ON THE LARGEST LITTLEWOOD–RICHARDSON COEFFICIENT

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Abstract. We study partitions which attain the largest Littlewood–Richardson coefficient. More precisely, we prove that the largest $c^{\lambda}_{\mu\nu}$ is attained at partitions such that $\mu \subseteq \nu$ and ν/μ is a disjoint union of squares. We conjecture that for $n \geq 16$, *all* largest $c^{\lambda}_{\mu\nu}$ must satisfy this property. We confirm this conjecture numerically, for $16 \leq n \leq 45$.

1. Introduction

According to Metropolis [Met80, p. 463], it was Paul Stein's idea to use the MANIAC computer to investigate problems in "discrete mathematics with emphasis on combinational theory." In Bivins et al. [BMSW54], the first publication that resulted from this investigation, the authors computed character tables of S_{15} and S_{16}, which took 5 and 12 hours, respectively. In what can only be viewed as gross understatement, the authors commented:

> "The experience was quite encouraging, and suggests that it would be profitable to apply electronic computer techniques to a large class of quite complicated problems in algebra and group theory," [BMSW54, p. 212].

The story of this paper started with a footnote 9 at the end of [BMSW54], where the authors asked for the largest dimension $\mathrm{D}(n)$ of the irreducible S_n module:

$$\mathrm{D}(n) \;:=\; \max_{\lambda \vdash n} f^{\lambda} \quad \text{where} \quad f^{\lambda} \;:=\; \chi^{\lambda}(1) \;=\; |\operatorname{SYT}(\lambda)|,$$

where $\operatorname{SYT}(\lambda)$ is the set of *standard Young tableaux* of shape λ, see e.g. [Sag01, Sta99].

Soon after, Comét [Com60] used the *hook-length formula* to compute $\mathrm{D}(n)$ for all $n \leq 30$. Baer and Brock [BB68] extended the computation to $n \leq 36$ and asked for the shape of a partition $\lambda \vdash n$ on which the maximum is achieved. This led to a long series of papers computing the sequence further, see [McK76, KP92, VP10, DS23]. We refer to [OEIS, A003040] for the updated list of values $\{\mathrm{D}(n), n \leq 153\}$.

A theoretical investigation also followed, beginning with McKay's lower bound [McK76], and the Vershik–Kerov upper bound [VK85]. Famously, in connection to the study of *longest increasing subsequences* in random permutations, the asymptotic limit shape (*VKLS shape*), was determined by Logan–Shepp [LS77] and Vershik–Kerov [VK81]. Most recently, Aggarwal and Elboim [AE26] proved the *Vershik–Kerov–Pass conjecture* that

$$\mathrm{D}(n) \;=\; \sqrt{n!} \cdot c^{-\sqrt{n}(1+o(1))} \quad \text{for some} \quad c > 1.$$

We refer to [Rom15] for an extensive discussion of the problem and connections to other areas.

Motivated by this problem, Richard Stanley [Sta23, Exc. 82(c)][1] asked the same questions about the largest *Littlewood–Richardson* (LR) *coefficient* :

$$\mathrm{C}(n) \;:=\; \max_{\lambda \vdash n} \max_{0 \leq k \leq n/2} \max_{\mu \vdash k} \max_{\nu \vdash n-k} c^{\lambda}_{\mu\nu}\,, \tag{1.1}$$

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[1]This exercise was added in 2017.

see [OEIS, A387851]. Stanley showed that the maximum is achieved at $k \sim \frac{n}{2}$, when

$$\mathrm{C}(n) \ = \ 2^{n/2 - O(\sqrt{n})}\,,$$

and asked about the asymptotic limit shape. This was resolved by Panova, Yeliussizov and the first author [PPY19a], who showed that all three partitions have VKLS shape. The authors also computed the values $\{\mathrm{C}(n), n \le 23\}$ and made many additional observations.

In [PPY19a, §4.7], the authors noted that the maximum in (1.1) is typically achieved on *nested partitions* $\mu \subseteq \nu \subseteq \lambda$. They conjectured that this always holds [PPY19a, Conj. 4.21], and observed that the *Lam–Postnikov–Pylyavskyy* (LPP) *inequality* implies that the nested property holds for all n on at *least one* largest LR coefficient.

