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Increasing trees and alternating permutations[1]

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Contents

§1. Introduction

In this article we consider some increasing trees, the number of which is equal to the number of alternating (updown) permutations, that is, permutations of the form $\sigma(1) < \sigma(2) > \sigma(3) < \dots$. It turns out that there are several such classes of increasing trees, each of which is interesting in itself. Special attention is paid to the study of various statistics on these trees, connected with the André polynomials and the Foata group on the one hand, and the Entringer numbers, which arise on grading alternating permutations from the first element, on the other. The proofs of most of the assertions are based on the construction of explicit bijections between classes of trees and the study of their properties. We also obtain new combinatorial identities for the Euler and Bernoulli numbers.

In §2 we give basic definitions and known theorems concerned with alternating permutations, binary and increasing trees. We define 0-1-2 trees, that is, trees such that at most two edges go out from any vertex, we establish a bijection between them and orbits of the action of the Foata group on binary trees, and consider various statistics.

In §3 we define even trees, that is, trees such that an even number of edges go out from each non-rooted vertex. We find an explicit bijection between them and alternating permutations, and establish a connection with the inversion polynomial and the Tutte dichromate of a complete graph.

[1] This translation incorporates some corrections agreed with the authors (Ed.).

In §4 we consider increasing trees with branchings on even levels and various statistics on them. We find a bijection between them and 0-1-2 trees, using some binary trees as an intermediate result.

In §5 we introduce embedded and weakly embedded trees, and study the statistics on them. It turns out that embedded trees are representatives of the orbits of the action of the Foata group on all increasing trees. We establish a connection with the Catalan and Bell numbers.

In §6 the authors consider geometric classes of trees, that is, sets of increasing trees such that if some tree is contained in it, then all trees isomorphic to it are also contained in it as unlabelled trees. We prove a general theorem on the equivalence of certain statistics on a given geometric class of trees, and find applications to the classes of trees listed above.

In §7 we give proofs of the theorems in §§3–6, and in §8 we formulate further perspectives of the development of this theme.

At the end of the paper, for the reader's convenience there are small tables of the classes of trees we have considered.

In the course of the article we shall keep to the following notation:

$[n] = \{1, 2, \dots, n\}$,

$[\bar{n}] = \{0, 1, 2, \dots, n\}$,

$\mathbb{N}$ is the set of natural numbers,

$\mathbb{Z}_+$ is the set of non-negative integers,

S_n is the group of permutations of elements of the set $[n]$.

Classes of increasing trees are denoted by upper case Roman letters, and the index denotes the number of edges in them. We shall denote bijections and statistics on trees by lower case Greek letters. All the notations are collected in a table at the end of the article.

The authors thank D. Foata and M.-P. Schutzenberger: this article was written under the influence of their results and ideas. The authors also thank V.I. Arnol'd for his interest in the theme of this work.

§2. Alternating permutations and 0-1-2 trees

2.1. Let us recall some well-known definitions (see [1], [2]).

A permutation $\sigma \in S_n$ is called *alternating* (or *updown*) if $\sigma(1) < \sigma(2) > \sigma(3) < \dots$. We denote the set of such permutations by Alt_n; $\mathrm{Alt}_{n,k} := \{\sigma \in \mathrm{Alt}_n | \sigma(1) = k\}$. We put $a_n := |\mathrm{Alt}_n|$, $a_{n,k} := |\mathrm{Alt}_{n,k}|$. The numbers $a_{n,k}$ are called the *Entringer numbers*.

Examples.

$$\begin{aligned}
&\mathrm{Alt}_1 = \{(1)\}; \quad \mathrm{Alt}_2 = \{(12)\}; \quad \mathrm{Alt}_3 = \{(132), (231)\};\\
&\mathrm{Alt}_4 = \{(1324), (1423), (2314), (2413), (3412)\};\\
&\mathrm{Alt}_5 = \{(13254), (14253), (14352), (15243), (15342),\\
&\qquad (23154), (24153), (24351), (25143), (25341),\\
&\qquad (34152), (34251), (35142), (35241), (45132), (45231)\}.
\end{aligned}$$

It is convenient to put the numbers $a_{n,k}$ in the form of a triangle, which is called the *Euler–Bernoulli triangle* (see Fig. 1). Its left-hand side, on which are the numbers $a_{2n+1,1} = a_{2n}$, is called the *Euler side*; the right-hand side, with the numbers $a_{2n,1} = a_{2n-1}$, is called the *Bernoulli side*. For the first few values of $a_{n,k}$ see Fig. 1.

$$
\begin{array}{ccccccccc}
 & & & & a_{1,1} & & & & \\
 & & & a_{2,2} & & a_{2,1} & & & \\
 & & a_{3,1} & & a_{3,2} & & a_{3,3} & & \\
 & a_{4,4} & & a_{4,3} & & a_{4,2} & & a_{4,1} & \\
a_{5,1} & & a_{5,2} & & a_{5,3} & & a_{5,4} & & a_{5,5} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 1 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 0 & & 1 & & 2 & & 2 & \\
5 & & 5 & & 4 & & 2 & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Fig. 1

***Proposition* 1.** *Suppose that* $1 \leqslant k < n$. *Then*

(1) $$a_{n,k} = a_{n,k+1} + a_{n-1,n-k}.$$

Proof. We construct the explicit bijection

$$\gamma\colon \mathrm{Alt}_{n,k+1} \sqcup \mathrm{Alt}_{n-1,n-k} \to \mathrm{Alt}_{n,k}$$

Suppose that $\sigma \in \mathrm{Alt}_{n,k+1}$, $\sigma' \in \mathrm{Alt}_{n-1,n-k}$. Suppose also that $\sigma_k = (k, k+1) \in S_n$, $\sigma'' = (n, n-\sigma'(1), \ldots, n-\sigma'(n-1))$. We put

$$\gamma(\sigma) = \sigma_k \circ \sigma, \qquad \gamma(\sigma') = \sigma'' \circ \sigma_{n-1} \circ \sigma_{n-2} \circ \cdots \circ \sigma_k.$$

Then

$$\mathrm{Alt}_{n,k} \setminus \gamma(\mathrm{Alt}_{n,k+1}) = \{\sigma \in \mathrm{Alt}_n,\ \sigma(1) = k,\ \sigma(2) = k+1\} = \gamma(\mathrm{Alt}_{n-1,n-k}),$$

which completes the proof of the proposition.

***Proposition* 2.**

(2) $$a_{n,k} = \sum_{i=0}^{[\frac{k-1}{2}]} (-1)^i \binom{k-1}{2i} a_{n-2i-1}, \qquad 1 \le k < n;$$

(3) $$a_{n,k} = \sum_{i=0}^{[(n-k-1)/2]} (-1)^i \binom{n-k}{2i+1} a_{n-2i-2}, \qquad 1 < k \le n.$$

Proof. We observe that (1) is a relation between adjacent numbers in the Euler–Bernoulli triangle. Applying (1) repeatedly, we express $a_{n,k}$ in terms of the numbers on the sides of the triangle. Clearly, $a_{n,1} = a_{n-1}$ and $a_{n,n} = 0$. Depending on the side of the triangle and the parity of n we obtain (2) and (3).

***Proposition* 3.** *We have the following relations:*

$$a_{2n+1} = \sum_{i=1}^{n} \binom{2n}{2i-1} a_{2i-1} a_{2n-2i+1}; \tag{4}$$

$$a_{2n+1} = \sum_{i=0}^{n} \binom{2n}{2i} a_{2i} a_{2n-2i}; \tag{5}$$

$$a_{2n} = \sum_{i=0}^{n-1} \binom{2n-1}{2i} a_{2i} a_{2n-2i-1}. \tag{6}$$

Proof. It is easy to see that $|\{\sigma \in \mathrm{Alt}_n, \sigma(j) = 1\}|$ is equal to $\binom{n-1}{j-1} a_{j-1} a_{n-j}$ for odd j, and equal to 0 for even j. Summing over odd j we obtain (5) and (6). Similarly (4) is obtained by considering the places where n can be.

Consider the exponential generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n/n!$, where $a_0 = a_{1,1} = 1$. Then from (4), (5), (6) we obtain the differential equation

$$2a'(x) = 1 + a(x)a(x). \tag{7}$$

We observe that $\tan(x) + \sec(x)$ satisfies (7). Taking account of the initial condition $a(0) = 1$, we immediately obtain

$$a(x) = \tan(x) + \sec(x). \tag{8}$$

The expression (8) can also be obtained without using (7). In fact, putting k equal to 1 and n in (2), (3) and taking account of the fact that $a_{n,1} = a_{n-1}$, $a_{n,n} = 0$, we obtain

$$a(x) \cdot \cos(x) = 1 + \sin(x), \tag{9}$$

from which (8) follows immediately.

We observe that $\tan(x)$ contains only odd terms, and $\sec(x)$ only even terms in the expansion. $E_n = a_{2n}$ are the Euler numbers, and $B_n = 2n \cdot 2^{-2n}(2^{2n} - 1)^{-1} a_{2n-1}$ are the Bernoulli numbers (see [3]).

The problem of listing alternating permutations was first solved by André (see [4], [5]). The numbers $a_{n,k}$ were introduced by Entringer (see [6], [7]). We shall be interested in combinatorial interpretations of these numbers, as numbers of certain trees.

2.2. We define a *tree* with n non-rooted vertices as a basic rooted tree in the complete graph K_{n+1} with vertices from the set $[\bar{n}]$, oriented from the root at 0 (see [8]–[10]).

