MONOTONE PARAMETERS ON CAYLEY GRAPHS OF FINITELY GENERATED GROUPS

MARTIN KASSABOV⋆ AND IGOR PAK⋄

Abstract. We construct a new large family of finitely generated groups with continuum many values of the following monotone parameters: spectral radius, critical percolation, and asymptotic entropy. We also present several open problems on other monotone parameters.

1. Introduction

1.1. **Main results.** There are several probabilistic parameters of Cayley graphs of finitely generated groups that capture "global properties" of the groups, i.e., independent of the generating sets (see below and §6). Typically, these parameters are *monotone*: as groups get "larger" the parameters increase/decrease, usually in a difficult to control way.

While there is an extensive literature on bounds for these parameters and their relations to each other, computing them remains challenging and the exact values are known in only few examples and special families. Our main result shows that these parameters can be as unwieldy as the finitely generated group themselves.

Theorem 1.1 (Main theorem)**.** *Let $G = \langle S\rangle$ be a finitely generated group, and let $f(G,S)$ denote one of the following parameters:*

- *spectral radius $\rho(G,S)$,*
- *asymptotic entropy $h(G,S)$,*
- *critical site percolation $p_c^{\mathrm{s}}(G,S)$,*
- *critical bond percolation $p_c^{\mathrm{b}}(G,S)$.*

Then, there is a family of 4-regular Cayley graphs $\big\{\mathrm{Cay}(G_\omega,S_\omega)\big\}$, such that the set of parameter values $\big\{f(G_\omega,S_\omega)\big\}$ has cardinality of the continuum.

The construction of this family of Cayley graphs is based on properties of an uncountable family of *decorated Grigorchuk groups*, the setting we previously considered in [KP13] (see also [KN06]), and related to the approach in [TZ19], see §7.1. See also §7.6 for an alternative approach to the theorem. We also obtain the following result of independent interest.

Theorem 1.2. *In notation of Theorem 1.1, for $k \ge 3$ the sets*

$$X_{\rho,k} := \big\{\rho(G,S) : |S| = k\big\}$$

have no isolated points on $(\alpha_k, 1]$, where $\alpha_k := \frac{2\sqrt{k-1}}{k}$.

Note that $X_{\rho,k} \subseteq [\alpha_k, 1]$, so the theorem implies that $X_{\rho,k}$ has no isolated points, except possibly at α_k. Here α_k is the spectral radius of the infinite k-regular tree, which can be viewed as the Cayley graph of a free product of k copies of $\mathbb{Z}_2$, when k is even this is also the Cayley graph on the free group on $k/2$ generators. A version of Theorem 1.2 for the asymptotic entropy was given by Tamuz and Zheng in [TZ19].

Date: April 23, 2024.

⋆Department of Mathematics, Cornell University, Ithaca, NY 14853, USA; martin.kassabov@gmail.com.

⋄Department of Mathematics, UCLA, Los Angeles, CA 90095, USA; pak@math.ucla.edu.

1.2. **Monotone parameters.** Let $f : \{(G,S)\} \to \mathbb{R}$ be a function on *marked groups*, i.e., pairs of finitely generated groups G and finite symmetric generating sets $S = S^{-1}$. We refer to f as *parameters* (see the discussion in §7.5). We say that f is *decreasing* if the following conditions hold:

$$(*) \qquad f(G/N, S') \ge f(G,S) \qquad \text{for all} \quad N \lhd G.$$

$$(**) \qquad (G_n, S_n) \to (G,S) \implies \limsup_{n\to\infty} f(G_n,S_n) \ge f(G,S).$$

Here $N \lhd G$ denotes a normal subgroup N of G, and S' is a projection of S onto G/N. In $(**)$, the convergence on the left[1] is in the Chabauty topology for marked groups, see §2.5.

We say that f is *increasing* if the inequalities $(*)$ and $(**)$ are reverted. We say that f is *monotone* if it is either increasing or decreasing. We say that f is *strictly decreasing* (respectively, *strictly increasing* and *strictly monotone*) if the inequality $(*)$ is strict, provided that N is nonamenable. Similarly, we say that f is *sharply decreasing* (respectively, *sharply increasing* and *sharply monotone*) if the inequality $(*)$ is strict if and only if N is nonamenable.

1.3. **Spectral radius.** Let $\mathrm{Cay}(G,S)$ denote the Cayley graph of a finitely generated group $G = \langle S\rangle$. The *spectral radius* $\rho(G,S)$ is defined as:

$$\rho(G,S) := \limsup_{n\to\infty} \frac{1}{|S|} \sqrt[n]{c(n)}\,,$$

where the *cogrowth sequence* $\big\{c(n) = c(G,S;n)\big\}$ is the number of words in the alphabet S which are equal to the identity e in G. Equivalently, this is the number of loops in $\mathrm{Cay}(G,S)$ of length n, starting and ending at e.

Famously, it was shown by Kesten [Kes59], that $\rho(G,S) = 1$ if and only if G is nonamenable, and that

$$\alpha_k = \tfrac{2\sqrt{k-1}}{k} \le \rho(G,S) \le 1\,, \quad \text{where} \quad k = |S|.$$

For $k = 2m$, the lower bound is attained on a free group $\mathbb{F}_m = \langle x_1^{\pm1}, \ldots, x_m^{\pm1}\rangle$.

Unfortunately, relatively little is known about the spectral radius in full generality, beyond these basic inequalities. Notably, it is open whether *every* $\alpha \in (0,1]$ is a spectral radius of *some* finitely generated group, cf. [BLM23, Question 4.2]. On the other hand, until this paper it was not known if there is a single example of a group with a transcendental spectral radius, a problem discussed in [Pak18, §2.4].

For several families of nonamenable groups, the exact value of $\rho(G,S)$ is known, see e.g. [GH97, Woe00] (see also [BLM23, E+14, Kuk99]). In all these cases the spectral radii are algebraic. For the surface group Γ_2 of genus 2, i.e.

$$\Gamma_2 = \langle a_1, a_2, b_1, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1\rangle,$$

after much effort in many papers, it was shown that $0.6624 \le \rho(\Gamma_2, S) \le 0.6629$, where S are standard generators as above. Sarnak's question whether this spectral radius is transcendental remains unresolved. For these and related results we refer to [GH97, §7] and references therein.

Proposition 1.3. *Spectral radius* $\rho : \{(G,S)\} \to (0,1]$ *is a decreasing parameter.*

Indeed, the property $(*)$ is straightforward, while the property $(**)$ uses the fact that $\rho(G,S)$ is a limit of a supermultiplicative sequence: $c(n+m) \ge c(n)c(m)$. The following result by Kesten shows that the spectral radius is sharply decreasing:

Lemma 1.4 ([Kes59, Lemma 3.1])**.** *Let* $G = \langle S\rangle$ *and let* N *be a normal subgroup of* G *and* S' *be the projection of* S *onto* G/N*. Then* $\rho(G,S) < \rho(G/N, S')$ *<u>if and only if</u>* N *is nonamenable.*

[1] The inequality $(**)$ states that f is *semi-continuous*, see Lemma 4.6; we include it in the definition to emphasize the direction of the inequality.

Note that the corresponding inequality for the Cheeger constant remains open (Conjecture 6.2). Note also that there is an alternative notion of the *cogrowth sequence*, where only reduced words are considered (equivalently, paths on the Cayley graph are not allowed to backtrack), see e.g. [CD21, §14.6]. Much of what we present translates easily to this setting; we omit it to avoid the confusion.

1.4. **Critical probabilities.** In the *Bernoulli site percolation*, the vertices of the graph $\mathrm{Cay}(G,S)$ are open with probability p and closed with probability $(1-p)$, independently at random. The *Bernoulli bond percolation* is defined analogously, but now the edges are open/closed. These notions are different, but closely related to each other, see below.

Denote by $\theta^{\mathrm{s}}(G,S,p)$ and $\theta^{\mathrm{b}}(G,S,p)$ the probability that the identity element e is in an infinite connected component in the site and bond percolation, respectively. We omit the superscript when the notation or results hold for both site and bond percolation, hoping this would not lead to confusion.

The *critical probability* is defined as follows:

$$p_c(G,S) \,:=\, \sup\big\{p \,:\, \theta(G,S,p)=0\big\}.$$

We refer to [BS96] for an introduction to percolation on Cayley graphs, to [BR06, Gri99, Wer09] for a thorough treatment of both classical and recent aspects, and to [Dum18] for a recent overview of the subject.

It is easy to see that $p_c = 1$ for all groups of linear growth (which are all virtually $\mathbb{Z}$). It was conjectured in [BS96, Conj. 2], that $p_c < 1$ for all groups of superlinear growth. Special cases of this conjecture have been established in a long series of papers, until it was eventually proved in [D+20]. The ultimate result, the remarkable "gap inequality" $p_c \le 1-\varepsilon$ for a universal constant $\varepsilon > 0$, was obtained in [PS23] for all groups of superlinear growth, see also [HT21]. This shows that Theorem 1.2 does not apply to critical probabilities.