We computed the values $\{\mathrm{C}(n), n \le 45\}$, see the Appendix. Our computations show that the PPY conjecture above fails already for $n = 11$, with $\lambda = (5,3,2,1)$, $\mu = (4,1)$, $\nu = (3,2,1)$, but $c^{\lambda}_{\mu\nu} = \mathrm{C}(11) = 3$. However, our results suggest that a stronger claim holds for sufficiently large n:

Conjecture 1.1. *For all $n \ge 16$, every maximum in (1.1) is achieved on partitions $\mu \subseteq \nu \subseteq \lambda$, such that skew shape ν/μ is a disjoint union of squares. Moreover, for $n \ge 28$, this maximum is unique up to the transposition*[2] *and conjugation of partitions: $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu} = c^{\lambda'}_{\mu'\nu'} = c^{\lambda'}_{\nu'\mu'}$.*

This says that μ can be obtained from ν by removing some of its corners. In this paper we present both theoretical and computational evidence in favor of the conjecture. Our main result in the following weak version of the conjecture, which, by the second part of the conjecture, is equivalent to the first part:

Theorem 1.2. *For all $n \ge 1$, at least one maximum in (1.1) is achieved on partitions $\mu \subseteq \nu \subseteq \lambda$, such that skew shape ν/μ is a disjoint union of squares.*

The new part here is the second condition; without it, this is proved in [PPY19a, Cor. 4.19]. For example, for $n = 40$, we have $\mathrm{C}(40) = 1484 = c^{\lambda}_{\mu\nu}$, where $\lambda = (9,7,6,5,4,3,2,2,1,1)$, $\mu = (6,4,3,2,2,1)$, $\nu = (6,5,4,3,2,1,1)$. Here ν/μ is a disjoint union of 4 squares, and this is unique largest LR coefficient up to conjugation.

Remark 1.3. In the example above, we also have $f^{\lambda} = \mathrm{D}(40) \approx 5.9 \cdot 10^{22}$ is the largest dimension as above. This is too unusual to be a mere coincidence, given that there are exactly two partitions: λ and λ' with the largest dimension f^{λ}, out of $p(40) = 37338$ integer partitions of $n = 40$. The same phenomenon holds for a few other values of n. Unfortunately, our dataset is too small to make predictions whether this happens infinitely often. The phenomenon is partly explained by [PPY19a, Thm 1.7], which says that for all maximal LR coefficients, we must have $f^{\lambda} > \sqrt{n!}\,c^{-n}$ for some explicit $c > 1$.

2. Schur positive inequalities

For standard definitions and notation in algebraic combinatorics and symmetric functions, see [Sag01, Sta99]. Recall that the *LR coefficients* are defined as structure constants in the ring Λ of symmetric functions:

$$s_{\mu} \cdot s_{\nu} \ = \ \sum_{\lambda} c^{\lambda}_{\mu\nu}\, s_{\nu}\,,$$

and note that $c^{\lambda}_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$. Famously, LR coefficients $c^{\lambda}_{\mu\nu} \in \mathbb{N}$ have a combinatorial interpretation which we will not need in this paper.

For two symmetric functions $f, g \in \Lambda$, we write $f \geqslant_s g$ if $(f-g)$ is *Schur positive*, i.e. a linear combination of Schur functions with nonnegative coefficients. Recall the *LPP inequality*:

$$s_{\mu} \cdot s_{\nu} \ \leqslant_s \ s_{\mu\cap\nu} \cdot s_{\mu\cup\nu}\,,$$

[2] In our notation, the transposition can only give the largest LR coefficients, when μ and ν have the same size: $|\mu| = |\nu| = n/2$.