We define an *increasing tree* as a tree whose vertices increase along the edges. We denote the set of trees and the set of increasing trees with n non-rooted vertices by F_n and U_n respectively. A classic theorem of Cayley asserts that $|F_n| = (n+1)^{n-1}$ (see [1], [8], [9]). On the other hand, $|U_n| = n!$. In fact, we can add the vertex k to a tree $t \in U_{k-1}$ in exactly k different ways. Hence, by induction, we immediately deduce that $|U_n| = n \cdot (n-1) \cdot \ldots \cdot 1 = n!$.

We define a *binary tree* on a set X as a structure defined by induction:

a) it is empty if $X = \emptyset$;

b) it consists of an *upper vertex* $x_0 \in X$ and *left* and *right* branches, that is, binary trees on sets L and R respectively such that $X = L \cup R \cup \{x_0\}$ and $|X| = |L| + |R| + 1$ (see [11], for example).

In what follows, unless we say otherwise we shall consider binary trees on the set $[n]$ such that in each branch the upper vertex is less than all the vertices of this branch. We denote the set of such binary trees by B_n. We shall represent a binary tree as an increasing tree that joins each upper vertex to the upper vertices of the left and right branches by edges (if these branches are not empty), calling them *left* and *right* edges respectively.

It is important to note that a binary tree is not a tree in the traditional sense, but is a new structure which, as we see, takes an intermediate position between increasing trees and permutations.

Let us construct a simple bijection $\psi : S_n \to B_n$ by induction. Suppose we can associate with the permutations of some set $I \subset [n]$ a binary tree with vertex set I for all I such that $|I| < k$. Consider a permutation $\sigma : I \to I$, $|I| = k$. Let m be the minimal element in I. Then σ has the form $(\sigma' m \sigma'')$. We construct a new tree with vertex m whose left and right branches are trees corresponding to σ' and σ'' respectively (see Fig. 2). It is easy to see that by extending this process we obtain a binary tree (see Fig. 3). The fact that this map is bijective is also obvious (see [1], [12]). Hence, in particular, we immediately deduce that $|B_n| = n!$.

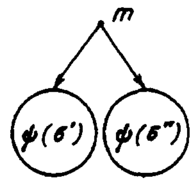


Fig. 2

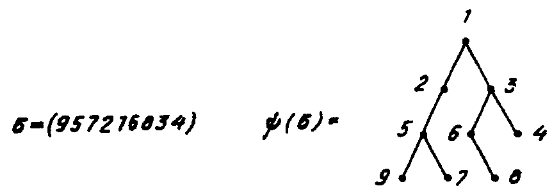


Fig. 3

We put $V_n = \psi(\mathrm{Alt}_n) \subset B_n$, $V_{n,k} = \psi(\mathrm{Alt}_{n,k}) \subset V_n$, hence $|V_n| = a_n$, $|V_{n,k}| = a_{n,k}$. It is easy to see that V_n consists of those binary trees with n vertices such that either 0 or 2 edges go out from each vertex, except possibly the extreme right vertex. We call them 0-2 *binary trees.* It is also obvious that a 0-2 binary tree $t \in V_{n,k}$ if its extreme left lower vertex is k.

Following established tradition (see [12], for example) we shall understand a *statistic* on a set X to be a grading, that is, a map $\tau : X \to \mathbb{Z}_+$. Two statistics (gradings) τ and τ' on sets X and X' respectively are said to be *equivalent* if for any $k \in \mathbb{Z}_+$, $|X_\tau(k)| = |X'_{\tau'}(k)|$, where $X_\tau(k) = \{x \in \xi; \tau(X) = k\}$.

In this language, on the set $S_n(\mathrm{Alt}_n)$ there is defined a statistic of the first element of a permutation, which is equivalent to the statistic on B_n (V_n) of the extreme lower left vertex. We denote the first statistic by ε, in honour of Entringer, and the second by $\psi(\varepsilon)$.

We now construct the important bijection $\varphi : U_n \to B_n$ (see [11], for example). Suppose, by induction, that we have already constructed binary trees corresponding to the branches of our increasing tree, obtained by removing the edges going from 0. We order the roots of these branches in ascending order and join them to the chain going left. We then join each of them by an edge going right to the corresponding binary tree (see Fig. 4). It is obvious that the resulting map φ is bijective (see Fig. 5).

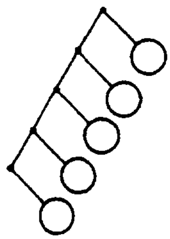


Fig. 4

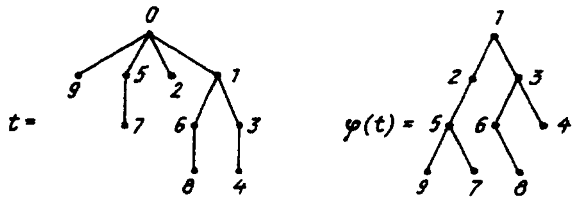


Fig. 5

The map φ enables us to carry over all facts about trees and statistics from increasing trees to binary trees and vice versa. For example, $\alpha = \varphi^{-1}(\psi(\varepsilon))$ is a grading on increasing trees from the maximal vertex joined to the root by an edge. However, we shall not consider $\varphi^{-1}(V_n)$. We note that the map $\varphi^{-1} \circ \psi : S_n \to U_n$ coincides with a map by means of records (left-to-right maximum) (see [12], and also [2]).

2.3. Let us define the *Foata group* F_n, acting on B_n. Suppose it is generated by generators $s_1, s_2, \ldots, s_{n-1}$ such that s_i acts on a tree $t \in B_n$ by replacing the right edge going from vertex i by the left, and the left by the right (if they exist).

It is easy to see that $s_i s_j = s_j s_i$, that is, the generators commute. Taking into account also that $s_i^2 = 1$, we immediately obtain $G_n \cong \mathbb{Z}_2^{n-1}$ (see [13], [14]). (Here we are giving a description of the Foata group that is somewhat different from the original.)

Let L_n be the set of increasing trees with n non-rooted vertices from which at most two edges go out. We call such trees 0-1-2 *increasing trees*. Then there is an obvious bijection $\pi : B_n/G_n \to L_{n-1}$ between orbits of the action of the Foata group and 0-1-2 increasing trees. The most important fact is that $|L_n| = a_{n+1}$. This was first formulated by Foata (see [13]) and was proved by Foata and Strehl (see [14], and also the bijection in [15]).

***Theorem* 1.** $|L_n| = a_{n+1}$.

For the reader's convenience we present the proofs of most theorems in §7. Here, however, we give a simple bijective proof due to Donaghey (see [16]).

Proof. Let us construct a map $\lambda : S_n \to B_n$ obtained by modifying the bijection ψ and the complementation operation $\iota : S_n \to S_n$, that is,

$$\iota(\sigma) := (n + 1 - \sigma(1), \dots, n + 1 - \sigma(n)), \qquad \sigma \in S_n.$$

Suppose that $\sigma : I \to I$, $I \subset [n]$ has the form $(\sigma' m_1 \sigma'')$ and $m_2 \in \sigma''$, where m_1 and m_2 are the minimal and maximal elements of I respectively. Then we must consider the vertex m_1 from which a left edge goes to $\lambda(\sigma')$ and a right edge goes to $\lambda(\sigma'')$, and form the bijection by induction, beginning with $I = [n]$. If $m_2 \in \sigma'$, then after the natural generalization of ι by the permutation $\sigma : I \to I$, $\iota(\sigma)$ will have the form $\iota(\sigma) = (\sigma' m_1 \sigma'')$, $m_2 \in \sigma''$ and we can proceed analogously. This completes the construction of the map λ (see Fig. 6).

$\sigma = (25371684)$

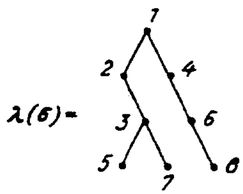


Fig. 6

We now prove that $\pi(\lambda(\mathrm{Alt}_n)) = L_{n-1}$. In fact, $\lambda(S_n)$ consists of those binary trees in which the maximal vertex of any subtree lies in its right branch. Obviously such binary trees are representatives of the orbits of the action of the Foata group. On the other hand, it is easy to show by induction that the inverse image L^{-1} of each such tree contains exactly one permutation $\sigma \in \mathrm{Alt}_n$ and $\tilde{\sigma} \in \iota(\mathrm{Alt}_n)$ (see [16]). Hence it follows immediately that $\pi \circ \lambda : \mathrm{Alt}_n \to L_{n-1}$, is bijective, which proves Theorem 1.

We put $LB_n = \varphi(L_n)$. It is easy to see that LB_n is the set of binary trees such that the chain of edges going from any vertex to the left of it consists of at most one edge. By Theorem 1, $|LB_n| = a_{n+1}$.

We define a *minimal descent* as a chain of edges in an increasing tree, the first of which goes out from the root and the last goes into the end vertex, and each of the edges goes into the minimal vertex of those into which the edges go that go out from this vertex. We denote by β the statistic on U_n of the end vertex of the minimal descent. We note that $\varphi(\beta)$, the statistic on B_n of the lower right vertex, goes over to the statistic $\varphi(\alpha) = \psi(\varepsilon)$ under the action by the element $(s_1 \cdot s_2 \cdot \ldots \cdot s_{n-1}) \in G_n$.