Famously, it was shown in [GS98], that

$$p_c^{\mathrm{b}}(G,S) \,\le\, p_c^{\mathrm{s}}(G,S) \,\le\, 1-\big(1-p_c^{\mathrm{b}}\big)^{k-1} \quad \text{where} \quad k=|S|.$$

It is also known that bond percolation can be simulated by site percolation, but not vice versa, see [GZ24]. In general, these critical percolations do not coincide. For example, the celebrated Kesten's theorem states that $p_c^{\mathrm{b}} = \frac{1}{2}$ for the square grid (Cayley graph of $\mathbb{Z}^2$ with standard generators). By comparison, $p_c^{\mathrm{s}} \approx 0.592746$ in this case [NZ01], although the exact value is not known.

In all known examples when the critical probabilities $p_c^{\mathrm{s}}(G,S)$ and $p_c^{\mathrm{b}}(G,S)$ are computed exactly, they are always algebraic, cf. [Koz08, SZ10]. For example, it is known that

$$p_c^{\mathrm{s}}(G,S) \,=\, p_c^{\mathrm{b}}(H,R) \,=\, 1 \,-\, 2\sin\left(\tfrac{\pi}{18}\right) \,\in\, \overline{\mathbb{Q}}\,,$$

where

$$G \,=\, \langle x,y \mid x^3=y^3=(xy)^3=1\rangle, \quad S \,=\, \{x,x^{-1},y,y^{-1}\},$$
$$H \,=\, \langle a,b,c \mid a^2=b^2=c^2=(abc)^2=1\rangle, \quad R \,=\, \{a,b,c\}.$$

In this case, $\mathrm{Cay}(G,S)$ is the *Kagomé lattice* and $\mathrm{Cay}(H,R)$ is the *hexagonal lattice*, see e.g. [BR06, §5.5]. It was asked in [PS08], if there exist critical percolations $p_c(G,S)$ that are transcendental. Theorem 1.1 gives a positive answer to this question.

Proposition 1.5. *Critical probabilities* $p_c^{\mathrm{s}}, p_c^{\mathrm{b}} \,:\, \{(G,S)\} \to (0,1]$ *are decreasing parameters.*

The proof of this result is straightforward. The following result by Martineau and Severo shows that critical probabilities are strictly decreasing:

Lemma 1.6 ([MS19])**.** *Let* $G = \langle S \rangle$ *and let* $N \neq \mathbf{1}$ *be a normal subgroup of* G*. Let* S' *be the projection of* S *onto* G/N*. Then:*

$$p_c^{\mathrm{s}}(G,S) \,<\, p_c^{\mathrm{s}}(G/N,S') \quad \text{and} \quad p_c^{\mathrm{b}}(G,S) \,<\, p_c^{\mathrm{b}}(G/N,S').$$

Note that there is no assumption that N is nonamenable, and, in fact, the main result in [MS19] is stated in the greater generality of quasi-transitive group actions on graphs. This result resolved a well-known open problem by Benjamini and Schramm [BS96, Question 1]. Thus, for example, we have $1 > p_c(\mathbb{Z}^2, S') > p_c(\mathbb{Z}^3, S)$, so the critical probabilities are not sharply decreasing.

1.5. **Entropy.** Let G be a finitely generated group, and let $\mu : G \to \mathbb{R}_{\geq 0}$ be a probability distribution. *Shannon's entropy* is defined as

$$\mathrm{H}(\mu) \,:=\, -\sum_{g\,\in\,\mathrm{supp}(\mu)} \mu(g)\,\log\mu(g).$$

As before, let $G = \langle S \rangle$, where $S = S^{-1}$. Denote by $\mu_n(g) := \mathbb{P}[x_n = g]$ the distribution of the simple random walk $\{x_n\}$ on $\mathrm{Cay}(G,S)$ starting at identity $x_0 = e$. Finally, let

$$h(G,S) \,:=\, \lim_{n\to\infty} \frac{\mathrm{H}(\mu_n)}{n}$$

denote the *asymptotic entropy*, see [KV83]. Recall that $h(G,S) > 0$ if and only if $\mathrm{Cay}(G,S)$ has the *non-Liouville property* (existence of non-constant bounded harmonic functions). Equivalently, $h(G,S) = 0$ if and only if the random walk $\{x_n\}$ has trivial Poisson boundary, ibid.

We note that there are solvable groups of exponential growth with positive asymptotic entropy; the *lamplighter group* $\mathbb{Z}_2 \wr \mathbb{Z}^d$ for $d \geq 3$ is the most famous example [KV83, §6] (see also [Pete23, §9.1] and an introduction in [Tab17]). Note also that the asymptotic entropy is known explicitly only in a few cases as it is so hard to compute. For example, it was computed in [Bis92, Prop. 2.11], that

$$h(\mathbb{F}_m, S) \,=\, \frac{2m-2}{2m}\,\log(2m-1).$$

Here $\mathbb{F}_m = \langle z_1^{\pm 1}, \ldots, z_m^{\pm 1} \rangle$ is a free group with the standard generating set. Note that the asymptotic entropy is transcendental in this case: $h(\mathbb{F}_m, S) \notin \overline{\mathbb{Q}}$.

Proposition 1.7. *Asymptotic entropy* $h : \{(G,S)\} \to \mathbb{R}_{\geq 0}$ *is an increasing parameter.*

The proof of this result is straightforward. The following result by Kaimanovich shows that the asymptotic entropy is strictly increasing:

Lemma 1.8 ([Kai02, Thm 2])**.** *Let* $G = \langle S \rangle$ *and let* N *be a normal subgroup of* G*. Suppose* N *is nonamenable, and let* S' *be the projection of* S *onto* G/N*. Then* $h(G,S) > h(G/N,S')$*.*

Since there are amenable groups with positive asymptotic entropy, we conclude that the asymptotic entropy is not sharply increasing. Also, note that the corresponding inequality for the speed of random walk $\{x_n\}$ remains open (Conjecture 6.5). Let us mention other (closely related) entropy notions, such as the *Connes–Størmer entropy* (see e.g. [Bis92]), and the *tree entropy* [Lyo05]. These can also be viewed as probabilistic parameters.

1.6. **Proof outline.** The proof of Theorem 1.1 is extremely general and is based on a group theoretic construction and a set theoretic argument. Formally, Theorem 1.1 is an immediate consequence from following two complementary lemmas.

Lemma 1.9 (combined lemma)**.** *All parameters* f *in Theorem 1.1 are strictly monotone.*

The lemma is a combination of Propositions 1.3, 1.5, 1.7 and Lemmas 1.4, 1.6, 1.8.

Lemma 1.10 (main lemma)**.** *Let* f *be a strictly monotone parameter. Then, there is a family* $\{(G_\omega, S_\omega)\}$ *of marked groups, s.t.* $\{f(G_\omega, S_\omega)\}$ *has cardinality of the continuum.*

The proof of Lemma 1.10 is in turn an easy consequence of the following result of independent interest.

Lemma 1.11 (main construction)**.** *There exists a family* $\{(G_J, S_J) : J \in 2^{\mathbb{N}}\}$ *of* 4*-generated marked groups, satisfying the following:*

- (strict monotonicity) *For every subset* $J \subsetneq J'$*, there is a surjection of marked groups* $G_{J'} \to G_J$*, s.t. the kernel of the projection is nonamenable.*
- (continuity) *For every sequence* $\{J_n \subset J : n \in \mathbb{N}\}$ *s.t.* $J_n \to J$ *in the Tychonoff topology of* $2^{\mathbb{N}}$*, we have* $(G_{J_n}, S_{J_n}) \to (G_J, S_J)$ *in the Chabauty topology.*

Equivalently, the continuity condition says that there exists a function $\lambda : \mathbb{N} \to \mathbb{N}$, such that the ball of radius n in the Cayley graph of (G_J, S_J), depend only on $J \cap \{1, \ldots, \lambda(n)\}$.

1.7. **Paper structure.** In Sections 2 and 3 we recall definitions of marked groups, Grigorchuk groups and their convergence. Most readers familiar with the area should be able to skip this section. In Section 4 we give the proof of Lemmas 1.11 and 1.10, which complete the proof of the main Theorem 1.1. In Section 5, we prove Theorem 1.2. We conclude with a discussion of other monotone parameters in Section 6 and final remarks in Section 7.

2. Marked groups and their limits

We recall standard definitions of marked groups. We stay close to [KP13] from which we heavily borrow the notation and some basic results.

2.1. **Basic definitions and notation.** Denote $[n] = \{1, \ldots, n\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Let $\mathbb{R}_{\geq 0} = \{x \geq 0\}$, $\overline{\mathbb{Q}}$ the algebraic numbers, and $\mathbb{Q}_p$ the p-adic numbers. We use $(\log a)$ to denote the natural logarithm.

