

BOUNDS ON KRONECKER COEFFICIENTS VIA CONTINGENCY TABLES

IGOR PAK* AND GRETA PANOVA†

ABSTRACT. We present both upper and lower bounds for the Kronecker coefficients and the reduced Kronecker coefficients, in term of the number of certain contingency tables. Various asymptotic applications and generalizations are also presented.

1. INTRODUCTION

By now the Algebraic Combinatorics is so well developed, it is easy to become overwhelmed with the abundance of tools, formulas, techniques and applications. Yet the celebrated *Kronecker coefficients* stand apart for being both unapproachable and deeply mysterious. Not only they are provably hard to compute or even decide whether they are vanishing, there are no good bounds for them except in a handful of special cases. In this paper we present general upper bounds for Kronecker coefficients by using recent work on 3-dimensional *contingency tables*.

Kronecker coefficients $g(\lambda, \mu, \nu)$ are defined as structure constants in products of S_n -characters:

$$\chi^\mu \cdot \chi^\nu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu) \chi^\lambda,$$

where $\lambda, \mu, \nu \vdash n$ (see §2 for the background).

Theorem 1.1. *Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:*

$$g(\lambda, \mu, \nu) \leq \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}$$

In particular, when $\ell m r \leq n$, this gives $g(\lambda, \mu, \nu) \leq 4^n$. The bound in the theorem is often much sharper than the *dimension bound* $g(\lambda, \mu, \nu) \leq \min\{f^\lambda, f^\mu, f^\nu\}$, which is the only known general upper bound for Kronecker coefficients. For example, when $n = \ell^3$, $\lambda = \mu = \nu = (\ell^2, \dots, \ell^2)$, ℓ times, the dimension bound gives only

$$g(\lambda, \mu, \nu) \leq f^\lambda = e^{\frac{1}{3}n \log n + O(n)}.$$

Our tool is the following general upper bound:

Lemma 1.2. *Let $T(\lambda, \mu, \nu)$ be the number of 3-dimensional contingency tables with margins $\lambda, \mu, \nu \vdash n$. Then $g(\lambda, \mu, \nu) \leq T(\lambda, \mu, \nu)$.*

This bound is very weak in full generality; e.g. for $\lambda = \mu = \nu = (1^n)$ we have $g(\lambda, \mu, \nu) = 0$ while $T(\lambda, \mu, \nu) = (n!)^2$. However, the bound in the lemma does seem to work well for partitions into few parts. The theorem follows from the analysis of known upper bounds on $T(\lambda, \mu, \nu)$ in some special cases plus majorization technology (see §3).

In the last part of the paper, we compare our upper bound with the upper and lower bounds coming from counting *binary contingency tables*. Let us single out the following curious asymptotic inequality:

Theorem 1.3. *Let $\mathcal{L}_n = \{\lambda \vdash n, \lambda = \lambda'\}$. We have:*

$$\sum_{\lambda \in \mathcal{L}_n} g(\lambda, \lambda, \lambda) \geq e^{cn^{2/3}} \text{ for some } c > 0.$$

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*Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: pak@math.ucla.edu.

†Department of Mathematics, USC, Los Angeles, CA 90089. Email: gpanova@usc.edu.

In §7, we present the analogue of Theorem 1.1 for the *reduced Kronecker coefficients*. We conclude the paper with final remarks and open problems in §8.

2. BASIC DEFINITIONS AND NOTATION

2.1. Standard notation. We use $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = \mathbb{R}_{\geq 0}$, and $[n] = \{1, 2, \dots, n\}$.

2.2. Partitions and Young tableaux. We use standard notation from [Mac] and [S2, §7] throughout the paper.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a *partition* of size $n := |\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 1$. We write $\lambda \vdash n$ for this partition. The *length* of λ is the number ℓ of parts of λ , denoted $\ell(\lambda)$. Denote by $p(n)$ the number of partitions $\lambda \vdash n$.

A *Young diagram* of shape λ is an arrangement of squares $(i, j) \in \mathbb{N}^2$ with $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. We denote by λ' the conjugate (transposed) partition. Special partitions include the *rectangular shape* $(a^b) = (a, \dots, a)$, b times, the *hooks shape* $(k, 1^{n-k})$, the *two-row shape* $(n-k, k)$, and the *staircase shape* $\rho_\ell = (\ell, \ell-1, \dots, 1)$.

A *plane partition* $A = (a_{ij})$ of n is an arrangement of integers $a_{ij} \geq 1$ of a Young diagram shape which sum to n and weakly decrease along rows and columns. Denote by $p_2(n)$ the total number of such plane partitions.

For $\lambda \vdash n$, denote by $f^\lambda = \chi^\lambda(1)$, which is also equal to the number $\text{SYT}(\lambda)$ of standard Young tableaux of shape λ . We have the *hook-length formula*:

$$(HLF) \quad f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook-length* of the square $u = (i, j)$.

2.3. Kronecker coefficients. The *Kronecker coefficients* $g(\lambda, \mu, \nu)$, $\lambda, \mu, \nu \vdash n$, can be computed as follows:

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \chi^\mu(\sigma) \chi^\nu(\sigma),$$

where χ^α denotes the character for the irreducible S_n representation indexed by α . From here it is easy to see that:

$$g(\lambda, \mu, \nu) = g(\mu, \lambda, \nu) = g(\lambda, \nu, \mu) = \dots \quad \text{and} \quad g(\lambda, \mu, \nu) = g(\lambda', \mu', \nu).$$

Also, for all $f^\nu \leq f^\mu \leq f^\lambda$ we have:

$$(2.1) \quad g(\lambda, \mu, \nu) \leq \frac{f^\mu f^\nu}{f^\lambda} \leq f^\nu,$$

see e.g. [Isa, Ex. 4.12] and [PPY, Eq. (3.2)].

3. CONTINGENCY TABLES

3.1. The number of 2-dimensional tables. Let $\lambda, \mu \vdash N$ be two integer partitions, where $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_n)$. A *contingency table* with *margins* (λ, μ) is an $m \times n$ matrix of non-negative integers whose i -th row sums to λ_i and whose j -th column sums to μ_j . We denote by $\mathcal{T}(\lambda, \mu)$ the set of all such matrices, and let $T(\lambda, \mu) := |\mathcal{T}(\lambda, \mu)|$. Finally, denote by $\mathcal{P}(\lambda, \mu) \in \mathbb{R}^{m \times n}$ the polytope of *real* contingency tables, i.e. table with row and column sums as above, and non-negative real entries.

Counting $T(\lambda, \mu)$, even approximately, is famously a difficult problem, both mathematically and computationally. In fact, even a change in a single row and column sum can lead to a major change in the count, see [B3, DLP]. Three-dimensional tables are even harder to count, see [B4, DO]. We refer to [B3, DG, FLL] for an introduction to the subject and further references in many areas.

