pure scalar*

$$\gamma \;=\; \prod_{x \in \{t_1, \ldots, t_k, e'\}} q_{hx}^{-1} \;\cdot\; \prod_{\substack{y \in S \\ y < e'}} q_{ye'}^{-1}, \tag{9.2}$$

with S the multiset of tail vertices. In other words, here we remove the support of the finished-clow vertices and one copy each of $h, t_1, \ldots, t_k, e'$. Indeed, only the cross-block inversions with left endpoint in P remain after Lemma 5.2: the head h gives no column inversion; each column t_i cancels against its row occurrence except for $q_{ht_i}^{-1}$ (the closing arc at h); and the row e' of $a_{t_k e'}$ leaves $q_{he'}^{-1}$ and one $q_{ye'}^{-1}$ per tail vertex $y < e'$ (whether y lies in the defected clow or a later one). No factor of (9.2) depends on the order of $\sigma(\boldsymbol{C})$, so γ is common.

By (9.1), with $\operatorname{weight}_{\boldsymbol{q}}(C_1)\operatorname{weight}_{\boldsymbol{q}}(C_2)\cdots$, P and γ common, we have:

$$\sum_{\boldsymbol{C}} \operatorname{sign}(\boldsymbol{C})\ \operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) \ = \ [\operatorname{weight}_{\boldsymbol{q}}(C_1)\ \operatorname{weight}_{\boldsymbol{q}}(C_2)\cdots]\cdot P\cdot\gamma\cdot \sum_{\boldsymbol{C}} \operatorname{sign}(\boldsymbol{C})\ \operatorname{weight}_{\boldsymbol{q}}\big(\sigma(\boldsymbol{C})\big).$$

The sign-preserving bijection $\boldsymbol{C}\mapsto\overline{\boldsymbol{C}}$ identifies the inner sum with the type-3 sum of order $m<n$. As in the proof of Lemma 9.6, we have $\operatorname{weight}_{\boldsymbol{q}}(\overline{\boldsymbol{C}}) = \gamma_0\, a_{he'}\ \operatorname{weight}_{\boldsymbol{q}}(\sigma(\boldsymbol{C}))$, with $\gamma_0, a_{he'}$ common. Thus, the sum above vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$ by Lemma 9.6. Cancelling $\gamma_0\, a_{he'}$, gives:

$$\sum_{\boldsymbol{C}} \operatorname{sign}(\boldsymbol{C})\ \operatorname{weight}_{\boldsymbol{q}}(\sigma(\boldsymbol{C})) \ = \ 0 \quad \text{ in } \quad \mathcal{B}_n^{\boldsymbol{q}}\,.$$

Multiplying on the left by $\operatorname{weight}_{\boldsymbol{q}}(C_1)\operatorname{weight}_{\boldsymbol{q}}(C_2)\cdots P\,\gamma$, we conclude that the original sum vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$, proving the lemma. □

Example 9.4. On the support $\{1,2,3,4,4,5,6,7,8,9\}$, take the type-1 family whose defected clow is first and begins $(18274\cdots4\cdots)$, so $h=1$, $t_1t_2t_3=827$, and $e'=e=4$. There are no finished clows, so Π is empty and the common prefix word is $P = a_{18}a_{82}a_{27}a_{74}$ (the arcs $1\to8\to2\to7\to4$). Each weight word in this family begins with P. Indeed, this follows from the fact that the second 4 and the vertices $3,5,6,9$ can be distributed in every admissible way. Now, since 3 is the only tail vertex below $e'=4$, formula (9.2) gives the common scalar

$$\gamma \ = \ q_{18}^{-1}\,q_{12}^{-1}\,q_{17}^{-1}\,q_{14}^{-1}\ \cdot\ q_{34}^{-1}\,.$$

For the member

$$\boldsymbol{C} \ = \ (1827435469)\,, \qquad \sigma(\boldsymbol{C}) \ = \ a_{43}a_{35}a_{54}a_{46}a_{69}a_{91}\,,$$

one checks $\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C}) = P\cdot\gamma\cdot\operatorname{weight}_{\boldsymbol{q}}\big(\sigma(\boldsymbol{C})\big)$.

Hence the signed sum factors as $P\gamma\cdot\sum_{\boldsymbol{C}}\operatorname{sign}(\boldsymbol{C})\ \operatorname{weight}_{\boldsymbol{q}}(\sigma(\boldsymbol{C}))$, whose inner factor is the type-3 sum (with its prefix removed) on $\{1,3,4,4,5,6,9\}$ and vanishes by Lemma 9.6.

Lemma 9.5. *For any integer $n\geq 1$, the signed $\boldsymbol{q}$-weighted sum of the clow sequences of type 2 supported on a fixed multiset with n elements that share a common prefix*

$$C_1C_2\cdots(he'\cdots)\cdots$$

vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$.

The proof is completely analogous to the proof of Lemma 9.3.