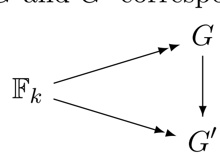
We use both 1 and e to denote the identity in the group, and we use $\mathbf{1}$ to denote the trivial group. Let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ denote the group of integers modulo m, and let $\mathbb{F}_k$ denote the free group on k generators. Throughout the paper, all generating sets will be finite and *symmetric*: $S = S^{-1}$. We use $[x, y] = x^{-1}y^{-1}xy$ to denote the commutator of elements x and y.

2.2. **Definition of marked groups and their homomorphisms.** All groups we will consider will have ordered finite generating sets of the same size k. Whenever we mention a group G, we mean a pair (G, S) where $S = \{s_1, \ldots, s_k\}$ is a ordered generating set of G of size k. Although Cayley graph does not depends on the order of generators, the order is crucial for our results. We call these *marked groups*, and k will always denote the size of the generating set.

By a slight abuse of notation, will often drop S and refer to a *marked group* G, when S is either clear from the context or not relevant. Note that we will also have groups that are not marked; we hope this does not lead to confusion.

Throughout the paper, the homomorphisms between marked groups will send one generating set to the other. Formally, let (G, S) and (G', S') be marked groups, where $S = \{s_1, \ldots, s_k\}$ and $S' = \{s'_1, \ldots, s'_k\}$. Then $\phi : (G, S) \to (G', S')$ is a *marked group homomorphism* if $\phi(s_j) = s'_j$, and this map on generators extends to the (usual) homomorphism between groups: $\phi : G \to G'$.

An equivalent way to think of marked groups is as epimorphisms $\mathbb{F}_k \twoheadrightarrow G$ and $\mathbb{F}_k \twoheadrightarrow G'$. In this picture, maps between marked groups G and G' correspond to commutative diagrams:



This means that for every two marked groups, there is at most one homomorphism $G \to G'$ which is necessary surjective.

2.3. **Products of groups.** The *direct product* of groups G and H is denoted $G \oplus H$, rather than more standard $G \times H$. This notation allows us to write infinite product as $\bigoplus G_i$, where all but finitely many terms are trivial and we will typically omit the index of summation.

We denote by $\prod G_i$ the (usually uncountable) group of sequences of group elements, without any finiteness conditions. Of course, when the index set is infinite, the groups $\bigoplus G_i$ and $\prod G_i$ are not finitely generated.

Finally, let $H \wr G = G \ltimes H^\ell$ denotes the permutation wreath product of the groups, where $G \subseteq \Sigma_\ell$ is a permutation group of ℓ letters.

2.4. **Products of marked groups.** Fix $I \subseteq \mathbb{N}$, and let $\{(G_i, S_i), i \in I\}$ be a sequence of marked groups with generating sets $S_i = \{s_{i1}, \ldots, s_{ik}\}$. Define the $(\Gamma, S) = (\bigotimes G_i, S_i)$ to be the subgroup of $\prod G_i$ generated by diagonally embedding the generating sets of each G_i, i.e., $\bigotimes G_i = \langle s_1, \ldots, s_k \rangle$, where $s_j = \{s_{ij}\} \in \prod G_i$.

Note that Γ comes with canonical epimorphisms $\zeta_i : \Gamma \twoheadrightarrow G_i$. Often the generating sets will be clear from the context and will simply use $\Gamma = \bigotimes G_i$. When the index set contains only 2 elements we denote the product by $G_1 \otimes G_2$.

The product $\bigotimes G_i$ can be defined by universal properties and it is the "smallest" marked group which surjects onto each G_i. Thus, this is equivalent to the categorical product in the category of marked groups.

2.5. **Limits of groups.** We say that the sequence of marked groups $\{(G_i, S_i) : i \in I\}$ *converges in the Chabauty topology*, to a group (G, S) if for any n there exists $m = m(n)$ such that such that for any $i > m$ the ball of radius n in G_i is the same as the ball of radius n in G. We write $\lim_{i\to\infty} G_i = G$. For Cayley graphs $\mathrm{Cay}(G_i, S_i)$ rooted at identity, this is equivalent to the *Benjamini–Schramm graph convergence*, see e.g. [Lov12, §19.1].

Equivalently, this can be stated as follows: if $R_i = \ker(\mathbb{F}_k \twoheadrightarrow G_i)$ and $R = \ker(\mathbb{F}_k \twoheadrightarrow G)$ then

$$\lim_{i\to\infty} R_i \cap B_{\mathbb{F}_k}(n) \, = \, R \cap B_{\mathbb{F}_k}(n).$$

Equivalently, for a fixed n and sufficiently large i the sets $R_i \cap B_{\mathbb{F}_k}(n)$ and $R \cap B_{\mathbb{F}_k}(n)$ must coincide.

Lemma 2.1 ([KP13, Lemma 4.6])**.** *Let $\{G_i\}$ be a sequence of marked groups which converge to a marked group G, and define $\Gamma := \bigotimes G_i$. Then there is an epimorphism $\pi : \Gamma \twoheadrightarrow G$. Moreover, the kernel of π is equal to the intersection $\Gamma \cap \bigoplus G_i$.*

Lemma 2.1 allows us to think of G as the *group at infinity* for Γ.

Example 2.2. We note that many group properties do not survive in the limit. For example, it is easy to construct examples of amenable groups with a nonamenable limit. In fact, classic Margulis's (constant degree) expander constructions are Cayley graphs of $G_p = \mathrm{PSL}(2, \mathbb{Z}_p)$ have girth $\Omega(p)$ and virtually free group limit $\mathrm{PSL}(2, \mathbb{Z})$. See e.g. [HLW06, §11] and [Lub95, §7.3] for more on this and further references.

3. Grigorchuk groups

We now recall some definitions and results on Grigorchuk groups. Again, we stay close to notation and definition in [KP13], and note that these results can be found throughout the literature.

3.1. **Free Grigorchuk group.** The *free Grigorchuk group* $\mathcal{G}$ with presentation

$$\mathcal{G} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle$$

will play a central role throughout the paper, as all our groups and also all Grigorchuk groups are homomorphic images of $\mathcal{G}$. This group is the free product of a group of order 2 and elementary abelian group of order 4, i.e. $\mathcal{G} \simeq \mathbb{Z}_2 * \mathbb{Z}_2^2$. It contains free subgroups and is nonamenable.

3.2. **Family of Grigorchuk groups.** Below we present variations on standard results on the Grigorchuk groups $\mathbf{G}_\omega$. Rather than give standard definitions as a subgroup of $\mathrm{Aut}(\mathrm{T}_2)$, we define $\mathbf{G}$ via its properties. We refer to [Gri05, GP08, dlH00] for a more traditional introduction and most results in this subsection.

Definition 3.1. Let $\varphi : \mathcal{G} \twoheadrightarrow \mathcal{G}$ denote the automorphism of order 3 of the group G which cyclicly permutes the generators b, c and d, i.e.,

$$\varphi(a) = a, \quad \varphi(b) = c, \quad \varphi(c) = d, \quad \varphi(d) = b.$$

Let $\pi : \mathcal{G} \twoheadrightarrow H$ be an epimorphism, i.e., suppose group H comes with generating set consisting of 4 involutions $\{a, b, c, d\}$ which satisfy $bcd = 1$. By $F(H)$ we define the subgroup of $H \wr \mathbb{Z}_2 = \mathbb{Z}_2 \ltimes (H \oplus H)$ generated by the elements A, B, C, D defined as

$$A = (\xi; 1, 1), \quad B = (1; a, b), \quad C = (1; a, c) \quad \text{and} \quad D = (1; 1, d),$$

where $\xi^2 = 1$ is the generator of $\mathbb{Z}_2$. It is easy to verify that A, B, C, D are involutions which satisfy $BCD = 1$, which allows us to define an epimorphism $\widetilde{F}(\pi) : \mathcal{G} \to F(H)$.

The construction can be twisted by the powers automorphism φ

$$\widetilde{F}_x(\pi) := \widetilde{F}(\pi \circ \varphi^{-x}) \circ \varphi^x.$$

An equivalent way of defining the group $F_x(H)$ is as the subgroups generated by

$$\begin{aligned}
A_0 &= (\xi; 1, 1), & B_0 &= (1; a, b), & C_0 &= (1; a, c) & D_0 &= (1; 1, d),\\
A_1 &= (\xi; 1, 1), & B_1 &= (1; a, b), & C_1 &= (1; 1, c) & D_1 &= (1; a, d),\\
A_2 &= (\xi; 1, 1), & B_2 &= (1; 1, b), & C_2 &= (1; a, c) & D_2 &= (1; a, d),
\end{aligned}$$

Here all groups H are marked, i.e., come with an epimorphism $\mathcal{G} \twoheadrightarrow H$. This allows us to slightly simplify the notation as above.