3.2. The number of 3-dimensional tables. Let $\lambda, \mu, \nu \vdash n$. Denote by $T(\lambda, \mu, \nu)$ the number of 3-dimensional $\ell(\lambda) \times \ell(\mu) \times \ell(\nu)$ contingency tables with 2-dimensional sums orthogonal to three coordinates are given by λ, μ and ν , respectively. Denote by $\mathcal{P}(\lambda, \mu, \nu) \subset \mathbb{R}^{\ell m r}$, the corresponding polytope of real 3-dimensional contingency tables. Note that $\mathcal{P}(\lambda, \mu, \nu)$ has codimension $d = \ell + m + r - 3$. Denote by A a $d \times \ell m r$ matrix defining a subspace spanned by $\mathcal{P}(\lambda, \mu, \nu)$ via $A \cdot X = 0$, where X is a column vector of all x_{ijk} . A subset $J \subset [\ell] \times [m] \times [r]$ is called *free* if the elements $(ijk) \in J$ correspond to linearly independent columns of A .

Theorem 3.1 (Barvinok, Benson-Putnins and Shapiro, see §8.2). *Let $\ell = \ell(\lambda)$, $m = \ell(\mu)$, $r = \ell(\nu)$. Let $Z = (z_{ijk}) \in \mathcal{P}(\lambda, \mu, \nu)$ be the unique point maximizing a strictly concave function*

$$g(Z) := \sum_{i=1}^{\ell} \sum_{j=1}^m \sum_{k=1}^r (z_{ijk} + 1) \log(z_{ijk} + 1) - z_{ijk} \log z_{ijk}.$$

Denote

$$G(\lambda, \mu, \nu) := e^{g(Z)} \quad \text{and} \quad M(\lambda, \mu, \nu) := \min_J \prod_{(ijk) \in J} \frac{1}{1 + z_{ijk}},$$

where the minimum is over all free subsets $J \subset [\ell] \times [m] \times [r]$. Then:

$$T(\lambda, \mu, \nu) \leq G(\lambda, \mu, \nu) \cdot M(\lambda, \mu, \nu).$$

We discuss the history of the theorem in §8.2.

Corollary 3.2. *Let $\lambda = (n/\ell)^\ell$, $\mu = (n/m)^m$, $\nu = (n/r)^r$ be three rectangular partitions. Then:*

$$T(\lambda, \mu, \nu) \leq \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r - (\ell + m + r) + 3}$$

Proof. By the symmetry and uniqueness of $Z = (z_{ijk})$, we conclude that $z_{ijk} = n/(\ell m r)$ for all $1 \leq i \leq \ell$, $1 \leq j \leq m$, and $1 \leq k \leq r$. Then:

$$\begin{aligned} G(\lambda, \mu, \nu) &= e^{g(Z)} = \exp(\ell m r [(1 + n/\ell m r) \log(1 + n/\ell m r) - (n/\ell m r) \log(n/\ell m r)]) \\ &= \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}. \end{aligned}$$

By the definition of free subset J as above, we have $|J| \leq d = \ell + m + r - 3$. Since we can always take J of size d and all of them give equal products as in the theorem, we have:

$$M(\lambda, \mu, \nu) = \left(1 + \frac{n}{\ell m r}\right)^{-(\ell + m + r) + 3}.$$

Now Theorem 3.1 implies the result. □

3.3. Majorization for the number of contingency tables. Let $\lambda, \mu \vdash n$. The *dominance order* is defined as follows: $\lambda \trianglelefteq \mu$ if $\lambda_1 \leq \mu_1$, $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$, etc. For $\lambda \vdash n$ and a set of partitions \mathcal{L} , we write $\lambda \trianglelefteq \mathcal{L}$ if $\lambda \trianglelefteq \mu$ for all $\mu \in \mathcal{L}$. This is a special case of *majorization*, equivalent for partitions and studied extensively in many fields of mathematics and applications, see e.g. [MOA].

Theorem 3.3 (Barvinok). *Let $\lambda, \mu, \alpha, \beta \vdash n$, and suppose $\ell(\lambda) = \ell(\alpha)$, $\ell(\mu) = \ell(\beta)$, $\lambda \trianglerighteq \alpha$, $\mu \trianglerighteq \beta$. Then:*

$$T(\lambda, \mu) \leq T(\alpha, \beta).$$

The original proof by Barvinok is in [B1, Eq. (2.4)]. Alternatively, it can be deduced from [V4, Thm. 4.9]. We refer [Pak] (Note 36, 37 in the expanded version on the paper), for a explicit combinatorial proof and further references. The following is a helpful extension of Theorem 3.3.

Theorem 3.4. *Let $\lambda, \mu, \nu, \alpha, \beta, \gamma \vdash n$, and suppose $\ell(\lambda) = \ell(\alpha)$, $\ell(\mu) = \ell(\beta)$, $\ell(\nu) = \ell(\gamma)$, $\lambda \succeq \alpha$, $\mu \succeq \beta$, $\nu \succeq \gamma$. Then:*

$$T(\lambda, \mu, \nu) \leq T(\alpha, \beta, \gamma).$$

Proof. For a contingency table $T \in \mathcal{T}(\lambda, \mu, \nu)$, let $A = A(T) \in \mathcal{T}(\mu, \nu)$ be a partition of the 1-margins, i.e. of sums along lines parallel to x axis. For a table A and vector ξ let $T(\xi, A)$ denote the number of contingency tables with margins ξ and the vector of entries in A . Thus, projecting along the x axis and applying Theorem 3.3, we have:

$$T(\lambda, \mu, \nu) = \sum_{A \in \mathcal{T}(\mu, \nu)} T(\lambda, A) \leq \sum_{A \in \mathcal{T}(\mu, \nu)} T(\alpha, A) = T(\alpha, \mu, \nu).$$

Applying this two more times, we obtain:

$$T(\lambda, \mu, \nu) \leq T(\alpha, \mu, \nu) \leq T(\alpha, \beta, \nu) \leq T(\alpha, \beta, \gamma),$$

as desired. \square

Remark 3.5. Theorem 3.4 can be easily generalized to d -dimensional contingency tables. The proof by induction follows verbatim the proof above.

3.4. Majorization for upper bounds. In notation of Theorem 3.1, we can naturally extend the definition of $g(Z)$ to all $\alpha \in \mathbb{R}_+^\ell$, $\beta \in \mathbb{R}_+^m$, $\gamma \in \mathbb{R}_+^r$, as an optimization problem on the polytope $\mathcal{P}(\alpha, \beta, \gamma)$ of 3-dimensional real tables with margins α, β, γ . Similarly, define $G(\lambda, \mu, \nu)$ and $G(\alpha, \beta, \gamma)$ as in the theorem.