We denote by δ the statistic on U_n of the vertex from which an edge goes to n. It turns out that the statistics β and δ on L_n are equivalent to the statistic ε on Alt_n up to a shift. More precisely:

***Theorem* 2.** *The number of trees $t \in L_{n-1}$ such that $\beta(t) = n - k$ is equal to the number of trees $t \in L_n$ such that $\delta(t) = n - k - 1$, and is equal to $a_{n,k}$.*

This theorem was first formulated and proved by Poupart (see [17]). He was also the first to find the statistics β and δ. We also note that he studied the statistics β and δ on the set V_n/G_n (see [18]), where they are also equivalent up to a shift. We shall not study such trees in this paper. (In §6 we prove the equivalence of β and δ on all geometric classes of trees.)

2.4. Let us consider an important statistic ν on U_n of the number of end vertices. We denote by $d_{n,k}$ the number of 0-1-2 increasing trees with k end vertices. Then by Theorem 1 we have

$$d_{n,1} + d_{n,2} + \cdots + d_{n,[(n+1)/2]} = a_{n+1}, \tag{10}$$

since 0-1-2 trees with $2n - 1$ or $2n$ vertices cannot have more than n end vertices.

Consider the orbit O_t of the Foata group G_n corresponding to the 0-1-2 increasing tree t, that is, $O_t = \pi^{-1}(t) \subset B_n$. Then $|O_t| = 2^{m_t}$, where m_t is the number of generators s_i, $i \in [n-1]$, that act non-trivially, that is, m_t is the number of non-end vertices of the tree t. Thus, $m_t = n - \nu(t)$, that is, we have obtained the relation

$$\sum_{k=1}^{[\frac{n+1}{2}]} d_{n-1,k} \cdot 2^{n-k} = n!, \quad \text{since} \quad \sum_{t \in L_n} |O_t| = |B_n| = n!. \tag{11}$$

***Proposition* 4.** *We have the relations*

$$d_{n,1} = 1 \tag{12}$$

$$d_{n,k} = k \cdot d_{n-1,k} + (n + 3 - 2k) d_{n-1,k-1}, \qquad 1 < k \le (n-1)/2, \tag{13}$$

$$d_{2n,n+1} = a_{2n+1}/2^n. \tag{14}$$

Proof. The relation (12) is obvious, since there is exactly one tree with one end vertex, namely a chain of $n - 1$ edges. Suppose we have fixed a tree $t \in L_{n-1}$, $\nu(t) = k$. Then t obviously has $k - 1$ vertices from which two edges go out, $n - 2k - 1$ vertices from which one edge goes out, and k end vertices. Consequently, we can add a new nth vertex in $(n - 2k - 1) + k = (n - k - 1)$

ways, and k of the resulting trees have k vertices, and $n-2k-1-k+1$ end vertices. Hence (13) follows immediately. The proof of (14) is based on a consideration of the action of the Foata group G_n. In fact, if $t \in L_{2n+1}$ and $v(t) = n+1$, then either 0 or 2 edges go out from each vertex of t. Consequently, $O(t) \subset V_{2n+1}$, but $|O(t)| = 2^{(2n+1)-(n+1)} = 2^n$; hence (14) follows immediately.

We note that the relations (12) and (14) uniquely define a two-index sequence of numbers $d_{n,k}$. The existence of the numbers $d_{n,k}$ in studying ascents and descents on permutations connected with the Euler and André polynomials is considered in [15] (see also [2] and the references listed there). We shall not touch on this question here (however, see §8).

§3. Even trees

3.1. This is a new class of increasing trees, which we shall study.

Definition 1. An increasing tree is said to be *even* if an even number of edges go out from each non-rooted vertex.

We denote the set of even trees with n non-rooted vertices by E_n.

***Theorem* 3.** $|E_n| = a_n$.

For examples of even trees see the tables at the end of the article.

The number of even trees $t \in E_n$ such that $v(t) = k$ is equal to $d_{n,n-k+1}$.

***Theorem* 4.** *The number of even trees $t \in E_n$ such that $\delta(t) = k-1$ is equal to the number of $t \in E_n$ such that $\beta(t) = k$, and is equal to $a_{n,k}$.*

Suppose that $\sigma = (\sigma_1, \sigma_2 \dots \sigma_n) \in \mathrm{Alt}_n$, $\sigma_{i_1} = 1$ is the minimal element of the set $\{\sigma_1 \dots \sigma_n\}$, σ_{i_2} is the maximal element of the set $\{\sigma_{i_1+1} \dots \sigma_n\}$, σ_{i_3} is the minimal element of the set $\{\sigma_{i_2+1} \dots \sigma_n\}$, and so on up to $\sigma_{i_k} = \sigma_n$. We call the elements $\sigma_{i_1}, \sigma_{i_2}, \dots, \sigma_{i_k}$ *distinguished.*

***Theorem* 5.** *The number of trees $t \in E_n$ such that exactly k edges go out from the root is equal to the number of alternating permutations $\sigma \in \mathrm{Alt}_n$ with k distinguished elements.*

For the proofs of Theorem 5 and the first part of Theorem 4 see §7.

3.2. For the proof of Theorem 3 we need an additional construction, which is of interest in itself. We denote by R_n the set of integer sequences $(z_1, z_2, \dots, z_n)$ in which $1 \leqslant z_i \leqslant n+1-i$. Let $Z_n \subset R_n$ be the set of sequences such that $z_1 \leqslant z_2 > z_3 \leqslant z_4 > z_5 \leqslant \dots$. Obviously $|R_n| = n!$. We construct a bijection $\omega : S_n \to R_n$ such that $\omega(\mathrm{Alt}_n) = Z_n$.

Suppose that $\sigma \in S_n$. We put $\omega(\sigma) = (z_1, z_2, \dots, z_n)$, where $z_i = \#\{j \mid i \leqslant j \leqslant n,\ \sigma(j) \leqslant \sigma(i)\}$. It is easy to see that ω is a bijection; the inequalities for $\omega(\mathrm{Alt}_n)$ are verified directly, from which it follows at once that $|Z_n| = a_n$. We note that $\omega(\sigma)$ is the classic vector of inversions of σ (see [1], [11], [12], for example). In what follows it will be important that

$[\omega(\sigma)](1) = \sigma(1)$, that is, $\omega(\varepsilon)$ is also the statistic of the first element of the sequence. For examples of Z_n see the table.

We now construct a bijection $\mu : Z_n \to E_n$ by successively joining the vertices of $[\overline{n}]$ by edges. More precisely, the bijection is a result of the working of an algorithm consisting of $n+1$ steps, where the results of working the kth step ($k \leqslant n$) is a forest (that is, a graph without cycles) on the set of vertices $I_k \subset [n]$. We denote by J_k the set of roots of the connected components of the forest obtained after completing the kth step. At the initial moment we put $I_0 = J_0 = \varnothing$. Let $\widetilde{J}_k = J_k \cup \{0; n+1\}$. We represent $\widetilde{J}_k$ in the form $\{0 = j_{l+1} < j_l < \ldots < j_1 < j_0 = n+1\}$, where l is the number of roots of the forest after the kth step. We also consider the set

$$M := \{m \in [n] \setminus I_k \mid j_{2p+1} < m < j_{2p},\ p \in \mathbb{Z}_+\} = \{m_1 < m_2 < \cdots < m_s\}.$$

If k is even we put $i_{k+1} = m_{z_{k+1}}$, and if k is odd we put $i_{k+1} = m_{n-k+2-z_{k+1}}$. Suppose that $j_{2p+1} < i_{k+1} < j_{2p}$. Then we add to the forest the vertex i_{k+1} and the edges that go out from i_{k+1} and go to the vertices $j_1, j_2, \ldots, j_{2p}$. Now $I_{k+1} := I_k \cup \{i_{k+1}\}$, $J_{k+1} := J_k \cup \{i_{k+1}\} \setminus \{j_1, j_2, \ldots, j_{2p}\}$, and we can proceed to the next step. At the $(n+1)$th step we must add the vertex 0, join it to all the vertices of J, and complete the working of the algorithm.

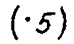
It is easy to see that $\mu(z) = \mu(z_1, \ldots, z_n)$ is actually an even tree, since at the kth step we join the vertex i_k to an even number of vertices, which are greater than i_k. Inequalities for $z \in Z_n$ guarantee the working of the algorithm, namely the existence of $m_{z_{k+1}}$ ($m_{n-k+2-z_{k+1}}$), that is, $\mu(z)$ is well defined as a result of the working of the algorithm. It is easy to define the action of the reverse algorithm, based on eliminating each time the root with maximal number. Hence we immediately obtain the bijection μ, which proves Theorem 3.

Example. Let $\sigma = (58261734) \in \mathrm{Alt}_8$, $\omega(\sigma) = (57241311) \in Z_8$.

After the 0th step we have $i_1 = 5$.