Proposition 3.2. *Each F_x is a functor form the category of homomorphic images of $\mathcal{G}$ to itself, i.e., a group homomorphism $H_1 \to H_2$ which preserves the generators induces, a group homomorphism $F_x(H_1) \to F_x(H_2)$.*

Proposition 3.3. *The functors F_x commutes with the products of marked groups, i.e.,*

$$F_x\left(\bigotimes H_j\right) = \bigotimes F_x(H_j).$$

Proof. This is immediate consequence of the functoriality of F_i and the universal property of the products of marked groups. Equivalently one can check directly from the definitions. □

Definition 3.4. One can define the functor F_ω for any finite word $\omega \in \{0, 1, 2\}^*$ as follows

$$F_{x_1 x_2 \ldots x_i}(H) := F_{x_1}(F_{x_2}(\ldots F_{x_i}(H) \ldots))$$

If ω is an infinite word on the letters $\{0, 1, 2\}$ by F^i_ω we will denote the functor F_{ω_i} where ω_i is the prefix of ω of length i.

In [Gri85], Grigorchuk defined a group $\mathbf{G}_\omega$ for any infinite word ω. One way to define these groups is by $\mathbf{G}_{x\omega} = F_x(\mathbf{G}_\omega)$, where x is any letter in $\{0, 1, 2\}$. The *first Grigorchuk group* is denoted $\mathbf{G} = \mathbf{G}_{(012)^\infty}$, which corresponds to a periodic infinite word, see e.g. [Gri85, Gri05].

3.3. **Contraction in Grigorchuk groups.** Let $\mathbf{G}_{\omega,i} = F^i_\omega(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial group with one element (with the trivial map $\mathcal{G} \twoheadrightarrow \mathbf{1}$).

Proposition 3.5 ([KP13, Prop. 5.9]). *There is a canonical epimorphism $\mathbf{G}_\omega \twoheadrightarrow \mathbf{G}_{\omega,i}$. For every i, the groups $\mathbf{G}_{\omega,i}$ are finite and naturally act on finite binary rooted tree of depth i and this action transitive on the leaves. These actions comes from the standard action of the Grigorchuk group on the infinite binary tree T_2.*

Here the group $F^i_\omega(H)$ is a subgroup of the permutational wreath product $H \wr_{X_i} \mathbf{G}_{\omega,i}$, where X_i is the set of leaves of the binary tree of depth i.

Lemma 3.6 ([KP13, Lemma 5.11]). *Let $\pi : \mathcal{G} \twoheadrightarrow H$ be an epimorphism, i.e., group H is generated by* 4 nontrivial *involutions which satisfy $bcd = 1$. If the word $\omega \in \{0,1,2\}^*$ does not stabilize, then the balls of radius $\le 2^m - 1$ in the groups $F^m_\omega(H)$ and $\mathbf{G}_\omega$ coincide.*

We conclude with an immediate corollary of the Proposition 3.5 and Lemma 3.6, which can also be found in [Gri11].

Corollary 3.7. *Let $\{\mathcal{G} \twoheadrightarrow H_i\}$ be any sequence of groups generated by $k = 4$ nontrivial involutions. Then the sequence of marked groups converges: $\lim_{i\to\infty} F^i_\omega(H_i) = \mathbf{G}_\omega$.*

4. Main construction

4.1. **A nonamenable group.** Our main construction uses that the free Grigorchuk group $\mathcal{G}$ is close to a free group and thus has many nonamenable quotients which are very different from the groups $\mathbf{G}_\omega$. One such quotient is generated by following matrices in $\mathrm{PSL}\big(2, \mathbb{Z}[\boldsymbol{i}, 1/2]\big)$, where $\boldsymbol{i}^2 = -1$,

$$a = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{i}/4 \\ 0 & -\boldsymbol{i} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} \boldsymbol{i} & 0 \\ 0 & -\boldsymbol{i} \end{pmatrix}.$$

A direct computation shows that $a^2 = b^2 = c^2 = d^2 = bcd = 1$, i.e., there is a (non-surjective) homomorphism $\iota : \mathcal{G} \to \mathrm{PSL}\big(2, \mathbb{Z}[\boldsymbol{i}, 1/2]\big)$.

Let $\mathcal{H} := \langle a, b, c, d\rangle$ denote the marked group generated by the above matrices; as always we consider it as marked group. The group $\mathcal{G}$ contains a normal subgroup of index (at most) 2 generated by $\{c, ad\}$,[2] which yields a subgroup $N := \langle c, ad\rangle$ of index 2 of $\mathcal{H}$. The generators of this subgroup are

$$c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad ad = \begin{pmatrix} 1 & -1/4 \\ 0 & 1 \end{pmatrix}.$$

It can be verified that $\langle c, ad\rangle = \mathrm{PSL}\big(2, \mathbb{Z}[1/2]\big)$, and thus we have $N = \mathrm{PSL}\big(2, \mathbb{Z}[1/2]\big)$. Another computations shows that

$$(ad)^4 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad h := [c, [d, [b, (ad)^4]]] = \begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}.$$

Lemma 4.1. *The element h normally generates the group N.*

Proof. Let K be the normal subgroup of N generated by h. Since N can be viewed as a lattice in $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{Q}_2)$, by Margulis's normal subgroup theorem, we have that subgroup K is either central or of finite index, see e.g. [Mar91, Ch. IV]. Moreover N has congruence subgroup property and h is not contained in any properly congruence subgroup. □

[2]The group $\mathcal{G}$ is 4 generated and the standard Schreier algorithm gives that any subgroup of index 2 is generated by 7 elements - in these case several of these generators are trivial, and other are redundant.

4.2. **Technical lemma.** Let r denote the element $[c,[d,[b,(ad)^4]]] \in \mathcal{G}$, so we have $\iota(r) = h$. Let $r_x = \varphi^x(r)$ be its twists by the automorphism φ described in Definition 3.1. The following lemma is a variation on our [KP13, Lemma 5.16], adjusted to this setting.

Lemma 4.2. *Let $\iota : \mathcal{G} \twoheadrightarrow \mathcal{H}$, and let $N \triangleleft \mathcal{H}$ be a normal subgroup generated by element $r_{x_{k+1}}$, defined as above. Then the kernel of the map $F^k_\omega(\mathcal{H}) \twoheadrightarrow \mathbf{G}_{\omega,k}$ induced by F^k_ω from the trivial homomorphism $\mathcal{H} \twoheadrightarrow \mathbf{1}$, contains $N^{\oplus 2^k}$. Moreover, there exists a word $\eta_{\omega,k} \in F$ of length less than $\leq C \cdot 2^k$, for some universal constant C, such that such the image of $\eta_{\omega,k}$ in $F^k_\omega(\mathcal{H})$ normally generates $N^{\oplus 2^k}$ and $\eta_{\omega,k}$ is trivial in $F^{k+1}_\omega(H)$, for every $\mathcal{G} \twoheadrightarrow H$.*

Proof. Consider the substitutions σ, τ (endomorphisms $\mathcal{G} \to \mathcal{G}$), defined as follows:

- $\sigma(a) = aca$ and $\sigma(s) = s$, for $s \in \{b, c, d\}$,
- $\tau(a) = c$, $\tau(b) = \tau(c) = a$ and $\tau(d) = 1$.

It is easy to see that for any $\eta \in \mathcal{G}$, the evaluation of $\sigma(\eta)$ in $F(\mathcal{G})$ is equal to

$$\big(1; \tau(\eta), \eta\big) \in \{1\} \times \mathcal{G} \times \mathcal{G} \subset \mathcal{G} \wr \mathbb{Z}_2 .$$

Define $w_i \in \mathcal{G}$ for $i = 0, \ldots, k$ as follows: $w_0 = r_{x_{k+1}}$ and $w_{i+1} = \sigma_{x_{k-i}}(w_i)$ where $\sigma_{x_i} = \varphi^{x_i}\sigma\varphi^{-x_i}$ the the twist of the substitution σ. Notice that all these words have the form $[c,[d,[b,*]]]$ because σ_{x_i} fixes b, c and d. Therefore $\tau_x(w_i) = 1$

By construction the word $\eta_{\omega,k} = w_k$ evaluates in $F^k_\omega(\mathcal{G})$ to $r_{x_{k+1}}$ in one of the copies of $\mathcal{G}$, viewed as a subgroup of $F^k_\omega(\mathcal{G}) \subset \mathcal{G} \wr_{X_k} \mathbf{G}_{\omega,k}$. Therefore the evaluation of w_k in $F^k_\omega(\mathcal{H})$ is in the kernel of $F^k_\omega(\mathcal{H}) \to \mathbf{G}_{\omega,k}$ is the elements h in one of the copies of $\mathcal{H}$. This together with the transitivity of the action of $\mathbf{G}_{\omega,k}$ on X_k shows that the kernel contains N^{2^k}. Finally the η is trivial in $F^k_\omega(\mathcal{G})$ since the $(ad)^4 = 1$ in $F(\mathcal{G})$ for any x. □

4.3. **Main construction.** Let $\mathcal{H}$ be the marked group described in §4.1. Denote G_i the marked group $F^i_{(012)^\infty}(\mathcal{H})$, which surjects onto $F^i_{(012)^\infty}(\mathbf{1}) = \mathbf{G}_{(012)^\infty,i}$ Using Corollary 3.7 we can see that G_i converge in the Chabauty topology to $\mathbf{G}_\omega$.