Theorem 3.6. *Let $\lambda \succeq \alpha$, $\mu \succeq \beta$, $\nu \succeq \gamma$, where $\lambda, \alpha \in \mathbb{R}_+^\ell$, $\mu, \beta \in \mathbb{R}_+^m$, and $\nu, \gamma \in \mathbb{R}_+^r$. Then:*

$$G(\lambda, \mu, \nu) \leq G(\alpha, \beta, \gamma).$$

Proof. As in the discrete case, it suffices to prove the result for 2-dimensional tables. The 3-dimensional case then follows verbatim the proof of Theorem 3.4.

Define $G(\lambda, \mu)$ and $M(\lambda, \mu)$ for 2-dimensional tables by setting $\nu = (1)$. To prove that $G(\lambda, \mu) \leq G(\alpha, \beta)$ and $M(\lambda, \mu) \leq M(\alpha, \beta)$, we again employ a standard majorization strategy inspired by the proof of Theorem 3.3.

First, it is easy to see that if two weakly decreasing sequences λ, α majorize one another, $\lambda \succeq \alpha$, then λ can be obtained from α by a finite sequence of operations adding vectors of the form $te(i, j)$, where $e(i, j)_r = 0$ for $r \neq i, j$ and $e(i, j)_i = 1, e(i, j)_j = -1$, see e.g. [MOA, §2]. Now it is enough to prove the inequality in the case when $\mu = \beta$ and $\lambda = \alpha + te(i, j)$, and apply it consecutively in the algorithm obtaining (α, β) from (λ, μ) by changing α to λ first, and then β to μ .

Let $w \in \mathbb{R}_+^{\ell^2}$ be the unique maximizer of $g(Z)$ for $G(\lambda, \mu)$, and let $\alpha = \lambda - te(i, j)$ and assume for simplicity that $i = 1, j = 2$. Consider the $2 \times \ell$ section with rows 1, 2, and let its column margins be a_1, \dots, a_ℓ . We have:

$$\sum_{i=1}^{\ell} (w_{1i} - w_{2i}) = \lambda_1 - \lambda_2 \geq 2t,$$

so the positive terms among $(w_{1i} - w_{2i})$ add up to at least $2t$. Assume for simplicity that the positive terms are for $i = 1, \dots, r$, and choose $0 \leq t_i \leq \frac{1}{2}(w_{1i} - w_{2i})$, so that $t_1 + \dots + t_r = t$. Let $z_{ij} = w_{ij}$ for $i \neq 1, 2$ or $j > r$, and let $z_{1j} = w_{1j} - t_j, z_{2j} = w_{2j} + t_j$. Then z has margins (α, μ) , and we will show that $g(z) \geq g(w)$.

To see this, let $f(x) = (1+x)\log(1+x) - x\log x$, and note that since it is concave we have that $f(a-x) + f(b+x)$ is increasing whenever $a > b$ and $x \in [0, \frac{a-b}{2}]$. Hence,

$$f(z_{1j}) + f(z_{2j}) = f(w_{1j} - t_j) + f(w_{2j} + t_j) \geq f(w_{1j}) + f(w_{2j})$$

for $j = 1, \dots, r$, and equal for the other indices. Thus, $g(z) \geq g(w)$. We have:

$$G(\alpha, \mu) = \max_{Z \in \mathcal{P}(\alpha, \mu)} \exp g(Z) \geq \exp g(z) \geq \exp g(w) = G(\lambda, \mu),$$

which completes the proof. \square

4. UPPER BOUNDS VIA GENERAL CONTINGENCY TABLES

4.1. Proof of the upper bound. We begin with the proofs of results in the introduction.

Proof of Lemma 1.2. Recall *Schur's theorem*¹ [S2, Exc. 7.78f], that:

$$\sum_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) s_\lambda s_\mu s_\nu = \prod_{i, j, k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda, \mu, \nu} T(\alpha, \beta, \gamma) m_\alpha m_\beta m_\gamma,$$

where s are the Schur functions and m are the monomial symmetric functions. Taking the coefficients in $x^\alpha y^\beta z^\gamma$ on both sides gives

$$T(\alpha, \beta, \gamma) = \sum_{\lambda \triangleright \alpha, \mu \triangleright \beta, \nu \triangleright \gamma} g(\lambda, \mu, \nu) K_{\lambda\alpha} K_{\mu\beta} K_{\nu\gamma} \geq g(\alpha, \beta, \gamma),$$

where $K_{\lambda\alpha} \geq 0$ is the *Kostka number*, and $K_{\alpha\alpha} = 1$ for all $\alpha \vdash n$. □

Proof of Theorem 1.1. Denote by $\alpha = (n/\ell)^\ell \in \mathbb{R}_+^\ell$, $\beta = (n/m)^m \in \mathbb{R}_+^m$, $\gamma = (n/r)^r \in \mathbb{R}_+^r$. Observe that $\lambda \triangleright \alpha$, $\mu \triangleright \beta$ and $\nu \triangleright \gamma$. By Lemma 1.2 and Theorem 3.1, we have:

$$g(\lambda, \mu, \nu) \leq T(\lambda, \mu, \nu) \leq G(\lambda, \mu, \nu) \cdot M(\lambda, \mu, \nu) \leq G(\alpha, \beta, \gamma).$$

Here in the last inequality we are using Theorem 3.6 and a trivial bound $M(\lambda, \mu, \nu) \leq 1$.

Adapting the calculation in the proof of Corollary 3.2 to vectors α, β, γ which are non-integral in general, we have:

$$G(\alpha, \beta, \gamma) \leq \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r},$$

which implies the result. □

4.2. Examples and applications. The bound in Theorem 1.1 is rather weak for large $(\ell m r)$. E.g., for $\ell m r \geq n^{3/2}$, we have the RHS $> \sqrt{n!}$, which is larger than the dimension bound. However, for $\ell m r = o(n)$ the bound is surprisingly strong.

Corollary 4.1. *Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Suppose $\ell m r = o(n)$ as $n \rightarrow \infty$. Then:*

$$g(\lambda, \mu, \nu) \leq \exp[\ell m r \log n + O(\ell m r)].$$

Let us give a somewhat stronger bound in the case when majorization for contingency tables (Theorem 3.4) can be applied directly.

Theorem 4.2. *Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) \leq \ell$, $\ell(\mu) \leq m$, and $\ell(\nu) \leq r$. Suppose further that $\ell, m, r \mid n$. Then:*

$$g(\lambda, \mu, \nu) \leq \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r - \ell - m - r + 3}$$

Proof. We modify the proof of Theorem 1.1 given above. In this case vectors $\alpha = (n/\ell)^\ell$, $\beta = (n/m)^m$, and $\gamma = (n/r)^r$ are all integral. By Lemma 1.2 and Theorem 3.4, we have:

$$g(\lambda, \mu, \nu) \leq T(\lambda, \mu, \nu) \leq T(\alpha, \beta, \gamma).$$

Now Corollary 3.2 implies the result. □

¹Sometimes this identity is called the *generalized Cauchy identity*.