where $\mu \cup \nu$ and $\mu \cap \nu$ are partitions defined by the union and intersection, respectively, of the corresponding Young diagrams. This inequality was introduced [LPP07], and studied extensively in recent years. See e.g. [CCPS26], for an advanced generalization. In terms of LR coefficients, the LPP inequality can be restated as

$$c^{\lambda}_{\mu\nu} \,\le\, c^{\lambda}_{\mu\cap\nu,\mu\cup\nu} \quad \text{for all} \quad \lambda, \mu, \nu. \tag{2.1}$$

To prove Theorem 1.2, we use the approach in [PPY19a], but this time we also employ the *Okounkov inequality* :

$$s_\mu \,\cdot\, s_\nu \,\leqslant_s\, s_{\lfloor(\mu+\nu)/2\rfloor} \,\cdot\, s_{\lceil(\mu+\nu)/2\rceil}\,,$$

where $\lfloor(\alpha_1, \alpha_2, \ldots)\rfloor := (\lfloor\alpha_1\rceil, \lceil\alpha_1\rfloor, \ldots)$ and $\lceil(\alpha_1, \alpha_2, \ldots)\rceil := (\lceil\alpha_1\rceil, \lceil\alpha_1\rceil, \ldots)$. A restricted version of this inequality was conjectured by Okounkov [Oko97]. In full generality, the result was obtained in [LPP07, Thm 11]. In terms of LR coefficients, the Okounkov inequality can be restated as

$$c^{\lambda}_{\mu\nu} \,\le\, c^{\lambda}_{\lfloor(\mu+\nu)/2\rfloor,\lceil(\mu+\nu)/2\rceil} \quad \text{for all} \quad \lambda, \mu, \nu. \tag{2.2}$$

Proof of Theorem 1.2. Let $c^{\lambda}_{\mu\nu} = \mathrm{C}(n)$ be a maximal LR coefficient. By (2.1), we also have $c^{\lambda}_{\mu\cap\nu,\mu\cup\nu} = \mathrm{C}(n)$. In other words, the first condition in the theorem holds for partitions $\mu \cap \nu \subseteq \mu \cup \nu \subseteq \lambda$.

For the second condition, by (2.2) applied to $c^{\lambda}_{\alpha\beta}$, where $\alpha := \mu \cap \nu$, $\beta := \mu \cup \nu$, we similarly have: $c^{\lambda}_{\lfloor(\alpha+\beta)/2\rfloor,\lceil(\alpha+\beta)/2\rceil} = \mathrm{C}(n)$. Note that partitions $\gamma := \lfloor(\alpha+\beta)/2\rfloor$, $\delta := \lceil(\alpha+\beta)/2\rceil$, satisfy $\gamma_i \le \delta_i \le \gamma_i + 1$ for all i. In other words, the skew shape δ/γ is a disjoint union of columns.

By the conjugation symmetry as in the conjecture, we have $c^{\lambda'}_{\gamma'\delta'} = \mathrm{C}(n)$. By (2.2), we also have $c^{\lambda}_{\lfloor(\gamma'+\delta')/2\rfloor,\lceil(\gamma'+\delta')/2\rceil} = \mathrm{C}(n)$. This gives the desired partitions $\tau := \lfloor(\gamma'+\delta')/2\rfloor$, $\pi := \lceil(\gamma'+\delta')/2\rceil$, such that $\tau \subseteq \pi \subseteq \lambda$ and π/τ is a disjoint union of squares. $\square$

3. Experimental results

Following [PPY19a], let

$$\mathrm{C}(n,k) \;:=\; \max_{\lambda\vdash n}\, \max_{\mu\vdash k}\, \max_{\nu\vdash n-k}\, c^{\lambda}_{\mu\nu}\,. \tag{3.1}$$

Note that $\mathrm{C}(n,k) = \mathrm{C}(n,n-k)$, and

$$\mathrm{C}(n) \;=\; \max_{0\le k\le n/2}\, \mathrm{C}(n,k)\,.$$

It was observed in [PPY19a, §4.8], that $\mathrm{C}(n,k) \le \mathrm{C}(n+1,k)$, and we also have *stability property* $\mathrm{C}(n,k) = \mathrm{D}(n)$ for $n \ge \binom{k+1}{2}$. A table of $\{\mathrm{C}(n,k)\}$ for $1 \le n \le 23$, is given in [PPY19a, App.]