Definition 4.3. Let $J \subseteq 2^{\mathbb{N}}$ be a fixed subset of $\mathbb{N}$. Denote

$$\widetilde{G}_J := \bigotimes_{i \in J} G_i \quad \text{and} \quad G_J := G_J \otimes \mathbf{G}_\omega .$$

By construction these are 4-marked groups that are quotients of $\mathcal{G}$, and s.t. $G_\varnothing = \mathbf{G}_\omega$.

Using the definition on G_J one can see that the group G_J can be defined as $\bigotimes_{i\in\mathbb{N}} \Gamma_{i,J}$ where

$$\Gamma_{i,J} = \begin{cases} G_i = F^i_{(012)^\infty}(\mathcal{H}) & \text{if } i \in J, \\ \mathbf{G}_{(012)^\infty,i} = F^i_{(012)^\infty}(\mathbf{1}) & \text{if } i \notin J. \end{cases}$$

Lemma 4.4. *The map $I \to G_I$ defines a continuous map from $2^{\mathbb{N}}$ to the space $\{G_J\}$ of marked groups.*

Proof. This follows from the convergence $G_i \to \mathbf{G}_\omega$ in Chabauty topology and the observation that the ball of radius n in $\bigotimes \Gamma_i$ depend only on the balls of radius n in Γ_i for each i – for any n, there exists N such that the ball of radius n in G_k, $\mathbf{G}_{(012)^\infty,k}$ and $\mathbf{G}_{(012)^\infty}$ coincide for $\kappa \geq N$. This implies that the ball of radius n in G_J coincide with the one in $\bigotimes_{i\in\mathbb{N}} \tilde{\Gamma}_{i,J}$ where

$$\tilde{\Gamma}_{i,J} = \begin{cases} G_i = F^i_{(012)^\infty}(\mathcal{H}) & \text{if } i \in J, i \leq N \\ \mathbf{G}_{(012)^\infty,i} = F^i_{(012)^\infty}(\mathbf{1}) & \text{if } i \notin J, i \leq N \\ \mathbf{G}_{(012)^\infty} & \text{if } i > N. \end{cases}$$

Therefore the ball of radius n in G_J depends only on $J \cap \{1, \ldots, N\}$, which implies that the function $J \to G_J$ is contunuous. □

For any set $J \subset J'$ there is a surjection $G_{J'} \twoheadrightarrow G_J$. The main step in proving Lemma 1.11 is show that if J is a proper subset of J' then this map has a large kernel.

Lemma 4.5. *The kernel of the map* $G_J \to \mathbf{G}_\omega$ *is contains to*

$$\bigoplus_{i\in J} N_i^{2^i} \subset \bigoplus_{i\in J} H_i^{2^i} \subset \bigoplus_{i\in I} G_J$$

Proof. We will use indiction on k to show that

$$\sum_{j\in J, j\leq k} N^{2^k} \subset G_J \cap \bigoplus_{i\leq k} \Gamma_{i,J}$$

The base case of the induction $k = 0$ is trivial. Using Lemma 4.2 we can see that element $\eta_{\omega,k}$ is trivial in G_i for $i > k$, therefore it corresponds to an element in $G_J \cap \bigoplus_{i\leq k} \Gamma_{i,J}$. If $k \in J$ this elements evaluates to a normal generators in of N^{2^k} inside $G_k = \Gamma_{k,J}$ and for $i \leq k$ and $i \in J$ it to some element in N^{2^i} inside $G_i = \Gamma_{i,J}$, which finishes the induction step. □

Proof of Lemma 1.11. The continuity is equivalent to Lemma 4.4. The existence of the surjection $G_{J'} \twoheadrightarrow G_J$ for $J \subset J'$ follows from the construction of the groups G_J. The strict monotonicity follows from Lemma 4.5 which gives that the kernel of $G_{J'} \twoheadrightarrow G_J$ contains N^{2^i} for any $i \in J' \setminus J$. □

4.4. **Proof of Lemma 1.10.** The first step is to show that for any decreasing parameter f, the mapping $J \to f(G_J)$ is a upper semi-continuous function $2^{\mathbb{N}} \to [0,1]$.

Lemma 4.6. *Suppose that* I_n *is decreasing sequence of subsets of* $\mathbb{N}$ *which converges to* $I = \cap I_n$ *in the Tychonoff topology. In other words, suppose for every* $n \in \mathbb{N}$, *there exists* $k = k(n)$ *s.t.* $I_m \cap [n] = I \cap [n]$ *for all* $m > k(n)$. *Then for any decreasing parameter* f *we have*

$$f(G_I) = \lim_{n\to\infty} f\big(G_{I_n}\big).$$

Proof. The mapping $I \to G_I$ defines a function from $2^{\mathbb{N}}$ to the space of marked groups, which is continuous with respect to the Tychnoff topology on $2^{\mathbb{N}}$ and the Chabauty topology on the space of marked groups. Therefore, the balls in the Cayley graphs of the groups G_{I_n} converge the ball of G_I, which implies that $f(G_I) \leq \lim_{n\to\infty} f(G_{I_n})$ by property (∗∗).

On other hand, we have that G_I is a quotient of G_{I_n} for each n. Therefore, $f(G_I) \geq f(G_{I_n})$ by property (∗). Passing to the limit gives $f(G_I) \geq \lim_{n\to\infty} f(G_{I_n})$. This completes the proof. □

One way to prove that the set of all possible values of f is large is to show that $f(G_I)$ is a strictly decreasing function with respect to the lex order of $2^{\mathbb{N}}$. Unfortunately this is not the case in general, however this become true if we restrict to sufficiently sparse subsets of $\mathbb{N}$:

Lemma 4.7. *Let* f *be a strictly decreasing parameter, then there exists a function* $\mu_f : \mathbb{N} \to \mathbb{N}$ *such that the following holds. For every infinite set* $M = \{m_1, m_2, \ldots, m_n, \ldots\}$ *with* $m_{n+1} > \mu_f(m_n)$, *the function* $J \to f(G_J)$ *is a strictly decreasing function from* $2^M \to \mathbb{R}$ *with respect to the lex total order on* 2^M.

Proof. Let n be an integer and let K be any subset of $[n-1] := \{1, \ldots, n-1\}$. Denote $K' = K \cup \{n\}$. Since $K \subsetneq K'$, by Lemma 1.11 the kernel of $G_{K'} \twoheadrightarrow G_K$ is nonamenable and the strict monotonicity of f implies that $f(G_K) > f(G_{K'})$. Let K_m denote the set $K_m = K \cup \{m, m+1, \ldots, \}$ for $m > n$. Clearly we have that $\cap_m K_m = K$ and the sets K_m converge to K from above. The semi-continuity of f (Lemma 4.6) implies that

$$f(G_K) = \lim_{m\to\infty} f(G_{K_m}).$$

Therefore, there exist $\Lambda_f(n, K) \in \mathbb{N}$ such that $f(G_{K'}) < f(G_{K_m})$ for all $m > \Lambda_f(n, K)$. Denote $\mu_f(n) := \max_K\{\Lambda_f(n, K)\} \in \mathbb{N}$, where the maximum is taken over all subsets of $[n-1]$.

Let $M = \{m_1, m_2, \dots, m_n, \dots\} \subset \mathbb{N}$ be an infinite set of integers, such that $m_{n+1} > \mu_f(m_n)$ for all n. Let J', J'' be subsets of J such that $J' < J''$ in the lex order on 2^M. By the definition of the lex order there exists k such that $J' \cap [m_k - 1] = J'' \cap [m_k - 1]$ and $m_k \in J''$ but $m_k \notin J'$. Using the notation from the previous paragraph with $n = m_k$ and $K = J' \cap [m_k - 1]$, we have $K' \subset J''$ and $J' \subset K_{j_{n+1}}$. By the choice of the function μ and the sequence m_k, we have that

$$f(G_{J'}) \geq f(G_{K_{M(n,K)}}) > f(G_{K'}) \geq f(G_{J''}),$$

where the first and the last inequality follow from the inclusions $J' \subset K_{M(n,K)}$ and $K' \subset J''$, and the the second inequality follows from the definition of the constant $\Lambda_f(n, K)$. This verifies that f is a strictly decreasing function on 2^M, as desired. □

Proof of Lemma 1.10. Let M be a sparse set satisfying the conditions in the previous theorem, so the set 2^M has cardinality of a continuum. By Lemma 4.7, the function $J \to f(G_J)$ is strictly decreasing function with respect to the total lex order on 2^M. Therefore the set of all possible values of f on the groups G_J has cardinality of the continuum. □

5. Isolated points

We now return to the set of spectral radii and asymptotic entropy discussed in Theorem 1.2.