Example 4.3. As in the introduction, let $\lambda = (\ell^2)^\ell$, where $\ell = n^{1/3}$. By the HLF, we have $f^\lambda = \exp\left[\frac{1}{3}n \log n + O(n)\right]$. Theorem 1.1 gives $g(\lambda, \lambda, \lambda) \leq 4^n$, while Theorem 4.2 improves this by a weakly exponential factor to

$$g(\lambda, \lambda, \lambda) \leq 8^{1-n^{1/3}} \cdot 4^n.$$

This is in sharp contrast with our result (see §6.2 below) that for $\lambda = \mu$ as above and *some* partition γ , we have:

$$g(\lambda, \lambda, \gamma) \geq \frac{(f^\lambda)^2}{\sqrt{p(n)n!}} = \exp\left[\frac{1}{6}n \log n + O(n)\right].$$

We also conjecture that our upper bound is tight up to the lower order terms in the example above:

Conjecture 4.4. *Let $\lambda = (\ell^2)^\ell$, where $\ell = n^{1/3}$, as above. Then:*

$$g(\lambda, \lambda, \lambda) \geq 4^{n-o(n)}.$$

Remark 4.5. We should warn the reader to avoid using the upper bound in Theorem 1.1 for relatively small n . For example, for $\lambda = (7, 5, 2, 2)$, $\mu = (7, 7, 2)$, $\nu = (8, 8)$, $\lambda, \mu, \nu \vdash 16$, we have $g(\lambda, \mu, \nu) = 4$. The dimension bounds are quite weak: $f^\lambda = 40040$, $f^\mu = 120120$, $f^\nu = 1430$. The bound in the theorem is even weaker: $g(\lambda, \mu, \nu) \leq (1 + \frac{24}{16})^{16} (1 + \frac{16}{24})^{24} \approx 4.91 \cdot 10^{11}$.

5. UPPER BOUNDS VIA BINARY CONTINGENCY TABLES

5.1. Number of binary contingency tables. Denote by $\mathcal{B}(\lambda, \mu, \nu)$ the set of 3-dimensional *binary* (0/1) contingency tables, and let $B(\lambda, \mu, \nu) = |\mathcal{B}(\lambda, \mu, \nu)|$. Denote by

$$\mathcal{Q}(\lambda, \mu, \nu) := \mathcal{P}(\lambda, \mu, \nu) \cap_{ijk} \{0 \leq z_{ijk} \leq 1\}$$

the intersection of the polytope of contingency tables with the unit cube.

Theorem 5.1 (Barvinok, see [B3, §3]). *Let $\ell = \ell(\lambda)$, $m = \ell(\mu)$, $r = \ell(\nu)$. Let $Z = (z_{ijk}) \in \mathcal{Q}(\lambda, \mu, \nu)$ be the unique point maximizing a strictly concave function*

$$h(Z) := \sum_{i=1}^{\ell} \sum_{j=1}^m \sum_{k=1}^r z_{ijk} \log \frac{1}{z_{ijk}} + (1 - z_{ijk}) \log \frac{1}{1 - z_{ijk}}.$$

Then:

$$B(\lambda, \mu, \nu) \leq e^{h(Z)}.$$

Example 5.2. Let $n = \ell^3$, $a = \ell^2$, $\lambda = (\ell^a)$. Consider $B(\lambda, \lambda, \lambda)$ as in the theorem by an $a \times a \times a$ cube with all marginals ℓ . By the symmetry, $z_{ijk} = n/a^3 = 1/n$ for all $1 \leq i, j, k \leq a$. This gives

$$h(Z) = a^3 \cdot \left[\frac{1}{n} \log n + (1 - 1/n) \log \frac{1}{1 - 1/n} \right] = n \log n + O(n)$$

and

$$B(\lambda, \lambda, \lambda) \leq \exp[n \log n + O(n)].$$

It is also known that this upper bound is asymptotically tight (see [B3, CM]).

5.2. Two upper bounds. Let $\lambda, \mu, \nu \vdash n$. As before, denote by $\mathcal{B}(\lambda, \mu, \nu)$ and $B(\lambda, \mu, \nu)$ the set and the number of 3-dimensional *binary* contingency tables, respectively.

Theorem 5.3 (see §8.3). *We have: $g(\lambda, \mu, \nu) \leq B(\lambda', \mu', \nu')$.*

Denote by n_i, m_i, r_i denote the number of part of size i in λ, μ, ν , respectively. Define the group $\Sigma = \Sigma(\lambda, \mu, \nu)$ as a product

$$\Sigma := S_{n_1} \times S_{n_1} \times \dots \times S_{m_1} \times S_{m_2} \times \dots \times S_{r_1} \times S_{r_2} \times \dots$$

Group Σ has a natural action on $B(\lambda, \mu, \nu)$ by a permutation of 2-dimensional layers with equal sums. Denote by $B^\circ(\lambda, \mu, \nu)$ the number of orbits under this action. Clearly,

$$\frac{1}{|\Sigma|} B(\lambda, \mu, \nu) \leq B^\circ(\lambda, \mu, \nu) \leq B(\lambda, \mu, \nu).$$

The following result is an improvement over the theorem above.

Theorem 5.4 (Bürgisser–Ikenmeyer, see also §8.3). *We have: $g(\lambda, \mu, \nu) \leq B^\circ(\lambda', \mu', \nu')$.*

Example 5.5. Let $\lambda = (\ell^k)$, where $n = \ell^3$, $k = \ell^2$. Then $B(\lambda, \lambda, \lambda) = 1$ since there is a unique $\ell \times \ell \times \ell$ binary table with all 2-dim sums ℓ^2 . It is easy to see directly (or deduce from Theorem 6.1 below), that $g(\lambda, \mu, \nu) = 1$ in this case. Note that in this case $|\Sigma| = (\ell!)^3$, but of course Theorem 5.4 does not improve over Theorem 5.3.

Example 5.6. Let $\lambda = (a^\ell)$, where $n = \ell^3$, $a = \ell^2$. By Theorem 5.3 and Example 5.2 above, we have:

$$g(\lambda, \lambda, \lambda) \leq B(\lambda', \lambda', \lambda') = \exp[n \log n + O(n)].$$

which is much weaker than both the dimension bound and our exponential bound given in Example 4.3. In addition, we have:

$$|\Sigma(\lambda', \mu', \nu')| = (a!)^3 = \exp[2n^{2/3} \log n + O(n^{2/3})].$$

Thus in this case, Theorem 5.4 does not improve over Theorem 5.3 again.

Remark 5.7. Note that Theorem 5.4 gives *some* general upper bound on the Kronecker coefficients $g(\lambda, \mu, \nu)$. Unfortunately, the number of orbits $B^\circ(\lambda, \mu, \nu)$ is not easy to analyze in general, while the total number $B(\lambda, \mu, \nu)$ of binary tables can be rather large (see examples above).

5.3. Hooks and double hooks. For a partition $\lambda \vdash n$, the *size of the Durfee square* is defined as $D(\lambda) = \max\{k : \lambda_k \geq k\}$. For example, $D(\lambda) = 1$ if and only if λ is a hook. Partition λ with $D(\lambda) = 2$ is called a *double hook* [Rem].