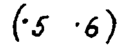
After the 1st step

$\tilde{J} = \{0, 5, 9\}, M = \{6, 7, 8\}, i_2 = 6;$



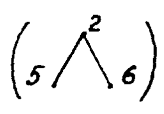
After the 2nd step

$\tilde{J} = \{0, 5, 6, 9\}, M = \{1, 2, 3, 4, 7, 8\}, i_3 = 2;$



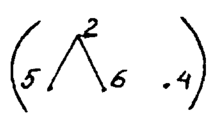
After the 3rd step

$\tilde{J} = \{0, 2, 9\}, M = \{3, 4, 7, 8\}, i_4 = 4;$



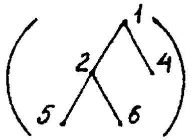
After the 4th step

$\tilde{J} = \{0, 2, 4, 9\}, M = \{1, 7, 8\}, i_5 = 1;$



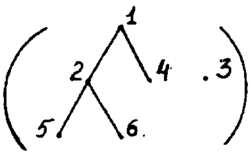
After the 5th step

$\tilde{J} = \{0, 1, 9\}, M = \{3, 7, 8\}, i_6 = 3;$



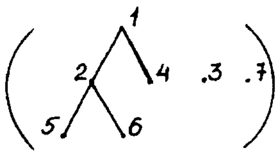
After the 6th step

$\tilde{J} = \{0, 1, 3, 9\}, M = \{7, 8\}, i_7 = 7;$

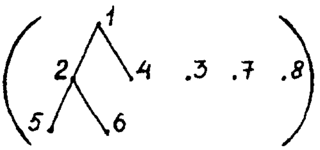


After the 7th step

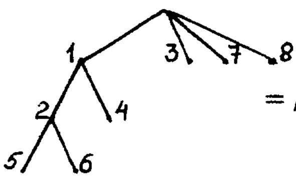
$\tilde{J} = \{0, 1, 3, 7, 9\}, M = \{8\}, i_8 = 8;$



After the 8th step



After the 9th step



$= \mu(\omega(\sigma))$

We now show that the bijection μ also proves the second part of Theorem 4. For this it is sufficient to verify that the statistic $\mu(\omega(\sigma))$ coincides with the statistic β on E_n. In fact, as we have already noted, in the construction of μ^{-1} we must discard the maximal root each time, that is, the statistic $\mu^{-1}(\beta)$ coincides with the statistic of the first element, which in turn coincides with $\omega(\varepsilon)$. Thus $\beta = \mu(\omega(\varepsilon))$, and the second part of Theorem 4 is completely proved.

In the remaining part of §3 we consider the connection between even trees and the polynomial that enumerates trees from the number of inversions and the Tutte dichromate of the complete graph K_n.

3.3. We recall that F_n consists of spanning rooted trees, oriented from the root to 0. Suppose we have fixed $t \in F_n$. We call a pair of vertices $i, j, 0 \leqslant i < j \leqslant n$, an *inversion* if the chain of edges going from the root to i passes through j. Let $\mathrm{inv}(t)$ denote the number of inversions of the tree t.

Let us consider $f_n(x) = \sum_{t \in F_n} x^{\mathrm{inv}(t)}$, the *inversion polynomial.* Obviously $f_n(1) = (n+1)^{n-1}$ is the number of all trees, and $f_n(0) = n!$ is the number of trees with no inversions, that is, increasing trees (see §2). A remarkable observation (see [1], §3.3.48) is that $f_n(-1) = a_n$ and, on the other hand, $f_n(2)$ is the number of connected graphs with $n+1$ vertices (see [19], [20]). Here once more we prove this from a single point of view.

Let $\widetilde{F}_n$ be the set of spanning rooted trees on the set $[\overline{n}]$, oriented from the root to an arbitrary vertex. Similarly $\widetilde{f}_n(x) = \sum_{t \in \widetilde{F}_n} x^{\mathrm{inv}(t)}$.

***Proposition* 5.**

$$\tilde{f}_n(x) = (1 + x + \cdots + x^n) f_n(x).$$

Proof. Let $F_n^k = \{t \in \widetilde{F}_n \text{ with root at } k\}$. We construct a bijection $\gamma_k : F_n^k \to F_n^{k+1}$ such that $\mathrm{inv}(\gamma_k(t)) = \mathrm{inv}(t) + 1$, $t \in F_n^k$. In fact, suppose that γ_k changes the vertices k and $k+1$ in t. Then the inversion (i, j) of the tree t is preserved, and exactly one new inversion $(k, k+1)$ is added.

Let us introduce some notation. Let $\mathfrak{p} = \{P_1, P_2, \ldots\}$ be a finite unordered collection of subsets $P_i \subset [n]$ such that $P_i \cap P_j = \emptyset$; $i \neq j$; $\cup P_i = [n]$; $p_i = |P_i| > 0$. Let $\mathfrak{B}_n$ be the set of all such collections $\mathfrak{p}$, and let $\mathfrak{B}_n^0$ be the set of all collections $\mathfrak{p} = \{P_1, P_2, \ldots\}$ such that $p_i = |P_i|$ is odd for all i.

Also let $(n)_x = 1 + x + \ldots + x^{n-1}, n \in \mathbb{N}$.

***Proposition* 6.** *We have the relations*

$$f_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} (k+1)_x f_k(x) f_{n-k-1}(x); \tag{15}$$

$$f_n(x) = \sum_{\mathfrak{p} \in \mathfrak{B}_n} (p_1)_x \cdot f_{p_1 - 1}(x) \cdot (p_2)_x \cdot f_{p_2 - 1}(x) \cdot \ldots \tag{16}$$

Proof. We split a tree into two subtrees, removing the edge going from the root to 0, after the removal of which 0 and n will be in different subtrees. It is completely obvious that the number of inversions of the tree is the sum of the number of inversions of the rooted subtrees, one of which has its root at 0. Summing over the number of vertices in the second subtree, we obtain (15). The relation (16) is obtained similarly if we split the tree into subtrees by removing all edges going out from the root.

We observe that $(k)_{-1} = 0$ when k is even, and $(k)_{-1} = 1$ when k is odd. This immediately proves that $f_n(-1) = a_n$. In fact, when $x = -1$ the relation (15) is a recurrence relation for the numbers $f_n(-1)$, which coincides with the relations (4)–(6). We show that $f_n(-1) = |E_n|$. This gives a new proof of Theorem 3. In fact, the condition on trees for the number of vertices in the branches to be odd is equivalent to the condition for the number of edges going out from each non-rooted vertex to be even. Now, acting as in the proof of Proposition 6, we deduce that for the number of even trees the recurrence relation (16) is satisfied when $x = -1$, that is, $f_n(-1) = |E_n|$. Let us formulate once more a relation that is apparently new:

$$a_n = \sum_{p \in \mathfrak{B}_n^0} a_{p_1 - 1} \cdot a_{p_2 - 1} \cdot \ldots . \tag{17}$$

We note that by means of the bijections γ_k (see the proof of Proposition 5) we can construct an involution that is fixed on increasing even trees and changes the parity of the number of inversions of the remaining trees. Concerning the polynomial $f_n(x)$ and the relation (15) see [20], [21]. When $x = 1$ the relation (15) turns into an identity of Abel type (see [22]).

3.4. We now turn to a formulation of results about the Tutte dichromatic polynomial of the complete graph $\chi(K_{n+1};\, 1, x) = g_n(x)$. By a theorem of Tutte (see [10], [23]) one of the equivalent definitions consists in the following.

We specify a lexicographical ordering on the edges of the complete graph $K_{n+1} : (i, j)$ is less than (i', j') $(i < j, i' < j')$ if $i < i'$ or $i = i'$, $j < j'$. Suppose we have fixed $t \in F_n$. We say that the edge (i, j) of the graph K_{n+1} is externally active with respect to t if $(i, j) \notin t$ and is a minimal edge in the unique cycle formed by the edges of the tree t and the edge (i, j). Let $e(t)$ be the number of edges externally active with respect to t, and $g_n(x) = \sum_{t \in F_n} x^{e(t)}$.

***Proposition* 7.** $f_n(x) = g_n(x)$.

Proof. It is sufficient to prove for $g_n(x)$ a recurrence relation analogous to (15). As earlier, we split the tree t into two subtrees t' and t'' by removing the edges going out from the root; then 0 and n lie in different subtrees. We observe that the edge (i, j), $0 < i < j$, cannot be externally active with respect to t if i and j lie in different subtrees, since otherwise the cycle contains an edge going out from 1, which is automatically less than the edge (i, j) in the lexicographical ordering. Hence $e(t) = e(t') + e(t'') + m(t)$, where m is the number of externally active edges of the form $(0, j)$ such that $0 \in t'$, $j \in t''$. For various removed edges and fixed t' and t'' the number $m(t)$ takes all values from 0 to $|t''| - 1$ at once. Summing over the number of vertices in the second subtree, we obtain $g_n(x) = \sum_{k=1}^{n} (1 + x + \ldots + x^k) \binom{n-1}{k-1} g_k(x) g_{n-k-1}(x)$, as required.

Hence we immediately obtain $g_n(-1) = a_n$. On the other hand, since $g_n(x) = \chi(K_{n+1};\, 1, x)$, where χ is the Tutte dichromate of the complete graph (see [10], [23]), from the properties of the dichromate we immediately deduce that $f_n(1 + x)$ is the generating function for the number of spanning connections of subgraphs of K_{n+1} over the number of edges. Proposition 7 in this form was first formulated by Bjorner in [40]. In particular, $f_n(2)$ is the number of all spanning connected graphs. We note that in [24] and [20] explicit bijections were constructed that prove the identity $f_n(x) = g_n(x)$ and the assertion that

$f_n(1+x)$ is the generating function for labelled connected graphs with $n+1$ vertices over the number of edges.

3.5. As an application we examine one more example where relations of the type (15) and (16) appear. Let FB_n be the set of all binary (not necessarily increasing) trees on the set $[n]$, and let $t \in FB_n$. Let $\operatorname{inv}(t)$ be the number of pairs (i,j), $1 \leqslant i < j \leqslant n$, such that the chain of edges going out from the root and going to i passes through j. We put $fb_n(x) := \sum_{t\in FB_n} x^{\operatorname{inv}(t)}$. Then $fb_n(1) = |FB_n| = n! \cdot \dfrac{1}{n+1}\dbinom{2n}{n}$ is the number of all binary trees (see [1], [9]), and $fb_n(0) = |B_n| = n!$ is the number of all increasing binary trees.