Lemma 5.1. *Let $k \geq 2$, and let Γ be a k-generated marked group which is not free as a marked group. Then there exists a sequence H_i of nonamenable k-generated marked groups such that H_i converge to an amenable marked group H of subexponential growth and for any i the projection*

$$\Gamma \otimes H_i \twoheadrightarrow \Gamma$$

has nonamenable kernel.

Remark 5.2. The condition that Γ is not free is necessary since if $\Gamma = \mathbb{F}_k$, then for every k-generated group H we have the map $\mathbb{F}_k \otimes H \twoheadrightarrow \mathbb{F}_k$ is an isomorphism.

Remark 5.3. For $k = 4$, if we drop the last condition in the lemma, one can simply take the group $G_i = F^i_{(012)^\infty}(\mathcal{H})$ constructed in Section 4. However, since all these groups are quotients of $\mathcal{G}$, we note that the last condition fails for $\Gamma = \mathcal{G}$.

As an immediate corollary of this lemma we obtain the following result.

Theorem 5.4. *Let f be a sharply decreasing parameter. Then, for any marked group (Γ, S) which is not free, there exists a sequence of marked groups $\{(\Gamma_i, S_i)\}$, s.t. $f(\Gamma_i, S_i) < f(\Gamma, S)$ and $\lim_{i\to\infty} f(\Gamma_i, S_i) = f(\Gamma, S)$. Therefore, the set $X_{f,k}$ of values of f on all marked groups with a generating set of size k has no isolated points, except possibly at $f(\mathbb{F}_k)$.*

Proof. This is an immediate consequence of Lemma 5.1, with $\Gamma_i = \Gamma \otimes H_i$ – the inequality $f(\Gamma_i, S_i) < f(\Gamma, S)$ follow from the strict property of f and the fact that the kernel of $\Gamma \otimes H_i \twoheadrightarrow \Gamma$ is nonamenable. The property $H_i \to H$ in the Chabauty topology implies that $\Gamma \otimes H_i \to \Gamma \otimes H$ in the in the Chabauty topology. Therefore, $\limsup_{i\to\infty} f(\Gamma \otimes H_i) \geq f(\Gamma \otimes H)$. However the sharpness of f implies that $f(\Gamma \otimes H) = f(\Gamma)$ since the kernel of $\Gamma \otimes H \twoheadrightarrow H$ is a subgroup of H, and therefore it is amenable. □

Proof of Lemma 5.1. We start with the following group theoretic result.

Claim 5.5. *For any nontrivial normal subgroup $\Delta \lhd \mathbb{F}_k$, there exist a finite k-generated marked group K such that the kernel $R = \ker(\mathbb{F}_k \twoheadrightarrow K)$ admits a surjective homomorphism $\psi : R \twoheadrightarrow \mathcal{G}$ where $\psi(\Delta \cap R)$ contains the generator a of $\mathcal{G}$.*

Proof of the Claim. Since Δ is nontrivial there exists a nontrivial $r \in \Delta \cap [\mathbb{F}_k, \mathbb{F}_k]$. Consider the 2-Frattini series for $\mathbb{F}_k$ defined by $T_0 = \mathbb{F}_k$ and $T_i = T_i^2[T_i, T_i]$. Each T_i is a finite index in $\mathbb{F}_k$ and $\bigcap T_i = 1$ since the free group is a residually 2-group. Therefore there exists an index i such that $r \in T_i \setminus T_{i+1}$ and by the choice of r we have that $i \geq 1$. We can take $K = \mathbb{F}_k/T_i$, the kernel R is T_i and is a free group of rank $(k-1)|K| + 1 > 3$. Since r is outside the 2-Frattini subgroup of R, there exists a homomorphism $R \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ which maps r to one of the generators. Finally ψ is the composition with the projection $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \twoheadrightarrow \mathcal{G}$. □

Let ψ be the homomorphism from the Claim applied to $\Delta = \ker(\mathbb{F}_k \twoheadrightarrow \Gamma)$. Denote by ψ_i the composition of ψ and the projection $\mathcal{G} \twoheadrightarrow F^i_{(012)^\infty}(\mathcal{H})$ and ψ_∞ the composition of ψ and the projection $\mathcal{G} \twoheadrightarrow \mathbf{G}$. Let L_i be in intersection of all conjugates of $\ker \psi_i$ in F_k (there are exactly $|K|$ of them since $\ker \phi_i$ is a normal subgroup of R, similarly define L_∞. By construction we have that $H_i = F_k/L_i$ fits into exact sequence

$$1 \to M_i \to H_i \to K \to 1$$

where M_i is naturally a subgroup of $F^i_{(012)^\infty}(\mathcal{H})^{\oplus |K|}$, and a similar statement for H_∞.

By construction there is a relation in Γ which maps to a generator of $\mathcal{G}$ therefore there exists a relation in Γ which maps to the word $\eta_{(012)^\infty, i}$ constructed in the proof of Lemma 4.2. This implies that the kernel of the map

$$\Gamma \otimes H_i \twoheadrightarrow \Gamma$$

contains N^{2^i} and is nonamenable.

The convergence $\mathcal{G} \twoheadrightarrow F^i_{(012)^\infty}(\mathcal{H}) \to \mathbf{G}$ in the Chabauty topology implies that H_i converge to H_∞ which is an amenable group. This completes the proof of the lemma. □

Proof of Theorem 1.2. The theorem follows immediately from Theorem 5.4 since the spectral radius is sharply decreasing by Lemma 1.4. □

6. Other monotone parameters

6.1. **Cheeger constant.** Let $|S| = k$ and define the *Cheeger constant* (also called *isoperimetric constant*):

$$\phi(G, S) := \inf_{X \subset G} \frac{|E(X, G \smallsetminus X)|}{k|X|},$$

where the infimum is over all nonempty finite X, and $E(X,Y)$ is the set of edges (x,y) in $\operatorname{Cay}(G,S)$ such that $x \in X$, $y \in Y$. Note that $0 \leq \phi(G,S) \leq 1$.

The following celebrated inequality relates the spectral radius and the Cheeger constant:

$$\tfrac{1}{2}\,\phi(G,S)^2 \;\leq\; 1 - \rho(G,S) \;\leq\; \phi(G,S).$$

This inequality was discovered independently by a number of authors in different contexts, see e.g. [Pete23, §7.2]. In particular, we have $\phi(G,S) > 0$ if and only if G is nonamenable.

Similar to other probabilistic parameters, computing the Cheeger constant is very difficult and the exact values are known only in a few special cases. For example, $\phi(\mathbb{F}_k, S) = \frac{k-1}{k}$ for a free group with standard generators. Additionally, the Cheeger constant is computed for nonamenable hyperbolic tessellations where it is always algebraic [HJL02] (see also [LP16, §5.3]).

Proposition 6.1. *Cheeger constant* $\phi : \{(G,S)\} \to \mathbb{R}_{\geq 0}$ *is an increasing parameter.*

The proof of this result is straightforward. By analogy with Lemma 1.4, one can ask if ϕ is strictly increasing? Perhaps even sharply increasing? Unfortunately, this remains open:

Conjecture 6.2. *Let $G = \langle S \rangle$, let N be a finitely generated normal subgroup of G, and let S' be the projection of S onto G/N. Then $\phi(G,S) < \phi(G/N, S')$ if and only if N is nonamenable.*

By analogy with the proof of our Theorem 1.1, the *if* part of the conjecture gives:

Proposition 6.3. *Conjecture 6.2 implies that there is a family of Cayley graphs* $\{(G_\omega, S_\omega)\}$, *such that the set of values* $\{\phi(G_\omega, S_\omega)\}$ *has cardinality of the continuum.*

The conclusion of the proposition remains an open problem in the area. The analogue of Theorem 1.2 for $X_{\phi,k}$ follow from the conjecture. We omit the details.