Lemma 9.6. *For any integer $n\geq 1$, the signed $\boldsymbol{q}$-weighted sum of the clow sequences of type 3 supported on a fixed multiset with n elements that share a common prefix*

$$(he'\cdots)\cdots$$

vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$.

Proof. As in the proof of Lemma 8.18, we may assume that the support is $\{1,2,\ldots,k,k,\ldots,n\}$ with the single repeated pair $e'=e=k$, and head $h=1$. Let

$$T_n \ = \ \sum_{\boldsymbol{C}} \operatorname{sign}(\boldsymbol{C})\ \operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C})$$

be the signed $\boldsymbol{q}$-weighted sum of all type-3 clow sequences supported on this multiset. Every such sequence has its defected clow of the form $(he'\cdots)$, so its weight begins with the letter $a_{he'}$, the arc $h\to e'$. We claim that this leading letter carries a $\boldsymbol{q}$-weight factor common to all terms. Indeed, in $\operatorname{weight}_{\boldsymbol{q}}(\boldsymbol{C})$ the leading position contributes, through (7.1), the inversions it forms with the rest of the word. It forms no column inversion, since its column index h is the head and hence the smallest column present; and it forms a row inversion with each later position of smaller row, contributing $\prod q_{re'}^{-1}$ over the rows $r<e'$ occurring after it. As all type-3 sequences in T_n share

the same support, this product is the same for every term. Writing $\mathrm{weight}_{\boldsymbol{q}}(\boldsymbol{C}) = \gamma_0\, a_{he'}\, w_{\boldsymbol{q}}(\boldsymbol{C})$, where $\gamma_0 = \prod_{r<e'} q_{re'}^{-1}$ is this common scalar, we factor the common left factor $\gamma_0\, a_{he'}$ out:

$$T_n \;=\; \gamma_0\, a_{he'}\, R_n\,, \qquad R_n \;=\; \sum_{\boldsymbol{C}} \mathrm{sign}(\boldsymbol{C})\, w_{\boldsymbol{q}}(\boldsymbol{C}).$$

This is the empty-block case ($k = 0$) of the common scalar γ of Lemma 9.3. Indeed, with no t_i between h and e' and no finished clows, the prefix is the single arc $a_{he'}$, the head-versus-block product of (9.2) disappears, and its second product $\prod_{y<e'} q_{ye'}^{-1}$ becomes γ_0.

We claim that

$$R_n \;=\; -\,\mathrm{Mdet}_{\boldsymbol{q}}(Q),$$

where $Q = (Q_{cr})$ is the $n \times n$ matrix obtained from A by replacing its first column with a copy of its k-th column:

$$Q_{1r} \;=\; a_{kr}\,, \qquad Q_{cr} \;=\; a_{cr} \quad \text{for all} \quad 2 \le c \le n\,.$$

Each entry of Q is one of the original entries a_{cr}, and the columns of Q form a sub-collection of the columns of A; hence the entries of Q satisfy the same column and crossing relations (3.2), so $Q \in \mathcal{B}_n^{\boldsymbol{q}}$ is again a $\boldsymbol{q}$-RQ matrix.

The same sign-reversing, weight-matching bijection as in the proof of Lemma 8.18 proves the claim. Indeed, deleting the distinguished repeated k that follows the head of the defected clow turns a type-3 sequence on $\{1, \ldots, k, k, \ldots, n\}$ into a cycle decomposition $\boldsymbol{C} \in \mathcal{S}_n$, the split arcs $h \to k \to (\cdot)$ collapsing to a single arc whose letter belongs to the first column of Q, and conversely; the $\boldsymbol{q}$-weights match under this bijection as in the proof of Lemma 7.4.

By Theorem 9.1 applied to Q, we have $\mathrm{Mdet}_{\boldsymbol{q}}(Q) = \mathrm{Cdet}_{\boldsymbol{q}}(Q)$ in $\mathcal{B}_n^{\boldsymbol{q}}$. The first and the k-th columns of Q are equal, both being the k-th column of A, and the $\boldsymbol{q}$-Cayley determinant of a $\boldsymbol{q}$-RQ matrix with two equal columns vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$ by Lemma 3.1. We conclude:

$$R_n \;=\; -\,\mathrm{Mdet}_{\boldsymbol{q}}(Q) \;=\; -\,\mathrm{Cdet}_{\boldsymbol{q}}(Q) \;=\; 0 \quad \text{in} \quad \mathcal{B}_n^{\boldsymbol{q}}\,.$$

Multiplying by the common left factor $\gamma_0\, a_{he'}$, and noting that scalar γ_0 is nonzero, we conclude that $T_n = \gamma_0\, a_{he'}\, R_n$ vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$. □

Together, Lemmas 9.3, 9.5 and 9.6 show that the signed $\boldsymbol{q}$-weighted sum of all clow sequences in $\mathcal{C}_n^{\circledcirc}$ vanishes in $\mathcal{B}_n^{\boldsymbol{q}}$, which proves Theorem 9.2.