In the Appendix, we present our extensive computations giving new values of $\{\mathrm{C}(n,k)\}$ for $24 \le n \le 45$. Our computations confirm the previously computed values, and satisfy stability, e.g. $\mathrm{C}(45,k) = \mathrm{D}(k)$ for all $0 \le k \le 9$, given that $45 = \binom{10}{2}$. We also computed triples of partitions which attained the maximal value $\mathrm{C}(n)$, for all $n \le 45$, and confirmed that Conjecture 1.1 hold in these cases, see a discussion below.

The runtimes are summarized in the following table. For $n \le 40$, the computation was done on a 4-core laptop. For $41 \le n \le 45$, the computation was done on 32 cores of the UCLA `Hoffman2` cluster under PyPy 3.11. This explains a big jump down from $n = 40$ to 41. Since the case $n = 45$ took over 20 hours, computing much beyond that point became infeasible.

Here $p(n) = |\{\lambda \vdash n\}|$ is the number of integer partitions of n, shown to give the first approximation of the size of the search space: to compute $\mathrm{C}(n,k)$ we need to search over $p(k)p(n-k)p(n)$ triples of partitions. See [OEIS, A000041] for further values and references.

n	$p(n)$	$\mathrm{C}(n)$	runtime
25	1958	45	28 sec.
30	5604	176	3:18 = 198 sec.
35	14883	423	54:18 = 3258 sec.
40	37338	1484	12:28:39 = 44919 sec.
41	44583	1768	1:33:29 = 5609 sec.
45	89134	4323	20:02:52 = 72172 sec.

4. Discussion

Let $\mathrm{M}(n) := |\{\lambda \vdash n \, : \, f^\lambda = \mathrm{D}(n)\}$ denote the number of irreducible S_n modules of the largest dimension. It is wide open whether $\mathrm{M}(n)$ is bounded, see [PPY19b, Remark 4.2]. Note that since there are only $p(n) = e^{O(\sqrt{n})}$ partitions of n, while the maximal dimension $\mathrm{D}(n) \approx \sqrt{n!}$, one would certainly expect few if any coincidences (except for conjugate partitions).

In fact, until relatively recently it was open whether *any* dimension, i.e., not necessarily maximal, can appear an unbounded number of times; this was proved by Craven [Cra08]. In the opposite direction, the bounds in [AE26] leave open whether $\mathrm{M}(n) = e^{o(\sqrt{n})}$.

Compare this with the second part of Conjecture 1.1. Again, asymptotically there are only $p(n)$ partitions, while the largest LR coefficients is much larger: $\mathrm{C}(n) \approx 2^{n/2}$, so one would not expect many coincidences. Note also that computing LR coefficients efficiently is a major open problem, see [Pan24, Conj. 5.14], compared with f^λ which has a nice product formula. Given this state of art, it is rather audacious to believe that the second part of the conjecture can be resolved in a foreseeable future.

The numerical evidence is helpful at this point. For example, we have a surprising coincidence $c^\lambda_{\mu\mu} = c^{\lambda'}_{\mu\mu} = c^\gamma_{\mu\mu} = c^{\gamma'}_{\mu\mu} = \mathrm{C}(20)$, where $\lambda = (6,5,3,3,2,1)$, $\mu = (4,3,2,1)$ and $\gamma = (6,4,4,3,2,1)$. This shows that the largest LR coefficient is not unique up to transposition and conjugation in this case. In our computations, the last time this happens is for $n = 27$, which is why we stated the second part of the conjecture for $n \geq 28$.