6.2. **Speed of simple random walks.** As in the introduction, let $\{x_n\}$ denote simple random walk on $\Gamma = \mathrm{Cay}(G, S)$ starting at $x_0 = e$. For all $z \in G$, denote by $|z|$ the distance from e to z in Γ. The *speed* (also called *drift* and *rate of escape*) of $\{x_n\}$ is defined as follows:

$$\sigma(G,S) \,:=\, \lim_{n\to\infty} \frac{\mathbb{E}[|x_n|]}{n}\,,$$

see e.g. [Pete23, §9.1] and [Woe00, §II.8]. Note that the limit is well defined and the same for almost all sample paths. Famously, it was shows in [KL07] that $\sigma(G, S) > 0$ if and only if $h(G, S) > 0$. We also have:

Proposition 6.4. *Spectral radius* $\phi : \{(G, S)\} \to \mathbb{R}_{\geq 0}$ *is an increasing parameter.*

The proof of this result is straightforward. Unfortunately, the following natural analogue of Lemma 1.8 remains open:

Conjecture 6.5. *Let* $G = \langle S\rangle$ *and let* N *be a finitely generated normal subgroup of* G*. Suppose* N *is nonamenable, and let* S' *be the projection of* S *onto* G/N*. Then* $\sigma(G, S) < \sigma(G/N, S')$*.*

By analogy with the proof of our Theorem 1.1 and Proposition 6.6 above, we have:

Proposition 6.6. *Conjecture 6.5 implies that there is a family of Cayley graphs* $\{(G_\omega, S_\omega)\}$, *such that the set of values* $\{\sigma(G_\omega, S_\omega)\}$ *has cardinality of the continuum.*

6.3. **Rate of exponential growth.** Let $b(n) := |\{z \in G : |z| \leq n\}|$ denote the number of elements at distance at most n, and define the *rate of exponential growth* as follows:

$$\gamma(G,S) \,:=\, \frac{1}{|S|} \lim_{n\to\infty} \frac{\log b(n)}{n}\,.$$

Clearly, $\gamma(G, S) \in [0, 1)$, and $\gamma(G, S) > 0$ is independent of the generating set S. See e.g. [Gri05, GP08, dlH00] for accessible introductions.

Proposition 6.7. *Rate of exponential growth* $\gamma : \{(G, S)\} \to \mathbb{R}_{\geq 0}$ *is an increasing parameter.*

The proof of this result is straightforward. Clearly, the rate of exponential growth is not strictly increasing. In fact, the notion of strictly increasing parameters for the rate of exponential growth is closely related to the notion of *growth tightness* introduced in [GH97, §2] and established in [AL02] for word hyperbolic groups. In [Ers04], Erschler was able to modify the notion of "strict monotonicity" to obtain the following natural analogue of Theorem 1.1 for the rate of exponential growth:

Theorem 6.8 ([Ers04])**.** *The set of rates of exponential growth* $\{\gamma(G, S)\}$ *has cardinality of the continuum.*

It is worth comparing this question with Grigorchuk's celebrated result in [Gri85], that there is a family of marked groups with incomparable growth functions and with the cardinality of the continuum. The proof of Theorem 6.8 is also somewhat related to Bowditch's elementary construction in [Bow98], of a family of marked groups with pairwise non-quasi-isometric Cayley graphs and the cardinality of the continuum. See also the most recent result by Louvaris, Wise and Yehuda [LWY24], which proves density on growth rates of subgroups of free groups.

7. Final remarks and open problems

7.1. **Historical notes.** The results of this paper (for the spectral radius) were announced over seven years ago.[3] We are happy that we could extend them to other monotone parameters, and create an alternative to the small cancellation theory approach (see below). In the meantime, our construction influenced [TZ19] mentioned in the introduction. While this creates a messy timeline, we would like to acknowledge that the asymptotic entropy version of Theorem 1.1 can be attributed to Tamuz and Zheng at least as much as to this work.

7.2. **Set theoretic considerations.** As in the introduction, let $X_{\rho,k} \subset (0,1]$ denote the set of spectral radii of marked groups with k generators, and let $X_\rho := \cup X_{\rho,k}$.

Open Problem 7.1. Prove that $X_\rho = (0,1]$.

Between ourselves, we disagree whether one should believe or disbelieve this claim. While our results seem to suggest a positive answer, they give no intuition whether X_ρ is closed or dense.

Now, Main Theorem 1.1 shows that $X_{\rho,4}$ has cardinality of the continuum. Our proof implies a stronger result, that $X_{\rho,4}$ has an embedding of the *Cantor set* (see Lemma 1.11). On the surface, this appears a stronger claim since there is a natural construction of the *Bernstein set* which has cardinality of the continuum and no embedding of the Cantor set, see e.g. [Kec95, Ex. 8.24].

Looking closer at our results, we prove in Theorem 1.2 that $X_{\rho,k} \smallsetminus \{\alpha_k\}$ has no isolated points in the interval $(\alpha_k, 1]$. If $X_{\rho,k}$ is closed, then it is a perfect Polish space that always contains the Cantor set, see e.g. [Kec95, Thm 6.2]. In our case, it is easy to see that $X_{\rho,k}$ is a projection of a Borel set, and thus *analytic*, see e.g. [Kec95, §14.A]. Recall that every analytic set that is a subset of a Polish space either is countable, or contains a Cantor set, see e.g. [Kec95, Ex. 14.13].

In other words, very general set theoretic arguments imply that set $X_{\rho,k}$ is either countable or contains a Cantor set, and thus has the cardinality of the continuum. This argument applies to other monotone parameters in Theorem 1.1, suggesting that containment of the Cantor set is unsurprising.

7.3. **Explicit constructions.** The proof of Lemma 1.11 is fundamentally set theoretic and does not allow an *explicit construction* of the Cayley graph with a transcendental spectral radius: $\rho(G,S) \notin \overline{\mathbb{Q}}$. Here we are intentionally vague about the notion of "explicit construction", as opposed to the setting in graph theory where it is well defined, see e.g. [HLW06, §2.1] and [Wig19, §9.2]. This leads to a host of interrelated open problems corresponding to different possible meanings in our context.

Question 7.2. *Is there a finitely presented group with a transcendental spectral radius? What about recursively presented groups? Similarly, what about graph automata groups?*

Despite our efforts, we are unable to resolve either of these questions using our tools.[4] Note that there is a closely related but weaker notion of *D-transcendental cogrowth series*, see e.g. [GH97, GP17]. We refer to [Pak18] for some context about problems of counting walks in graphs and further references.

7.4. **Asymptotic versions.** When the group is amenable, one can ask about the asymptotics of the return probability and the (closely related) *isoperimetric profile*. Similarly, when the Cayley graph has no non-constant bounded harmonic functions, one can ask about the asymptotics of the speed and entropy functions. In these setting, there is a large literature on both the exact and oscillating growth of these functions, too large to be reviewed here. We refer to [BZ21] for the recent breakthrough and many references therein.

[3] Martin Kassabov, *A nice trick involving amenable groups*, MSRI talk (Dec. 9, 2016), video and slides available at www.slmath.org/workshops/770/schedules/21638

[4] Naturally, we only tried a positive direction. Unfortunately, it took us several years to accept the defeat.

For the critical percolation, one can ask about asymptotics of the number of cuts, see e.g. [Tim07]. We note that this asymptotic version is always exponential for vertex-transitive graphs, and is largely of interest for families of graphs with nearly linear growth.

7.5. **Monotone properties.** The notion of *monotone properties* are modeled after standard notions of monotone and hereditary properties in probabilistic combinatorics, which describe set systems closed under taking subsets. Typical examples include properties of graphs that are invariant under deletion of edges or vertices, see e.g. [AS16, §6.3, §17.4]. Similarly, *parameters* are standard in graph theory, and describe any of the numerous quantitative graph functions, see e.g. [Bon95, Lov12].

7.6. **Small cancellation groups.** There is an alternative approach to monotone parameters coming from the small cancellation theory. Notable highlights include Bowditch's work [Bow98] mentioned in §6.3, and Erschler's Theorem 6.8.

After the results of this paper were obtained, Erschler showed us how to prove both Lemma 1.11 and Theorem 1.1 using the construction from [Ers04] combined with strictly monotone properties.[5] Moreover, Osin suggested that this approach could also be used to have groups satisfy additional properties, such as acylindrically hyperbolic, lacunary hyperbolic, and property (T).[6] It would be interesting to see how much further this construction can be developed. We note, however, that our approach is nearly self-contained (modulo some lemmas proved in [KP13]). Note also that Theorem 1.2 seems not attainable via small cancellation groups.

Acknowledgements. We are very grateful to Jason Bell, Nikita Gladkov, Slava Grigorchuk, Tom Hutchcroft, Vadim Kaimanovich, Anders Karlsson, Andrew Marks, Justin Moore, Tatiana Nagnibeda, Omer Tamuz, Romain Tessera and Tianyi Zheng, for many interesting remarks and help with the references. Special thanks to Anna Erschler and Denis Osin for telling us about the small cancelation group approach. Both authors we partially supported by the NSF.

References

[AS16] Noga Alon and Joel H. Spencer, *The probabilistic method* (Fourth ed.), John Wiley, Hoboken, NJ, 2016, 375 pp.

[AL02] Goulnara N. Arzhantseva and Igor G. Lysenok, Growth tightness for word hyperbolic groups, *Math. Z.* **241** (2002), 597–611.

[BLM23] Jason Bell, Haggai Liu and Marni Mishna, Cogrowth series for free products of finite groups, *Internat. J. Algebra Comput.* **33** (2023), 237–260.