Example 5.8. Let $\lambda = (m+1, 1^m)$, $n = 2m+1$, and note that $\lambda = \lambda'$. Observe that Theorem 5.3 in this case gives a very weak estimate since $B(\lambda, \lambda, \lambda) > m!$. This can be seen, e.g. by placing $(m+1)$ ones along a line and a permutation in S_m into an orthogonal 2-plane. Of course, Theorem 3.1 gives an even weaker bound, since $B(\lambda, \lambda, \lambda) < T(\lambda, \lambda, \lambda)$. On the other hand, $f^\lambda = \binom{2m}{m} = 2^n \Theta(1/\sqrt{n})$ is a better estimate. Surprisingly, Theorem 5.4 is much stronger. Indeed, the orbits of $\Sigma(\lambda, \lambda, \lambda)$ are characterized by the number of 1's on lines $(1, 1, *)$, $(1, *, 1)$, $(*, 1, 1)$ and the value at $(1, 1, 1)$. This gives $g(\lambda, \lambda, \lambda) \leq B^\circ(\lambda', \mu', \nu') = O(n^3)$. In fact, $g(\lambda, \lambda, \lambda) = 1$ in this case, see e.g. [Rem, Ros].

Proposition 5.9. *Let $\lambda, \mu, \nu \vdash n$ such that λ and μ are double hooks, and ν is a hook. Then:*

$$g(\lambda, \mu, \nu) \leq n^{450} p(n)^{400}.$$

Note that $g(\lambda, \mu, \nu)$ has a cumbersome combinatorial interpretation when ν is a hook, but no explicit bounds [Bla] (see also [BB]). The proof generalizes the approach in the example above.

Proof. By the symmetry and Theorem 5.4, we have: $g(\lambda, \mu, \nu) \leq B^\circ(\lambda, \mu, \nu')$. Denote $\lambda = (a, b, 2^p, 1^q)$, $\mu = (c, d, 2^u, 1^v)$, and $\nu' = (r, 1^{n-r})$. Let $\Sigma = \Sigma(\lambda, \mu, \nu) = S_p \times S_q \times S_u \times S_v \times S_{n-r}$, as in the theorem.

For a binary table $X = (x_{ijk}) \in B^\circ(\lambda, \mu, \nu')$, denote by $Y = (y_{ij})$, where $y_{ij} := x_{ij1}$, $1 \leq i \leq \ell(\lambda) = 2 + p + q$, and $1 \leq j \leq \ell(\mu) = 2 + r + s$. We think of Y as a block table in the following manner. In the first two rows of Y one can have 1's in either or both rows, giving four possibilities for a pair (y_{1j}, y_{2j}) , for $3 \leq j \leq 2 + u + v$. Same holds for pairs (y_{i1}, y_{i2}) , for $3 \leq i \leq 2 + a + b$. Action of Σ allows one to order these pairs as follows: first two 1's, then one in first row/column, then in second, then in neither. Do these separately for $3 \leq j \leq 2 + u$, $3 + u \leq j \leq 2 + u + v$, and $2 \leq i \leq 2 + a$, $3 + a + \leq i \leq 2 + a + b$, respectively. This divides Y into 9×9 block matrix, and within each of the 64 non-top-boundary block we have row/column sums ≤ 2 .

We are not done, however. Let $X' = (x'_{ij})$ be the projection of X onto the first two coordinates $x'_{ij} = x_{ij1} + x_{ij2} + \dots$, and let $Y' := X' - Y$. Repeat the above procedure for Y' . Overlay block for Y with blocks for Y' . Each 4-block overlaid with 4-blocks give 7 block, giving in total 15×15 block matrix. Now split X vertically as well into 2 blocks of size 1, $n - k$, giving a $15 \times 15 \times 2$ block matrix.

We are now ready to calculate the number of orbits $B^\circ(\lambda, \mu, \nu')$. There is one 2×2 corner block, giving 2^4 possibilities, and n^4 for the block above it. There are $2(14 + 14) = 56$ border blocks, giving at most n^{56} possibilities. Indeed, we already took into account the action of $S_p \times S_q \times S_u \times S_v$ onto the 28 border blocks on the first level, and the action of S_{n-r} reduces a unique configuration of 1's in each remaining block.

Now, for the remaining $2(14 \cdot 14) = 392$ blocks there are at most n^{392} ways to divide the remaining 1's. However, the configuration in each block is no longer unique. Rather, it counts the number of bipartite graphs with degrees at most 2. For the $t \times t$ block sums all equal to 2, this is exactly the number of partitions $p(t)$, since this is the number of conjugacy classes of S_t . Furthermore, the number is smaller when the sums are smaller.

Putting all these (very rough) upper bounds together, we get

$$B^\circ(\lambda, \mu, \nu') \leq 16 \cdot n^4 \cdot n^{56} \cdot n^{392} \cdot p(n)^{392} \leq n^{450} p(n)^{400},$$

as desired. \square

Remark 5.10. Proposition 5.9 gives a subexponential bound $g(\lambda, \mu, \nu) = e^{O(\sqrt{n})}$, which is smaller than the dimension bound. The proof of the theorem does not generalize to triple hooks or three double hooks as the number of configurations inside each block can become exponential. This upper bound is not tight and can be further improved to $g(\lambda, \mu, \nu) \leq 27n^6$ (in preparation). In fact, we conjecture that $g(\lambda, \mu, \nu)$ is at most polynomial when all three partitions have bounded Durfee squares.

5.4. Tensor squares. Let $\lambda = \nu \vdash n$, $\ell(\lambda) = \ell$, $\lambda_1 = m$, $\ell(\nu) = r$. By Theorem 5.3, we have:

$$g(\lambda, \lambda, \nu) = g(\lambda', \lambda, \nu') \leq B(\lambda, \lambda', \nu).$$

Proposition 5.11. *We have: $B(\lambda, \lambda', \nu) \geq f^\nu$.*

The proposition implies that the upper bound in Theorem 5.3 is weaker than the dimension bound for every self-conjugate λ .²

Proof of Proposition 5.11. The result follows from two inequalities:

$$f^\nu \leq \binom{n}{\nu_1, \dots, \nu_r} \leq B(\lambda, \lambda', \nu).$$

The first inequality is a trivial consequence of

$$f^\nu = \chi^\nu(1) \leq \phi^\nu(1) = \binom{n}{\nu_1, \dots, \nu_r},$$

where ϕ^ν is the character of the corresponding permutation representation. The second inequality follows from the following interpretation of the multinomial coefficient. Start with an $\ell \times m$ matrix $X = (x_{ij})$, where $x_{ij} = 1$ if $j \leq \lambda_i$, and $x_{ij} = 0$ otherwise. Consider all binary contingency tables $Y = (y_{ijk}) \in \mathcal{B}(\lambda, \lambda', \nu)$ which project onto X along the third (vertical) coordinate. Because the

²This case is motivated by the *tensor square conjecture* [PPV].

horizontal margins are equal to ν_i , the number of such Y is exactly the multinomial coefficient as above. \square

6. PYRAMIDS APPROACH

6.1. Lower bound. Let $\lambda, \mu, \nu \vdash n$. A 3-dimensional binary contingency table $X = (x_{ijk}) \in \mathcal{B}(\lambda, \mu, \nu)$ is called a *pyramid* if whenever $x_{ijk} = 1$, we also have $x_{pqr} = 1$ for all $p \leq i, q \leq j, r \leq k$. Denote by $\mathcal{Pyr}(\lambda, \mu, \nu)$ the set of pyramids with margins λ, μ, ν . Finally, let $\text{Pyr}(\lambda, \mu, \nu) := |\mathcal{Pyr}(\lambda, \mu, \nu)|$ denote the number of pyramids.