For the first part of the conjecture, there is a bit more hope that asymptotic methods can help, so it can be established without the second part. We need the following:

Proposition 4.1. *Suppose* $\mathrm{C}(n)$ *is unique in the set of values* $\{\mathrm{C}(n,k), 0 \leq k \leq n/2\}$*. Then the first part of Conjecture 1.1 holds.*

Proof. In notation of the proof of Theorem 1.2, suppose the nested property fails for the maximal LR coefficient $c^\lambda_{\mu\nu} = \mathrm{C}(n)$, i.e. suppose $\mu \not\subseteq \nu$. Then $c^\lambda_{\alpha\beta} = \mathrm{C}(n)$ is also maximal, and since $\mu \subset \alpha$, $\mu \neq \alpha$, we obtain a contradiction with the assumption: $\mathrm{C}(n, |\mu|) = \mathrm{C}(n, |\alpha|)$. Use the same argument for the two averaging operations used in the proof; we omit the details. $\square$

Now, observe that we have only $n/2$ numbers $\{\mathrm{C}(n,k), 0 \leq k \leq n/2\}$ distributed between 1 and $\mathrm{C}(n) \approx 2^{n/2}$. Heuristically, one would expect a runaway behavior of $\mathrm{C}(n)$. Thus, we speculate that the assumption of the proposition could in principle be proved asymptotically.

Of course, whether the table in the Appendix supports this approach is a subject for interpretation. For example, $\mathrm{C}(45) = \mathrm{C}(45, 22) = 4323$ is much larger than the second largest value $\mathrm{C}(45, 21) = 4044$. On the other hand, we note that $\mathrm{C}(42) = \mathrm{C}(42, 18) = 2074$, which is only a little larger than $\mathrm{C}(42, 21) = 2064$.

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Appendix A. Table of the largest Littlewood–Richardson coefficients

n	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
$\mathrm{C}(n)$	41	45	50	59	93	112	176	196	270	298	408	423	640	840	1096	1216	1484	1768	2074	2624	3450	4323

Table of $\mathrm{C}(n,k)$, for $0 \leq k \leq n/2$ and $24 \leq n \leq 45$. Green cells maximal values $\mathrm{C}(n)$.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
24	1	1	1	2	3	6	16	20	24	26	31	41	38										
25	1	1	1	2	3	6	16	20	24	29	40	45	44										
26	1	1	1	2	3	6	16	20	24	29	44	47	48	50									
27	1	1	1	2	3	6	16	20	24	36	48	59	58	58									
28	1	1	1	2	3	6	16	35	30	36	54	64	72	93	68								
29	1	1	1	2	3	6	16	35	45	38	60	76	88	111	112								
30	1	1	1	2	3	6	16	35	45	55	68	83	110	133	132	176							
31	1	1	1	2	3	6	16	35	45	60	88	96	112	139	149	196							
32	1	1	1	2	3	6	16	35	45	60	96	109	134	161	173	218	270						
33	1	1	1	2	3	6	16	35	45	60	96	137	142	168	195	228	298						
34	1	1	1	2	3	6	16	35	45	60	96	146	173	191	203	254	329	408					
35	1	1	1	2	3	6	16	35	45	60	96	165	210	213	254	276	377	423					
36	1	1	1	2	3	6	16	35	90	75	120	165	210	253	322	640	424	462	467				
37	1	1	1	2	3	6	16	35	90	111	136	179	249	266	378	744	840	566	571				
38	1	1	1	2	3	6	16	35	90	111	192	199	268	317	444	864	960	1096	716	693			
39	1	1	1	2	3	6	16	35	90	111	192	255	277	333	456	882	1080	1216	1168	795			
40	1	1	1	2	3	6	16	35	90	111	192	311	346	387	522	1002	1208	1352	1484	1424	1038		
41	1	1	1	2	3	6	16	35	90	111	192	311	414	423	540	1022	1328	1495	1636	1768	1536		
42	1	1	1	2	3	6	16	35	90	111	192	347	498	510	618	1160	1380	1666	2074	2045	1922	2064	
43	1	1	1	2	3	6	16	35	90	111	192	347	498	600	700	1186	1518	1806	2332	2624	2381	2614	
44	1	1	1	2	3	6	16	35	90	111	192	347	498	610	824	1344	1664	1974	2468	2882	2919	3280	3450
45	1	1	1	2	3	6	16	35	90	216	272	347	498	711	872	1552	2096	3250	2620	3149	3490	4044	4323