[BS96] Itai Benjamini and Oded Schramm, Percolation beyond $\mathbb{Z}^d$, many questions and a few answers, *Electron. Comm. Probab.* **1** (1996), no. 8, 71–82.

[Bis92] Dietmar Bisch, Entropy of groups and subfactors, *J. Funct. Anal.* **103** (1992), 190–208.

[BR06] Béla Bollobás and Oliver Riordan, *Percolation*, Cambridge Univ. Press, New York, 2006, 323 pp.

[Bon95] John A. Bondy, Basic graph theory: paths and circuits, in *Handbook of combinatorics*, Vol. 1, Elsevier, Amsterdam, 1995, 3–110.

[Bow98] Brian H. Bowditch, Continuously many quasi-isometry classes of 2-generator groups, *Comment. Math. Helv.* **73** (1998), 232–236.

[BZ21] Jérémie Brieussel and Tianyi Zheng, Speed of random walks, isoperimetry and compression of finitely generated groups, *Annals of Math.* **193** (2021), 1–105.

[CD21] Tullio Ceccherini-Silberstein and Michele D'Adderio, *Topics in groups and geometry – growth, amenability, and random walks*, Springer, Cham, 2021, 464 pp.

[Dum18] Hugo Duminil-Copin, Sixty years of percolation, in *Proc. ICM*, Vol. IV, World Sci., Hackensack, NJ, 2018, 2829–2856.

[D+20] Hugo Duminil-Copin, Subhajit Goswami, Aran Raoufi, Franco Severo and Ariel Yadin, Existence of phase transition for percolation using the Gaussian free field, *Duke Math. Jour.* **169** (2020), 3539–3563.

[5] Anna Erschler, personal communication.

[6] Denis Osin, personal communication.

[E+14] Murray Elder, Andrew Rechnitzer, Esaias J. Janse van Rensburg and Thomas Wong, The cogrowth series for BS(N, N) is D-finite, *Internat. J. Algebra Comput.* **24** (2014), 171–187.

[Ers04] Anna Erschler, Growth rates of small cancellation groups, in *Random walks and geometry*, Walter de Gruyter, Berlin, 2004, 421–430.

[GP17] Scott Garrabrant and Igor Pak, Words in linear groups, random walks, automata and P-recursiveness, *J. Comb. Algebra* **1** (2017), 127–144.

[GZ24] Nikita Gladkov and Aleksandr Zimin, Bond percolation does not simulate site percolation, preprint (2024), 9 pp.; available at tinyurl.com/3p7tmhuw

[Gri85] Rostislav I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, *Math. USSR-Izv.* **25** (1985), 259–300.

[Gri05] Rostislav Grigorchuk, Solved and unsolved problems around one group, in *Infinite groups: geometric, combinatorial and dynamical aspects*, Birkhäuser, Basel, 2005, 117–218.

[Gri11] Rostislav I. Grigorchuk, Some topics of dynamics of group actions on rooted trees (in Russian), *Proc. Steklov Inst. Math.* **273** (2011), 1–118.

[GH97] Rostislav Grigorchuk and P. de la Harpe, On problems related to growth, entropy, and spectrum in group theory, *J. Dynam. Control Systems* **3** (1997), 51–89.

[GP08] Rostislav Grigorchuk and Igor Pak, Groups of intermediate growth: an introduction, *Enseign. Math.* **54** (2008), 251–272.

[Gri99] Geoffrey Grimmett, *Percolation* (second ed.), Springer, Berlin, 1999, 444 pp.

[GS98] Geoffrey R. Grimmett and Alan M. Stacey, Critical probabilities for the site and bond percolation models, *Ann. Probab.* **26** (1998), 1788–1812.

[HJL02] Olle Häggström, Johan Jonasson and Russell Lyons, Explicit isoperimetric constants and phase transitions in the random-cluster model, *Ann. Probab.* **30** (2002), 443–473.

[dlH00] Pierre de la Harpe, *Topics in geometric group theory*, Univ. Chicago Press, Chicago, IL, 2000, 310 pp.

[HLW06] Shlomo Hoory, Nathan Linial and Avi Wigderson, Expander graphs and their applications, *Bull. AMS* **43** (2006), 439–561.

[HT21] Tom Hutchcroft and Matthew Tointon, Non-triviality of the phase transition for percolation on finite transitive graphs, preprint (2021), 62 pp.; to appear in *Jour. Eur. Math. Soc.*; `arXiv:2104. 05607`.

[Kai02] Vadim A. Kaimanovich, The Poisson boundary of amenable extensions, *Monatsh. Math.* **136** (2002), 9–15.

[KV83] Vadim A. Kaimanovich and Anatoly M. Vershik, Random walks on discrete groups: boundary and entropy, *Ann. Probab.* **11** (1983), 457–490.

[KN06] Martin Kassabov and Nikolay Nikolov, Cartesian products as profinite completions, *Int. Math. Res. Not.* **2006**, Art. ID 72947, 17 pp.

[KP13] Martin Kassabov and Igor Pak, Groups of oscillating intermediate growth, *Annals of Math.* **177** (2013), 1113–1145.

[KL07] Anders Karlsson and François Ledrappier, Linear drift and Poisson boundary for random walks, *Pure Appl. Math. Q.* **3** (2007), 1027–1036.

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Springer, New York, 1995, 402 pp.

[Kes59] Harry Kesten, Symmetric random walks on groups, *Trans. AMS* **92** (1959), 336–354.

[Koz08] Iva Kozáková, Critical percolation of free product of groups, *Internat. J. Algebra Comput.* **18** (2008), 683–704.

[Kuk99] Dmitri Kuksov, Cogrowth series of free products of finite and free groups, *Glasgow Math. J.* **41** (1999), 19–31.

[Lov12] László Lovász, *Large networks and graph limits*, AMS, Providence, RI, 2012, 475 pp.

[LWY24] Michail Louvaris, Daniel T. Wise and Gal Yehuda, Density of growth-rates of subgroups of a free group and the non-backtracking spectrum of the configuration model, preptint (2024), 37 pp.

[Lub95] Alexander Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Birkhäuser, Basel, 1994.

[Lyo05] Russell Lyons, Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.* **14** (2005), 491–522.

[LP16] Russell Lyons and Yuval Peres, *Probability on trees and networks*, Cambridge Univ. Press, Cambridge, UK, 2016, 422 pp.

[Mar91] Gregory A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, Berlin, 1991, 388 pp.

[MS19] Sébastien Martineau and Franco Severo, Strict monotonicity of percolation thresholds under covering maps, *Ann. Probab.* **47** (2019), 4116–4136.

[NZ01] Mark E. J. Newman and Robert M. Ziff, Fast Monte Carlo algorithm for site or bond percolation, *Phys. Rev. E* **64** (2001), 016706, 16 pp.

[Pak18] Igor Pak, Complexity problems in enumerative combinatorics, in *Proc. ICM Rio de Janeiro*, Vol. IV, World Sci., Hackensack, NJ, 2018, 3153–3180; an expanded version is available at `arXiv:1803.06636`.

[PS23] Christoforos Panagiotis and Franco Severo, Gap at 1 for the percolation threshold of Cayley graphs, *Ann. Inst. Henri Poincaré Probab. Stat.* **59** (2023), 1248–1258.

[Pete23] Gabor Pete, *Probability and Geometry on Groups*, monograph in preparation (June 8, 2023 version), 246 pp.; available at math.bme.hu/~gabor/PGG.pdf

[PS08] Gabor Pete and Mark Sapir, Percolation on transitive graphs, AIM workshop summary, May 2008; available at www.aimath.org/pastworkshops/percolationrep.pdf

[SZ10] Christian R. Scullard and Robert M. Ziff, Critical surfaces for general inhomogeneous bond percolation problems, *J. Stat. Mech.* (2010), P03021, 27 pp.

[Tab17] Jennifer Taback, Lamplighter groups, in *Office hours with a geometric group theorist*, Princeton Univ. Press, Princeton, NJ, 2017, 310–330.

[TZ19] Omer Tamuz and Tianyi Zheng, On the spectrum of asymptotic entropies of random walks, to appear in *Groups, Geometry, and Dynamics*; preprint (2019), 18 pp.; `arXiv:1903.01312`.

[Tim07] Ádám Timár, Cutsets in infinite graphs, *Combin. Probab. Comput.* **16** (2007), 159–166.

[Wer09] Wendelin Werner, *Percolation et modèle d'Ising* (in French), Soc. Math. de France, Paris, 2009, 161 pp.

[Wig19] Avi Wigderson, *Mathematics and computation*, Princeton Univ. Press, Princeton, NJ, 2019, 418 pp.; available at math.ias.edu/avi/book

[Woe00] Wolfgang Woess, *Random walks on infinite graphs and groups*, Cambridge Univ. Press, Cambridge, UK, 2000, 334 pp.