***Proposition* 8.**

$$fb_n(-1) = \begin{cases} a_n & \textit{if } n \textit{ is odd,} \\ 0 & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. We shall act as in the proof of Propositions 5 and 6. Consider the polynomial $f\widetilde{b}_n(x) = \sum x^{\operatorname{inv}(t)}$, where the summation is over all binary trees with upper vertex at 1. We have $f\widetilde{b}_n(x) = fb_n(x)(1+x+\ldots+x^{n-1})$. Similarly $fb_n(x) = \sum_{k=0}^{n-1} f\widetilde{b}_k(x)\cdot fb_{n-k-1}(x)$. Hence $fb_n(-1) = 0$ for even n, and $fb_n(-1)$ is equal to the number of even binary trees, that is, $fb_n(-1) = |V_n| = a_n$, for odd n (see §2). This completes the proof of the proposition.

§4. Trees with branchings on even levels

4.1. Let us introduce a new class of increasing trees.

Definition 2. The *level* of the vertex i in an increasing tree t is the number of edges in the chain of edges that go out from the root and go to the vertex i. *Trees with branches on even levels* are increasing trees whose branchings, that is, vertices from which more than one edge goes out, lie only on even levels (the root lies on the zero level, that is, on an even level). We denote the set of all such trees by W_n.

***Theorem* 6.** $|W_n| = a_{n+1}$.

We give a proof of it in this subsection by means of an explicit bijection.

***Theorem* 7.** *The number of trees $t \in W_n$ such that $\alpha(t) = k$ is equal to $a_{n+1,n-k+1}$ (where $\alpha(t)$ is the statistic of the maximal vertex joined to the root; see §2).*

***Theorem* 8.** *The number of trees $t \in W_n$ such that $\beta(t) = k$ is equal to the number of trees $t \in W_n$ such that $\delta(t) = k-1$ (where $\beta(t)$ is the statistic of the end of the minimal descent, and $\delta(t)$ is the statistic of the vertex joined by an edge to the vertex n; see §2).*

***Theorem* 9.** *The number of trees $t \in W_{n-1}$ with $n-2k+1$ end vertices on odd levels is equal to $d_{n,k}$.*

We note that the statistic found in Theorem 8 is apparently new. For the proof of Theorems 8 and 9 see §7. For examples see the table.

4.2. Let us construct an explicit bijection $\rho : W_n \to L_n$. For this we define $\varphi(W_n)$.

The *level* of the vertex i in a binary tree $t \in B_n$ is the number of edges going to the left in the chain of edges going out from the root to the vertex i. Consider the set $WB_n = \varphi(W_n) \subset B_n$ of binary trees $t \in B_n$ that have no edges going to the left and going out from a vertex on an odd level. It is easy to see that $WB_n = \varphi(W_n) \subset B_n$. We define $\overline{LB}_n \subset B_n$ as the set of binary trees of which the chain of edges going to the right from each vertex has length not greater than 1. Obviously $h : t \to h \cdot t$ specifies a bijection between $\overline{LB}_n$ and LB_n, where $h = (s_1 \cdot s_2 \cdot \ldots \cdot s_n) \in G_n$ (we need to replace right edges by left and vice versa).

We construct the bijection $\eta : WB_n \to \overline{LB}_n$. Then the required bijection is $\rho = \varphi \circ \eta \circ h \circ \varphi^{-1}$. Since edges cannot go to the left from vertices at odd levels, we suppose that η consists in replacing edges going to the right and going out from a vertex at an odd level by an edge going to the left (see Fig. 7). Then, of course, chains going to the right will have length not greater than 1, hence $\eta(WB_n) \subset \overline{LB}_n$. To construct $\eta^{-1}(t)$ we consider vertices x of the tree t to which a right edge goes and from which a left edge goes. Since by the definition of $\overline{LB}_n$ no right edge goes out from x, at all such vertices x we interchange a left edge and a right edge. Hence it is obvious that η is invertible, and consequently bijective. This proves Theorem 6.

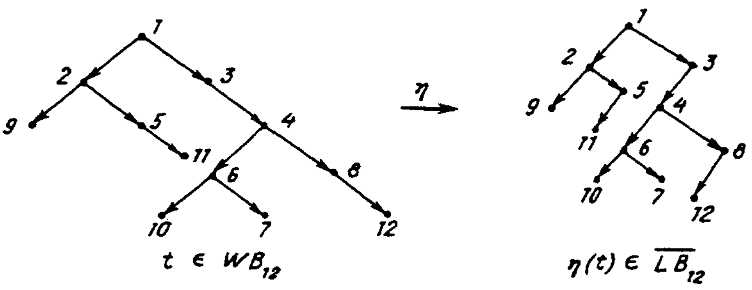


Fig. 7

We show that we have simultaneously proved Theorem 7. In fact, the bijection η does not change the extreme lower left vertex, that is, $\varphi(\alpha)(t) = \varphi(\alpha)(\eta(t))$, $t \in B_n$. But the bijection h obviously takes the statistic $\varphi(\alpha)$ into the statistic $\varphi(\beta)$ (see §2). Hence

$$\alpha(t) = \varphi(\alpha)(\varphi(t)) = \varphi(\alpha)(\eta \circ \varphi(t)) = \varphi(\beta)(\varphi^{-1} \circ h\eta \circ \varphi(t)) = \beta(\varphi(t)),$$

that is, the statistic α on W_n is equivalent to the statistic β on L_n, that is, Theorem 7 is a consequence of the already known Theorem 2.

In exactly the same way we can show that $n + 1 - 2\rho^{-1}(v)$ is the statistic of the number of end vertices at odd levels in a tree $t \in W_n$, and thus prove Theorem 9. Another proof of Theorem 9 by means of the recurrence relations (12), (13) (see §2) will be given in §7.

We also mention another important consequence of the construction for the bijection ρ.

***Theorem* 10.** *The number of trees $t \in W_n$ such that k edges go out from the root is equal to the number of trees $t \in L_n$ such that the length of the minimal descent is equal to k (see §2).*

Proof. In fact, as we mentioned above, the bijection ρ takes the set of vertices of the tree t to which an edge goes from the root into the set of vertices to which edges of the minimal descent go. Hence ρ takes the statistic of the number of edges going out from the root into the statistic of the length of minimal descent, which proves Theorem 10.

4.3. When the edges going out from the root are removed, a tree $t \in W_n$ splits into several trees. To this there corresponds the recurrence relation

$$a_{n+1} = \sum_{\mathfrak{p} \in \mathfrak{B}} a_{p_1 - 1} \cdot a_{p_2 - 1} \cdot \dots . \tag{18}$$

The relation (18) is more surprising than (17) (see §3). We shall return to it in the next section.

§5. Embedded and weakly embedded trees

5.1. Let t be a branch of an increasing tree $t_0 \in U_n$. We denote the minimal and maximal vertices of t by $\min(t)$ and $\max(t)$.

Definition 3. We say that an increasing tree $t \in U_n$ is *embedded* if on removal of the edges going out from any vertex $a \in t$ the resulting branches $t_1, t_2, \dots, t_k$ satisfy the *embedding condition*:

$$\min(t_1) < \min(t_2) < \dots < \min(t_k) \le \max(t_k) < \dots < \max(t_2) < \max(t_1).$$

We denote the set of embedded trees with n vertices by T_n.

***Theorem* 11.** $|T_n| = a_n$.

***Theorem* 12.** *The number of embedded trees $t \in T_n$ such that $\alpha(t) = k$ is equal to the number of embedded trees $t \in T_n$ such that $\delta(t) = n - k$, and is equal to $a_{n,k}$ (see §2).*

We prove Theorems 11 and 12 by establishing an explicit bijection between T_n and L_{n-1}.

For examples of embedded trees see Table VII.

5.2. Consider the set $TB_n \subset B_n$ of binary trees t such that for any vertex a the maximal vertex of its left branch is less than the maximal vertex of its right branch. We show that $TB_n = \varphi(B_n)$.

In fact, if a is a vertex of a tree $t \in T_n$, then the set of vertices of the right branch of the vertex $\min(t_1) \in \varphi(t)$ is precisely $\varphi(t_1)$ (see Definition 3), and the set of vertices of its left branch is $\bigcup_{i=2}^{k}\varphi(t_i)$, hence from the embedding condition we immediately obtain

$$\max(\varphi(t_1)) = \max(t_1) > \max(t_2)\max(\cup_{i=2}^{k}\varphi(t_i)).$$

Consequently, $\varphi(T_n) \subset TB_n$.

Conversely, suppose that $t \in U_n$ and $\varphi(t) \in TB_n$. Then, using the definition of TB_n for the vertex $\varphi(\min(t_1))$, where $t_1, t_2, \ldots, t_k$ are the branches going out from $a \in T$, so that $\min(t_1) < \min(t_2) < \ldots < \min(t_k)$, we have $\max\left(\bigcup_{i=2}^{k}\varphi(t_n)\right) < \max(t_1)$. Similarly, for the vertex $\min(\varphi(t_2))$ we have $\max\left(\bigcup_{i=3}^{k}\varphi(t_n)\right) < \max(t_2)$, and so on, for the vertex $\min(\varphi(t_{k-1}))$ we have $\max\left(\bigcup_{i=k}^{k}\varphi(t_n)\right) < \max(t_{k-1})$. Hence there immediately follows the embedding condition $\max(t_k) < \ldots < \max(t_2) < \max(t_1)$, that is, $t \in T$.