Remark 9.7. When all $q_{ij} \leftarrow 1$, the whole section specializes to §8.2. Indeed, the relations (3.2) become (1.5), the $\boldsymbol{q}$-weight $\mathrm{weight}_{\boldsymbol{q}}$ reduces to weight, and Theorems 9.1 and 9.2 become Lemmas 8.1 and 8.4. Setting all $q_{ij} \leftarrow q$ recovers the q-right-quantum case, but since the proof do not significantly simplify we opted for the general case.

10. Final remarks and open problems

10.1. For a *skew-symmetric matrix* $A = -A^T$, computing the Pfaffian plays the same role as the determinant of the usual matrix. Numerically, this is an equivalent problem since $\det(A) = \mathrm{pf}(A)^2$. However, computing the Pfaffian as a polynomial of matrix entries is different from computing the permanent. A variation on the Mahajan–Vinay algorithm was given in [MSV04], see also [Rote01, Urb10]. It would be interesting to see if this algorithm can be extended to compute the Pfaffian of right-quantum matrices.

10.2. In [MV99], Mahajan and Vinay wrote that their algorithm was partly inspired by combinatorial proofs of the *Cayley–Hamilton theorem* (CHT) by Rutherford, Straubing and Zeilberger [Rut64, Str83, Zei85]. Most recently, Ikenmeyer [Ike25] constructed a new ABP for the determinant. He suggested that his approach is even closer to Straubing's and Zeilberger's combinatorial arguments. It would be interesting to see if there is a proof of our Main Theorem 1.1 in the style of Ikenmeyer's construction, as it would potentially avoid using our involved combinatorial arguments.

In fact, several (possibly, equivalent) versions of a $\boldsymbol{q}$-CHT are already present or implicit in the literature. Indeed, both the RQ and q-RQ versions of the CHT are given in [CF08, CFR09, CFRS14]. For $\boldsymbol{q}$-RQ matrices, one can take a fully noncommutative version of the CHT by Gelfand and Retakh for quasideterminants [GR92, GGRW05], translated into the language of $\boldsymbol{q}$-Cayley determinants following [GR91, ER99, KP07]. Alternatively, one can use the "$\boldsymbol{q} = 1$" principle by Konvalinka–Pak [KP07, Lemma 12.3], generalizing the "$q = 1$" principle by Foata–Han [FH08], to derive the $\boldsymbol{q}$-CHT directly from the RQ version.

10.3. For an irreducible character $\chi^\lambda : S_n \to \mathbb{Z}$, where λ is a partition of n, an *immanant* of the matrix $A = (a_{ij})$ is defined as follows:

$$\operatorname{Imm}_\lambda(A) \,:=\, \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, a_{1\sigma_1} \,\cdots\, a_{n\sigma_n}\,.$$

We refer to [GJ92] for an accessible introduction to immanants and their applications to algebraic combinatorics. Note that the immanants generalize both the permanent and the determinant, for the trivial and sign characters, respectively. The algorithmic and complexity properties of immanants have also been extensively studied, see [Bür00, §7] for an overview, and [Cur21, MR26] for some notable recent results.

Given the negative results in the papers cited above, it would be interesting to see if one could extend our approach to obtain positive results for computing quantum immanants on quantum matrices. As a starting point, we refer to Okounkov [Oko96] and Konvalinka [Kon12], for two different definitions of quantum immanants.

10.4. Arguably, matrix inversion is the real reason the noncommutative determinant can be computed efficiently. This is the philosophy underlying the theory of quasideterminants, see [GGRW05]. Matrix inversion exists, in particular, when there is an R-matrix which satisfied the quantum Yang–Baxter equation, see [ESS00, ER99]. We speculate that the determinantal identities in Theorem 4.1 are in fact consequences of algebraic properties guaranteed by the R-matrices. More concretely, we conjecture that Theorem 4.1 can be extended to all generalized Belavin–Drinfeld triples [ESS00], see also [Hod15] for the introduction.

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Use of AI. During the preparation of this manuscript, the authors used Claude (Anthropic, Claude Opus 4.8) to make numerous experiments in the combinatorics of words, in our search for the desired sign-reversing involution. We also used it to make figures and proof check the examples. Additionally, we used ChatGPT (OpenAI, GPT-5.5) for historical background and references. All AI-generated suggestions were reviewed and, where appropriate, modified before inclusion in the manuscript. More often than not, they proved false or unhelpful, and were simply ignored.

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(Igor Pak) Department of Mathematics, UCLA, Los Angeles, CA 90095.
Email address: pak@math.ucla.edu

(Daniel Soskin) Department of Mathematics, UCLA, Los Angeles, CA 90095.
Email address: dsoskin@math.ucla.edu