Theorem 6.1 (§8.3). *We have: $\text{Pyr}(\lambda', \mu', \nu') \leq g(\lambda, \mu, \nu)$.*

Example 6.2. In notation of Example 5.6, the unique binary table is a pyramid. This implies that $g(\lambda, \mu, \nu) = 1$ in this case.

Proposition 6.3. *Let $\lambda, \mu, \nu \vdash n$, s.t. $\ell(\lambda) = \ell(\mu) = \lambda_1 = \mu_1 = \ell$. Then $\text{Pyr}(\lambda, \mu, \nu) > 0$ if and only if $\lambda = \mu'$ and $\nu = (n)$.*

Proof. For the ‘if’ part, observe that $\text{Pyr}(\lambda, \lambda', (n)) = 1$. This follows by the definition of a pyramid which becomes a partition in the plane (x_{**1}) .

For the ‘only if’ part, let $X = (x_{ijk}) \in \mathcal{B}(\lambda, \mu, \nu)$. Then $x_{\ell 1 1} = x_{1 \ell 1} = 1$ since the last margins $\lambda_\ell = \mu_\ell = 1$. This implies that $x_{i 1 1} = x_{1 i 1} = 1$ for all $1 \leq i \leq \ell$. On the other hand, since the first margins $\lambda_1 = \mu_1 = \ell$, this implies that $x_{i 1 2} = x_{1 j 2} = 0$. Thus, $x_{i j 2} = 0$ for all $1 \leq i, j \leq \ell$, and $\nu = (n)$. But then $x_{i j 1} = 1$ if and only if $1 \leq j \leq \lambda_i$ for all $1 \leq i \leq \ell$. This implies $\lambda = \mu'$, as desired. \square

Recall the *Saxl conjecture* [PPV], which states that $g(\rho_\ell, \rho_\ell, \nu) \geq 1$ for all $\nu \vdash n = |\rho_\ell| = \ell(\ell+1)/2$, where $\rho_\ell = (\ell, \ell-1, \dots, 1)$ is the *staircase shape*. The Proposition 5.11 implies that Theorem 6.1 cannot be applied to the Saxl conjecture in any nontrivial special case. More generally, the *tensor square conjecture* [PPV] is also unreachable with this approach.

6.2. Explicit construction. It was shown by Stanley [S3] that

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} g(\lambda, \mu, \nu) = \sqrt{n!} e^{-O(\sqrt{n})}$$

In [PPY], we refined this to

$$\frac{f^\lambda f^\mu}{\sqrt{p(n)} n!} \leq \max_{\nu \vdash n} g(\lambda, \mu, \nu) \leq \min\{f^\lambda, f^\mu\},$$

where $p(n)$ is the number of partitions of n . Stanley’s result follows from an easy asymptotic formula:

$$\max_{\lambda \vdash n} f^\lambda = \sqrt{n!} e^{-O(\sqrt{n})}$$

(cf. [VK2]). While we know the asymptotic shape maximizing f^λ , see [VK1], we do not know any explicit construction of $\lambda, \mu, \nu \vdash n$ which satisfy $g(\lambda, \mu, \nu) \geq \exp \Omega(n \log n)$. Here by an *explicit construction* (see e.g. [Wig]), we mean a complexity notion (there is a deterministic poly-time algorithm for generating the triple).³ In fact we do not have a randomized algorithm either in this case, only the existence results as above.

The current best explicit construction of triples in [PP2, Thm 1.2] has $g(\lambda, \mu, \nu) = \exp \Theta(\sqrt{n})$, based on a technical proof using both algebraic and analytic arguments in the case $\lambda = \mu = (\ell^\ell)$, for $\nu = (k, k)$, $n = \ell^2 = 2k$. See also [MPP] which gives precise asymptotics in this case. It is a major challenge to improve upon this (relatively weak) bound. Below we give an elementary explicit construction of a better lower bound.

Theorem 6.4. *There is an explicit construction of $\lambda, \mu, \nu \vdash n$, such that:*

$$g(\lambda, \mu, \nu) = \exp \Omega(n^{2/3}).$$

³For the purposes of this section, a naive combinatorial notion of an ‘explicit construction’ will suffice.

To understand this result, observe the following:

Proposition 6.5. *For some $\lambda, \mu, \nu \vdash n$, we have*

$$\text{Pyr}(\lambda, \mu, \nu) = \exp \Theta(n^{2/3}).$$

Proof. Observe that the number $p_2(n)$ of *plane partitions* of n satisfies:

$$p_2(n) := \sum_{\lambda, \mu, \nu \vdash n} \text{Pyr}(\lambda, \mu, \nu).$$

Recall that the number of triples of margins $\lambda, \mu, \nu \vdash n$ is

$$p(n)^3 = \exp \Theta(\sqrt{n}), \quad \text{while} \quad p_2(n) = \exp \Theta(n^{2/3}),$$

see e.g. [FS, §VIII.24-25]. This implies the result. \square

Lemma 6.6 ([V1, Ex. 3.3]). *For $\alpha = (7, 4, 2) \vdash 13$ we have $\text{Pyr}(\alpha, \alpha, \alpha) = 2$.*

Proof of Theorem 6.4. Fix $N = 13$, and consider two distinct pyramids $X, X' \in \mathcal{Pyr}(\alpha, \alpha, \alpha)$, where $\alpha = (7, 3, 2) \vdash N$ as in the lemma above. Denote $\ell = \ell(\alpha) = 3$.