We now show that TB_n is the set of representatives of the orbits of the action of the Foata group G_n on B_n. Suppose that $t \in B_n$, $h_I = s_{i_1} \cdot s_{i_2} \cdot \ldots$, where $I = \{i_1, i_2, \ldots\}$ is the set of vertices i_j of the tree t at which the maximal vertex of the left branch is greater than the maximal vertex of the right branch. Then it is obvious that $\widetilde{t} = h_I \cdot t \in TB_n$, and if the trees t_1 and t_2 lie in different orbits of the action of the Foata group, then $\widetilde{t_1} \neq \widetilde{t_2}$. Thus we have established a bijection π^{-1} between L_n and TB_n, which consists in the fact that we must regard edges as right or left depending on which maximal vertex of such a branch is the greater. Hence $\xi = \varphi^{-1} \circ \pi^{-1}$ is the required bijection between L_n and T_{n+1}. We note that the map $\pi : TB_n \to L_n$ is trivial and consists in neglecting the structure of right and left edges (see §2).

Thus Theorem 11 is completely proved. We show that we have also proved the second part of Theorem 12. For this we verify that $\xi(\delta) = \delta$. In fact, if in a tree $t \in T_n$ an edge goes from a vertex a to a vertex n, then by the embedding condition for the vertex a no other edges go out from it. Consequently, from the definition of the bijection φ, the edge (a, n) belongs to the tree $\varphi(t)$, and hence to the tree $\xi^{-1}(t)$, that is, $\xi(\delta) = \delta$, and the second part of Theorem 12 follows from Theorem 2. We shall prove the first part of Theorem 12 in §7. We also note that for any tree $t \in T_n$ we have $\beta(t) = n$, since the maximal vertex in a given branch always lies in the subbranch with minimal root, by the embedding condition.

5.3. Let us consider one more class of increasing trees.

Definition 4. Let Q_n be the set of increasing trees on the set of vertices $[\overline{n}]$ satisfying the embedding condition for all non-root vertices. We call such trees *weakly embedded.* Obviously, $T_n \subset Q_n$.

For greater clarity we represent each tree $t \in Q_n$ as a forest, obtained by removing edges going out from the root (see the examples in Table VII).

***Theorem* 13.** $|Q_n| = a_{n+1}$.

Proof. In fact, since the branches of a weakly embedded tree obtained by removing edges going out from the root are embedded trees, we have

$$|Q_n| = \sum_{\mathfrak{p}\in\mathfrak{B}_n} |T_{p_1-1}|\cdot|T_{p_2-1}|\cdots = \sum_{\mathfrak{p}\in\mathfrak{B}_n} a_{p_1-1}\cdot a_{p_2-1}\cdot = a_{n+1}$$

by Theorem 11 and the relation (17).

***Theorem* 14.** *The number of trees $t\in Q_n$ such that $\alpha(t) = n+1-k$ is equal to $a_{n+1,k}$.*

***Theorem* 15.** *The number of trees $t\in Q_n$ such that k edges go out from the root is equal to the number of trees $t\in L_n$ such that the length of minimal descent is equal to k.*

To prove the theorems we construct a bijection $\zeta : Q_n \to W_n$. Let $t\in Q_n$ be a weakly embedded tree; $t_1 \in T_{p_1-1}$, $t_2\in T_{p_2-1}$, $\ldots$ is an unordered collection of embedded trees obtained from t by removing edges going out from the root; $\mathfrak{p} = \{P_1, P_2, \ldots\} \in \mathfrak{B}_n$ is the corresponding unordered collection of sets of vertices of the trees t_i (see above). We now number the vertices of the tree $\rho^{-1}(\xi^{-1}(t_i))$ by elements of the set P_i, $\{\min(P_i)\}$, $i=1,2,\ldots$. We now consider the tree $\zeta(t)$, which has a root at 0, from which edges go to the vertices $\min(P_i)$, from each of which one edge goes out to $\rho^{-1}(\xi^{-1}(t_i))$ (see Fig. 8). Obviously $\xi : Q_n \to W_n$ is a bijection, since ζ is, and the invertibility of ζ follows from the bijectivity of ρ and ζ. This gives us a new proof of Theorem 13. We note that $\xi\circ\rho\circ\zeta$ is a bijection between Q_n and T_{n+1}.

We now show that $\zeta(\alpha) = \alpha$. In fact, it follows from the construction of the bijection ζ that the set of vertices joined to the root by an edge, that is, having level equal to 1, does not change. Thus Theorems 14 and 15 follow from Theorems 7 and 10 on trees with branchings on even levels.

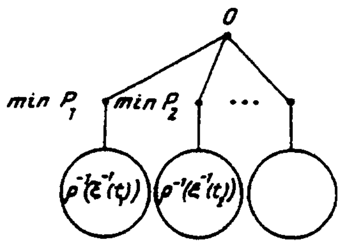


Fig. 8

5.4. To conclude this section we consider two other classic sequences of numbers.

The *Bell number* b_n is the number of partitions of $[n]$ into unordered parts, that is, $b_n = |\mathfrak{B}_n|$. The *Catalan number* c_n is the number of binary unlabelled trees. More precisely, c_n is the number of orbits of the natural action of the permutation group S_n on the set of all binary trees FB_n (see §3.5). For explicit formulae and different combinatorial interpretations of the Bell and Catalan numbers see [1], [12], [22].

We call edges (i_1, j_1) and (i_2, j_2) of an increasing tree $t \in U_n$ *embedded* if $0 < i_1 < i_2 < j_2 < j_1$.

***Proposition* 9.** *The number of weakly embedded trees $t \in Q_n$ without branchings at non-rooted vertices is equal to b_n. Among them the number of trees without embedded edges is equal to c_n.*

As a simple consequence of Proposition 9 we obtain $c_n \leqslant b_n \leqslant a_{n+1}$. We note that $c_n = b_n = a_{n+1}$ when $n \leqslant 3$, and also $c_4 = 14 < b_4 = 15 < a_5 = 16$ (see the table).

§6. Geometric classes of trees

6.1. We prove a general theorem about equivalent statistics on geometric classes of trees.

Definition 5. We call a set of increasing trees $H \subset U_n$ *geometric* if with any tree $t \in H$ it also contains a tree $t' \in U_n$ isomorphic to t as a rooted unlabelled tree. In other words, H is geometric if and only if $t \in H$ implies that $\mathrm{Or}_t \cap U_n \subseteq H$, where Or_t is the orbit containing t of the transformation group S_n, which acts on F_n by permutations of the vertices.

It is easy to see that the classes of 0-1-2 trees, even trees, and trees with branchings at even levels are geometric, but the classes of embedded and weakly embedded trees are not geometric.

To formulate the main theorem we introduce some notation. Suppose that $t \in U_n$. The *minimal descent vector* of t is the collection of numbers $x(t) = (x_1, x_2, \dots, x_l)$ such that the minimal descent (see §2) of t consists of the edges $(0,1), (1,x_1), (x_1,x_2), \dots, (x_{l-1}, x_l)$, and $\beta(t) = x_l$ is the end of the minimal descent (see §2). The n-descent is the unique path going from the root to the vertex n. Its length m is by definition equal to the level of the vertex n. The *n-descent vector* of t is the collection of numbers $y(t) = (y_1, y_2, \dots, y_m)$ such that the n-descent of t consists of the edges $(0, y_1), (y_1, y_2), \dots (y_{m-1}, y_m), (y_m, n)$. Clearly, $x(t)$ and $y(t)$ are multidimensional statistics on the set U_n.

***Theorem* 16.** *Let H be a geometric set of increasing trees, $H \subset U_n$. Then the statistics of the n-descent and minimal descent vectors are equivalent up to a shift. More precisely, the number of trees $t \in H$ such that $x(t) = (v_1, v_2, \dots, v_k)$ is equal to the number of trees $t \in H$ such that $y(t) = (v_1 - 1, v_2 - 1, \dots, v_k - 1)$.*

Proof. To prove this theorem we construct an explicit bijection $\theta : H \to H$ such that $y(\theta(t)) = (x_1 - 1, x_2 - 1, \dots, x_k - 1)$, where $(x_1, x_2, \dots, x_k) = x(t)$.

Suppose that $t \in H$, $x(t) = (x_1, x_2, \dots, x_k)$. Consider the tree t' on the set $\{1, 2, \dots, n+1\}$ obtained by replacing the vertex 0 by the vertex 1, the vertex 1 by the vertex x_1, the vertex x_1 by the vertex x_2, and so on, the vertex x_{k-1} by the vertex x_k, and the vertex x_k by the vertex $n+1$. Let $\theta(t)$ be the tree obtained by lowering the numbers of all vertices of t' by one. Obviously, $y(\theta(t)) = (x_1 - 1, x_2 - 1, \dots, x_k - 1)$. We now show that $\theta(t) \in H$. For this it is sufficient to show that $\theta(t) \in U_n$, since the set H is geometric, and the fact that t, t' and $\theta(t)$ are isomorphic is clear from the construction. Suppose that the tree t' contains an edge (x_i, z) such that $z < x_i$. Then t contains an edge (x_{i-1}, z) and an edge (x_{i-1}, x_i), which is impossible, since $z < x_i$ and the edge (x_{i-1}, x_i) is contained in the minimal descent of t. Thus t' is an increasing tree, and $\theta(t) \in U_n$. The fact that θ is invertible, and hence bijective, is obvious, which completes the proof of the theorem.

Theorem 16 is due to Schutzenberger and is a special case of Assertion 5.1 in [25]. The proof of Theorem 16 also follows [25].

6.2. We particularly note two important corollaries of Theorem 16.

***Corollary* 1.** *Let H be a geometric set of increasing trees, $H \subset U_n$. Then the number of trees $t \in H$ such that $\beta(t) = k$ is equal to the number of trees $t \in H$ such that $\delta(t) = k - 1$.*

Proof. In fact, as we observed above, if $x(t) = (x_1, x_2, \dots, x_l)$, $y(t) = (y_1, y_2, \dots, y_m)$, then $\beta(t) = x_l$, $\delta(t) = y_m$, from which we immediately obtain the assertion of the corollary.

Thus we have proved Poupard's theorems on the equivalence up to a shift of the statistics β and δ on L_n and V_n/G_n (see Theorem 2 in §2), Theorem 8 (see §4), and the equivalence of the first and second parts of Theorem 4 (see §3).

***Corollary* 2.** *Let H be a geometric set of increasing trees, $H \subset U_n$. Then the number of trees $t \in H$ for which the length of minimal descent is equal to k is equal to the number of trees $t \in H$ for which the vertex n is at level k.*

Thus, in the case $H = L_n$ we have found the statistic of the level of a vertex of L_n, equivalent to the statistic on L_n of the length of minimal descent, which by Theorems 10 and 15 is in turn equivalent to the statistics on W_n and Q_n of the number of edges going out from the root. To emphasize the importance of this statistic, we formulate a theorem on a statistic equivalent to it on the set of alternating permutations Alt_{n+1}.

We call an element $\sigma(i)$, $2 \leqslant i \leqslant n$, *extreme* in the permutation $\sigma \in S_n$ if $\sigma(j) < \sigma(i)$ for all $1 \leqslant j < i$ or $\sigma(j) > \sigma(i)$ for all $1 \leqslant j < i$, that is, the elements of the permutation to the left of $\sigma(i)$ are either all less than $\sigma(i)$ or all greater than $\sigma(i)$. Let $\kappa(\sigma)$ be the number of extreme elements in the permutation σ.

***Theorem* 17.** *The number of alternating permutations $\sigma \in \mathrm{Alt}_{n+1}$ such that $\kappa(\sigma) = k$ is equal to the number of 0-1-2 trees $t \in L_n$ for which the length of minimal descent is equal to k.*

We prove Theorem 17 in §7 by means of recursive relations found for this statistic.

6.3. To conclude this section we say something about the geometric approach to problems of listing permutations and trees.

A class of permutations, or a statistic on permutations, is called geometric if it is defined just by ascents and descents in the permutation, that is, $\sigma(i) < \sigma(i+1)$ or $\sigma(i) > \sigma(i+1)$. In particular, the classic Euler statistics of the number of descents and the McMahon statistic of the sum of places of descents are geometric (see [1], [2], [12]). A systematic study of geometric classes of permutations and statistics was apparently first started in [2] and then continued in a number of other works. Since we cannot give a survey of this cycle of articles, we mention in this connection a long series of papers by Carlitz and his co-authors, concerned with generalizations of the numbers $a_{n,k}$ for other regular configurations of ascents and descents, the original approach by Stanley by means of Möbius functions for binomial partially ordered sets (see [26], [12]), and also the rich factual material collected in Chapter 4 of the monograph [1] (see the list of literature there). Concerning the listing of the number of trees in geometric classes there is also a large quantity of results, mainly connected with the application of the technique of differential equations (see [27], [1], for example). We mention here the advantage of the bijective approach, which enables one to formulate and prove non-geometric results about permutations in terms of geometric classes of trees, and vice versa.

§7. Proofs of theorems

To start with we note that we have constructed a complete system of bijections between Alt_n and L_{n-1}, E_n, W_n, T_n, Q_{n-1} (see the figure in Table I). Hence we immediately obtain the assertions of Theorems 1, 3, 6, 11, 13. In studying the properties of bijections, as we showed in §§3–5, we obtained the proofs of Theorems 7, 9, 14, 15, and also the second parts of Theorems 4 and 12. Next, Theorem 16, together with Theorem 8 and the first part of Theorem 4 which follow from it, were proved in §6. For a proof of Theorem 2 see [17]. Here we give the proofs of the remaining Theorems 5 and 17 and the first part of Theorem 12, and we also give a second proof of Theorem 9.

7.1. *Proof of Theorem* 4. We denote by $c_{n,k}$ the number of trees $t \in E_n$ with k end vertices. We have the relation

$$c_{n,k} = \sum_{l=1}^{[(n-1)/2]} \sum_{m=1}^{k-1} \binom{n-1}{2l} c_{2l,m} \cdot c_{n-2l-1,k-m} + c_{n-1,k-1}. \tag{19}$$

In fact, we split a tree $t \in E_n$ into two subtrees t' and t'' by removing the edge (0, 1). Let $2l+1$ be the number of vertices in the subtree t' containing the vertex 1, and m the number of end vertices of t'. Then, if $l > 0$, there are in all $n-2l-1$ vertices in t'', of which $k-m$ are end vertices. Analysing the cases $l = 0$ and $l > 0$ separately and summing over l and m, we obtain (19).

We denote by $b_{n,k}$ the number of trees $t \in L_{n-1}$ with k non-end vertices. Then

$$b_{n,k} = \sum_{l=1}^{n-2} \sum_{m=1}^{k-1} \binom{n-2}{l} b_{l,m} \cdot b_{n-l-1,k-m} + b_{n-1,k-1}. \tag{20}$$

The relation (20) is proved similarly, except that in this case we remove all the edges going out from the root, and consider separately the cases when there are one or two such edges. We now recall that $b_{n,k} = d_{n,n-k+1}$ by the definition of the sequence $d_{n,k}$ (see §2). Now, taking account of the explicit expressions for the binomial coefficients and the relation (12), we can derive (19) from the relation (20) for the sequence $b_{n,k}$. Since (19) gives the sequence $c_{n,k}$ uniquely up to the initial values, we have $b_{n,k} = c_{n,k}$. This completes the proof of Theorem 4, since in an even tree any non-end and non-rooted vertex is a branching.

7.2. *Proof of Theorem* 5. We denote by $c_{n,k}$ the number of trees $t \in E_n$ such that k edges go out from the root. Then we have the relation

$$c_{n,k} = \sum_{l=1}^{[(n-1)/2]} \binom{n-1}{2l} a_{2l} \cdot c_{n-2l-1,k-1}. \tag{21}$$

In fact, we split the tree t into two subtrees t' and t'' by removing the edge (0, 1), and suppose that t'' contains the vertex 0 and t' contains exactly $2l+1$ vertices. Now, summing over all t' and t'', we obtain the relation (21).

We denote by $b_{n,k}$ the number of alternating permutations with k distinguished elements. We derive the relation (21) for the sequence $b_{n,k}$. Let $\sigma_{i_1} = 1$ be the first distinguished element in the permutation σ. We represent σ in the form $(\sigma' 1 \sigma'')$, where $\sigma' = (\sigma_1 \sigma_2 \dots \sigma_{i_1-1})$, $\sigma'' = (\sigma_{i_1+1} \dots \sigma_n)$. Then $i_1 = 2l+1$, and all the distinguished elements other than σ_{i_1} lie in σ''. Summing over σ' and σ'', we immediately obtain the relation (21) for the sequence $b_{n,k}$, which completes the proof of Theorem 5.

7.3. *Proof of Theorem* 9. We denote by $c_{n,k}$ the number of trees $t \in W_{n-1}$ with $n-2k+1$ end vertices at odd levels. Obviously $c_{n,1} = 1$, since a tree $t \in U_{n-1}$ with $n-1$ end vertices is precisely the tree t_0 containing $n-1$ edges going out from the root. We now prove the relations (13) for the sequence $c_{n,k}$. Consider $t \in W_{n-1}$ with k vertices at even levels. Then it has $k-1$ non-end vertices at odd levels and $n-k-(k-1) = n-2k+1$ end vertices at odd levels. Consequently, we can add a new vertex in $k+(n-2k+1)$ different ways, where in k cases the number of end vertices at odd levels is equal to $n-2k+2$, and in $n-2k+1$ cases it is equal to $n-2k$. Hence $c_{n,k} = k \cdot c_{n-1,k} + (n-2k+2) \cdot c_{n-1,k-1}$, and we have proved (13) for the sequence $c_{n,k}$, which completes the proof of Theorem 9.

7.4. *Proof of the first part of Theorem* 12. Let $t \in L_n$. An *antimaximal descent* in t is a chain of edges, of which the first goes out from the root, and the last goes into a non-branching vertex of t (that is, a vertex from which at most one edge goes out) such that the edge going out from the vertex i goes into a branch containing the minimal of the maximal vertices of the branch that go out from i. We denote by $\overline{\beta}$ the statistic of the end of the antimaximal descent. We show that $\overline{\beta} = [\pi \circ \varphi](\alpha) = \xi^{-1}(\alpha)$. In fact, as we showed in §2, $\varphi(\alpha)$ is the statistic of the end of the maximal chain going left. But by definition $TB_n = \varphi(T_n)$, and the maximal of the vertices i lies in the right half-branch, hence $\overline{\beta} = \xi^{-1}(\alpha)$. We denote by $b_{n,k}$ the number of trees $t \in L_{n-1}$ such that $\overline{\beta}(t) = \alpha(\xi(t)) = k-1$. We show that $b_{n,k} = a_{n,k}$. For this we introduce the recursive relations $c_{n,1} = a_{n-1}$ and

$$c_{n,k} = \sum_{l=0}^{k-2} \sum_{m=0}^{n-k-1} \binom{k-2}{l} \binom{n-k-1}{m} c_{n-l-m-2,k-l-1} \cdot a_{l+m+1} \tag{22}$$

for $c_{n,k} = a_{n,k}$ and $c_{n,k} = b_{n,k}$ independently.