Let $s \geq 1$. Consider a matrix

$$Y = (y_{ijk}) \in \mathcal{Pyr}(\theta_{s-1}, \theta_{s-1}, \theta_{s-1})$$

given by $y_{ijk} = 1$ for all $i + j + k \leq s + 1$, and $y_{ijk} = 0$ otherwise, so $\theta_{s-1} = ((\binom{s}{2}, \binom{s-1}{2}, \dots, \binom{2}{2}))$. Replace each 1 by an all-1 matrix of size $\ell \times \ell \times \ell$, each 0 with $i + j + k = s + 2$ with X or X' , and the remaining 0's by an all-0 matrix of the same size. There are $2^{s(s+1)/2} = \exp \Omega(s^2)$ resulting pyramids which all have the same margins $(\lambda^{(s)}, \lambda^{(s)}, \lambda^{(s)})$, where

$$n_s := |\lambda^{(s)}| = \ell^3 \cdot (1 + 3 + \dots + s(s-1)/2) + N \cdot \binom{s+1}{2} = \Theta(s^3).$$

By Theorem 6.1, this gives a lower bound

$$g((\lambda^{(s)})', (\lambda^{(s)})', (\lambda^{(s)})') \geq \text{Pyr}((\lambda^{(s)}), \lambda^{(s)}, (\lambda^{(s)})) = \exp \Theta(n_s^{2/3}),$$

as desired. \square

6.3. Summation bounds. Now Theorem 1.3 can be obtained as a corollary of Theorem 6.1 and the asymptotic approach above.

Proof of Theorem 1.3. A plane partition A is called *totally symmetric* if the corresponding pyramid has an S_3 -symmetry, see Case 4 in [Kra, S1] and [OEIS, A059867]. Denote by \mathcal{A}_n the set of totally symmetric plane partitions of n . By the symmetry, the margins of $A \in \mathcal{A}_n$ are triples $(\lambda, \lambda, \lambda)$, where $\lambda \in \mathcal{L}_n$. From Theorem 6.1, we have:

$$\sum_{\lambda \in \mathcal{L}_n} g(\lambda, \lambda, \lambda) \geq \sum_{\lambda \in \mathcal{L}_n} \text{Pyr}(\lambda, \lambda, \lambda) \geq |\mathcal{A}_n|.$$

It is easy to see by an argument similar to the proof of Theorem 6.4 above (or from the explicit product form of the GF), that:

$$|\mathcal{A}_n| = \exp \Omega(n^{2/3}).$$

This completes the proof. \square

Recall that $|\mathcal{L}_n| = \exp \Theta(n^{1/2})$, see e.g. [OEIS, A000700]. It was shown in [BB] that $g(\lambda, \lambda, \lambda) \geq 1$ for all $\lambda \in \mathcal{L}_n$. This gives only the $\exp \Omega(n^{1/2})$ lower bound for the LHS in Theorem 1.3. Of course, we believe a much stronger bound lower holds:

Conjecture 6.7. *Let $\mathcal{L}_n := \{\lambda \vdash n, \lambda = \lambda'\}$. We have:*

$$\sum_{\lambda \in \mathcal{L}_n} g(\lambda, \lambda, \lambda) = \exp \left[\frac{1}{2} n \log n + O(n) \right].$$

In the conjecture, the upper bound follows from equation (2.1). We refer to [PPY, §3] for partial motivation behind this conjecture.

7. BOUNDS FOR THE REDUCED KRONECKER COEFFICIENTS

The *reduced Kronecker coefficients* were introduced by Murnaghan in 1938 as the stable limit of *Kronecker coefficients*, when a long first row is added:

$$(7.1) \quad \bar{g}(\alpha, \beta, \gamma) := \lim_{n \rightarrow \infty} g(\alpha[n], \beta[n], \gamma[n]),$$

where

$$\alpha[n] := (n - |\alpha|, \alpha_1, \alpha_2, \dots), \quad \text{for all } n \geq |\alpha| + \alpha_1,$$

see [Mur1, Mur2]. They generalize the classical *Littlewood–Richardson (LR–) coefficients*:

$$\bar{g}(\alpha, \beta, \gamma) = c_{\beta\gamma}^\alpha \quad \text{for } |\alpha| = |\beta| + |\gamma|,$$

see [Lit]. As such, they occupy the middle ground between the Kronecker and the LR–coefficients. We apply the results above to give an upper bound for the reduced Kronecker coefficients with few rows. Note that only other upper bound we know for $\bar{g}(\alpha, \beta, \gamma)$ is the upper bound for the maximal value given in [PP4].

Theorem 7.1. *Let $\alpha \vdash a$, $\beta \vdash b$, $\gamma \vdash c$, such that $\ell(\alpha) = \ell$, $\ell(\beta) = m$, and $\ell(\gamma) = r$. Denote $N := a + b + c$. Then:*

$$\bar{g}(\alpha, \beta, \gamma) \leq \sum_{n=0, n=N \bmod 2}^{\min\{a,b,c\}} \mathbf{E}(\ell m r, n) \cdot \mathbf{E}(\ell m, v - c) \cdot \mathbf{E}(\ell r, v - b) \cdot \mathbf{E}(m r, v - a),$$

where

$$\begin{aligned} \mathbf{E}(s, w) &:= \left(1 + \frac{s}{w}\right)^w \left(1 + \frac{w}{s}\right)^s \quad \text{for } w > 0, \\ \mathbf{E}(s, 0) &:= 1, \quad \text{and } v := \frac{1}{2}(N - 3n). \end{aligned}$$

The proof follows the previous pattern. We begin with the following combinatorial lemma.

Lemma 7.2.

$$g(\alpha, \beta, \gamma) \leq \mathbf{R}(\alpha, \beta, \gamma),$$

where

$$\mathbf{R}(\alpha, \beta, \gamma) = \sum_{\Phi(\alpha, \beta, \gamma)} \mathbf{T}(\lambda, \mu, \nu) \mathbf{T}(\pi, \rho) \mathbf{T}(\sigma, \tau) \mathbf{T}(\eta, \zeta),$$

and

$$\Phi(\alpha, \beta, \gamma) := \{(\lambda, \mu, \nu, \pi, \rho, \sigma, \tau, \eta, \zeta) \in \mathcal{P}^9, (\alpha, \beta, \gamma) = (\lambda, \mu, \nu) + (\pi, \rho, \emptyset) + (\sigma, \emptyset, \tau) + (\emptyset, \eta, \zeta)\}.$$

Proof. The result follows from the identity given recently in [BR], and by taking the coefficients in $\mathbf{x}^\alpha \mathbf{y}^\beta \mathbf{z}^\gamma$ on both sides:

$$\begin{aligned} & \sum_{(\alpha, \beta, \gamma) \in \mathcal{P}^3} \bar{g}(\alpha, \beta, \gamma) s_\alpha(\mathbf{x}) s_\beta(\mathbf{y}) s_\gamma(\mathbf{z}) \\ &= \left[\prod_{i,j,k=1}^{\infty} \frac{1}{1 - x_i y_j z_k} \right] \left[\prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j} \right] \left[\prod_{i,k=1}^{\infty} \frac{1}{1 - x_i z_k} \right] \left[\prod_{j,k=1}^{\infty} \frac{1}{1 - y_j z_k} \right] \\ &= \sum_{(\alpha, \beta, \gamma) \in \mathcal{P}^3} \mathbf{R}(\alpha, \beta, \gamma) m_\alpha(\mathbf{x}) m_\beta(\mathbf{y}) m_\gamma(\mathbf{z}). \end{aligned}$$

Here the nine partitions in the definition of $\Phi(\alpha, \beta, \gamma)$ come from a combinatorial interpretation of one 3-dimensional and three 2-dimensional contingency tables. \square

Proof of Theorem 7.1. In notation of Lemma 7.2, denote $n := |\lambda| = |\mu| = |\nu|$. Note that $n \leq a, b, c$, and has the same parity as $N = a + b + c$. Observe that n determines the remaining six partitions sizes: $\pi, \rho \vdash v - c$, $\sigma, \tau \vdash v - b$, and $\eta, \zeta \vdash v - a$. By assumption as in the theorem, we have an upper bound on the number of rows of all nine partitions: $\ell(\lambda), \ell(\pi), \ell(\sigma) \leq \ell$, $\ell(\mu), \ell(\rho), \ell(\eta) \leq m$, and $\ell(\nu), \ell(\tau), \ell(\zeta) \leq r$. In notation of the theorem, from the proof in the previous section we obtain:

$$\mathbb{T}(\lambda, \mu, \nu) \leq G((n/\ell)^\ell, (n/m)^m, (n/r)^r) = \mathbb{E}(\ell m r, n).$$

The same holds for the remaining pairs of partitions:

$$\mathbb{T}(\pi, \rho) \leq G(((v-c)/\ell)^\ell, ((v-c)/m)^m) = \mathbb{E}(\ell m, v-c),$$

$$\mathbb{T}(\sigma, \tau) \leq G(((v-b)/\ell)^\ell, ((v-b)/r)^r) = \mathbb{E}(\ell r, v-c),$$

$$\mathbb{T}(\eta, \zeta) \leq G(((v-a)/m)^m, ((v-a)/r)^r) = \mathbb{E}(m r, v-c).$$

Note that in the case of empty partitions of zero, we have a unique zero contingency table, giving $\mathbb{T}(\cdot) = \mathbb{E}(\cdot, 0) = 1$, by definition in the statement of the theorem. Putting all these bounds into the main inequality in Lemma 7.2, implies the result. \square

Example 7.3. Suppose $a = N/6$, $b = N/3$ and $c = N/2$, with $\ell = m = r = N^{1/3}$. Then the sum over n in Theorem 7.1 maximizes at $n = a = N/6$. Thus $v = N/2$, and we have:

$$\begin{aligned} \bar{g}(\alpha, \beta, \gamma) &\leq (n/2) \cdot \mathbb{E}(N, N/6) \cdot \mathbb{E}(N^{2/3}, 0) \cdot \mathbb{E}(N^{2/3}, N/6) \cdot \mathbb{E}(N^{2/3}, N/3) \\ &\leq \left(1 + \frac{1}{6}\right)^N (1+6)^{N/6} \cdot \exp O(N^{2/3}) \\ &\leq \exp [cN + O(N^{2/3})]. \end{aligned}$$

where $c = (\log 7/6) + (\log 7)/6 \approx 0.4785$.

Note that in the limit (o) it suffices to take $n \geq |\alpha| + |\beta| + |\gamma|$, see [BOR, V2]. Since the number of rows increases only by 1 under the limit, Theorem 1.1 still gives an exponential bound for $\bar{g}(\alpha, \beta, \gamma)$, but with a much larger constant.

8. FINAL REMARKS AND OPEN PROBLEMS

8.1. There are two distinct motivations behind our work. First, the Kronecker coefficients are famously difficult and mysterious, full of open problems such as the *Saxl Conjecture* [PPV] (see also [BBS, Ike]). This means that there are very few strong results and those that exist are not very general, so our general bounds can prove helpful in applications.

More importantly, the Kronecker coefficients are famously #P-hard to compute, and NP-hard to decide if they are nonzero, so one should not expect a closed formula, see [IMW, Nar, PP1]. Furthermore, it is a long standing open problem [S3] to find a combinatorial interpretation for Kronecker coefficients, so it is not even clear *what* we are counting. Thus, good general bounds is the next best thing one could hope for.

This paper is meant to be the first in a series of papers on the subject of bounds on Kronecker coefficients; the forthcoming papers are based on Sra–Khare–Tao majorization tools [PPP] and Féray–Śniady character bounds [D+]. We should mention that this paper is a substantially truncated version of the arXiv preprint [PP3] which has upper bounds on Kronecker coefficients via Kostka numbers and via multi-LR coefficients combined with LR-bounds from [PPY]. Unfortunately, all these bounds are weaker than Theorem 3.1 except sometimes for the lower order terms.

8.2. Counting contingency tables, both general and binary, is a very large subject with numerous approaches and many techniques. We refer to [B4] and [Wor] for recent broad survey (the latter is only in the binary case). Theorems 3.1 and 5.1 are taken from Barvinok [B4, §3]. They are based on results of Barvinok [B2, B3], Benson-Putnins [Ben] and Shapiro [Sha].

8.3. The history of theorems 5.3 and 6.1 is a bit confusing. The upper bound in Lemma 1.2 is new, yet an easy consequence of the standard symmetric functions identities. The binary version can be proved in a similar way (see [PP3, §7.1]), and has been rediscovered multiple times. Notably, Theorem 5.3 follows from [JK, Lemma 2.9.16] and [Man] (in a different context). Theorem 6.1 is a special case of [V3, Cor. 3.5] and a version of Theorem 5.3 were given in [V3]. Both theorems are proved in this form in [IMW, Lemma 2.6]. Finally, Theorem 5.4 is given in [BI, Thm. 4.1] in an equivalent form. It implies Theorem 5.3, of course.

8.4. In the context of §6, we believe in the following claims.

Conjecture 8.1. *Denote by $a(n)$ and $b(n)$ the number of triples (λ, μ, ν) , $\lambda, \mu, \nu \vdash n$, s.t. $B(\lambda, \mu, \nu) \geq 1$ and $\text{Pyr}(\lambda, \mu, \nu) \geq 1$, respectively. Then:*

$$\frac{a(n)}{p(n)^3} \rightarrow 1 \quad \text{and} \quad \frac{b(n)}{p(n)^3} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, we believe that the Kronecker coefficients are non-vanishing a.s.

Conjecture 8.2. *Denote by $c(n)$ the number of triples (λ, μ, ν) , such that $\lambda, \mu, \nu \vdash n$ and $g(\lambda, \mu, \nu) \geq 1$. Then:*

$$\frac{c(n)}{p(n)^3} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

These conjectures are motivated by Conjecture 8.3 in [PPV] which states that $\chi^\lambda[\mu] \neq 0$ a.s. In the opposite direction, it is known that the Kostka numbers $K(\lambda, \mu) = 0$ a.s. This is equivalent to the former *Wilf's conjecture* that the probability $P(\lambda \leq \mu) \rightarrow 0$ for uniform random $\lambda, \mu \vdash n$. This conjecture was resolved in [Pit].

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