Let $t \in L_{n-1}$. If $\overline{\beta}(t) = 0$, then exactly one edge goes out from the root, and consequently $b_{n,1} = a_{n-1}$. Now suppose that $\overline{\beta}(t) > 0$, that is, exactly two edges go out from the root of t ($t \in L_{n-1}$). We split t into two subtrees t' and t'' by removing these edges. Suppose that the vertex n is contained in t'. Then clearly the vertex $k-1$ is contained in t'' (by the definition of an antimaximal descent). We denote by l the number of vertices of t'' less than $k-1$, and by m the number of vertices of t' greater than $k-1$. Summing over all t' and t'', we immediately obtain the relation (22) for the sequence $b_{n,k}$.

Let $\sigma \in \mathrm{Alt}_n$, and let $k = \varepsilon(\sigma)$ be the first element of the permutation σ. Clearly, $a_{n,1} = a_{n-1}$ (see §1). Now suppose that $k > 1$, $\sigma_{i_1} = 1$, $\sigma_{i_2} = n$, $i = \min(i_1, i_2)$. We represent σ in the form $(\sigma'\sigma_i\sigma'')$, where $\sigma' = (\sigma_1, \ldots, \sigma_{i-1})$, $\sigma'' = (\sigma_{i+1}, \ldots, \sigma_n)$. We denote by l the number of elements σ_j in σ'' such that $1 < \sigma_j \leqslant k$, and by m the number of elements σ_j in σ'' such that $k < \sigma_j < n$. We show that by summing over all σ' and σ'' we obtain the relation (22) for the sequence $a_{n,k}$. For this we observe that when $i = i_1$ the summation will run over l and m such that $n-l-m$ is even, and when $i = i_2$ over l and m such that $n-l-m$ is odd. Thus we have proved identical recurrence relations for $a_{n,k}$ and $b_{n,k}$, which completes the proof of the theorem.

7.5. *Proof of Theorem* 17. We denote by $c_{n,k}$ the number of trees $t \in W_n$ such that exactly k edges go out from a vertex. We have the relation

$$c_{n,k} = \sum_{l=1}^{n-1} \binom{n-1}{l} c_{n-l-1,k-1} \cdot a_l + c_{n-1,k-1}. \tag{23}$$

In fact, we split t into two subtrees t' and t'' by removing the edge (0, 1). Suppose that t'' contains $l+1$ vertices, and t' contains the vertex 0. Analysing the two cases $l = 0$ and $l > 0$ separately and summing over all t' and t'', we obtain the relation (23).

We now prove the relation (23) for the sequence $b_{n,k}$, where $b_{n,k}$ is the number of permutations $\sigma \in \mathrm{Alt}_n$, $\kappa(\sigma) = k$, $\sigma_{i_1} = 1$, $\sigma_{i_2} = n$, $i = \max(i_1, i_2)$. We represent σ in the form $(\sigma'\sigma_i\sigma'')$, where $\sigma' = (\sigma_1, \dots, \sigma_{i-1})$, $\sigma'' = (\sigma_{i+1}, \dots, \sigma_n)$. We observe that σ' contains $k-1$ extreme elements but σ'' does not contain them, and also that when $i = i_1$ the number i is even, and when $i = i_2$ it is odd. Analysing the cases $i = n$ and $i < n$ separately and summing over σ' and σ'', we immediately obtain the relation (23) for the sequence $b_{n,k}$, in which the summation goes over $l = n - i$. This completes the proof of Theorem 17, since by Theorem 10 the statistic on L_n of the length of minimal descent is equivalent to the statistic on W_n of the number of edges going out from the root.

7.6. *Proof of Proposition* 9. We construct the surjection $\gamma : Q_n \to \mathfrak{B}_n$ as follows. Let $t \in Q_n$ be a weakly embedded tree, and let $P_1, P_2, \dots$ be the sets of vertices of its subtrees obtained by removing all the edges going out from the root. We put $\gamma(t) = \mathfrak{p} = (P_1, P_2 \dots) \in \mathfrak{B}_n$. We show that γ brings about a bijection between weakly embedded trees without branchings at non-rooted vertices and $\mathfrak{B}_n$. In fact, if one of the subtrees of t obtained by removing all the edges going out from the root contains vertices of the set $P = \{p_1 < p_2 < \dots < p_l\}$, then only one edge can go from p_1 to p_2, one from p_2 to p_3, and so on. Thus a weakly embedded tree t without branchings at non-rooted vertices is restored uniquely from $\mathfrak{p} \in \mathfrak{B}_n$, $\gamma(t) = \mathfrak{p}$, which implies the first part of Proposition 9.

The proof of the second part of Proposition 9 is based on a different construction. Let $\{(p_1, q_1), (p_2, q_2), \dots, (p_l, q_l)\}$ be the set of edges of a weakly embedded tree t not going out from the root, where $p_1 < p_2 < \dots < p_l$, which is possible if t has no branchings. The condition for the lack of embedded edges in this language means that $q_1 < q_2 < \dots < q_l$ (if $q_i > q_{i+1}$, then the edges (p_i, q_i) and (p_{i+1}, q_{i+1}) are embedded). We now consider a path on the lattice $\mathbb{Z}^2$ with start $(0, 0)$ and end $(2n, 0)$ (for the definition see [1], for example), having the form

$$\begin{gathered}(0,0), (1,1), (2,2), \dots, (q_1, q_1), (q_1 + 1, q_1 - 1), \dots, \\ (q_1 + p_1, q_1 - p_1), (q_1 + p_1 + 1, q_1 - p_1 + 1), , \dots, (q_2 + p_1, q_2 - p_1), \dots, \\ (q_2 + p_2, q_2 - p_2), \dots, (q_l + p_l, q_l - p_l), \dots, (n + p_l, n - p_l), \dots, (2n, 0).\end{gathered}$$

It is well known that the number of such paths is equal to the Catalan number (see [1], §5.2, for example), which completes the proof of Proposition 9.

§8. Remarks, open problems, and further perspectives

8.1. We should be interested in obtaining direct combinatorial proofs of identities of type (1) for statistics equivalent to the Entringer statistic on various classes of increasing trees.

8.2. Springer [28] and Arnol'd [29], [30] have proposed generalizations of the numbers $a_{n,k}$ to other systems of roots. The authors do not know of an interpretation of them in the form of the numbers of certain trees. It would be of particular interest to obtain a generalization of Proposition 7 and the identity $\chi(A_n;\ 1,\ -1) = a_{n-1}$. In this case, obviously, we need to consider the Tutte dichromate of the corresponding system of roots. The authors intend to return to this question later.

8.3. In [31] the authors consider the "unlabelled analogue" of the bijection ρ of the present work. The role of increasing trees is played by plane trees with a hanging root, which, as is known (see [1]), have the Catalan number. On them we can also introduce the action of an "unlabelled analogue" of the Foata group, the number of orbits of which is equal to the number of plane trees with branchings on even levels and is equal to the Motzkin number (see [1], [32], [33], for example). It is possible that there are unlabelled analogues of other classic combinatorial objects.

8.4. Let $N(n)$ be the subgroup of $GL(n, \mathbb{F}_2)$ consisting of upper triangular matrices with ones on the diagonal. Kirillov remarked in [34] that the number of classes of conjugate elements of $N(n)$, like the number of orbits of the coadjoint representation of $N(n)$, is equal to 1, 2, 5, 16, 61 for small n, that is, it coincides with a_n for $n \leqslant 5$. However, for $n = 6$ we obtain $275 > 272 = a_6$ conjugacy classes (see [35]). If it turned out that the number of *some* conjugacy classes of $N(n)$ is equal to a_n, it would be interesting to generalize this fact to all fields $\mathbb{F}_q$ and to other systems of roots.

8.5. Kreweras [36] established a connection between the polynomial $f_n(x)$ (see §3) and majorizing sequences. An explicit involution on majorizing sequences, which proves that $f_n(-1) = a_n$, various generalizations, and a connection with the theory of representations of the symmetric group were considered by the last two authors in [37].

8.6. Theorem 16 on the equivalence of the statistics β and δ on geometric classes of increasing trees is connected with the *Schutzenberger shift* (jeu de taquin), which plays a fundamental role in the theory of representations of the symmetric group (see [25], [38]). A generalization to any partially ordered set was obtained in [25]. Generalizations connected with the statistics α, ν and so on are not known to the authors.

8.7. "Multidimensional analogues" of classic combinatorial objects are well known. Thus, the role of ordinary trees is played by k-dimensional trees, that of binary trees by $(k+1)$-ary trees, and so on (see [9], for example). It would be interesting to obtain a multidimensional generalization of the numbers a_n and polynomials $f_n(x)$ in the spirit of the present article.

8.8. The authors do not know of a simple explicit bijection between T_n and Q_{n-1}. It is possible that the discovery of it would give a simple proof of Theorems 12–15.

8.9. Various authors at different times have studied q-analogues of the numbers a_n, connected with the statistic on Alt_n of the number of inversions of permutations. Extensive information about them is collected in [1]. It would be interesting to obtain q-analogues of the corresponding statistics on increasing trees. In this connection we mention the paper [39], in which a kind of q-analogue of Cayley's formula is presented.

8.10. It would be interesting to obtain a generalization of relations of the type (15) for Tutte polynomials of bichromatic and k-chromatic graphs (see [8], [9], for example), since in these cases there are simple multiplicative formulae for the number of spanning trees.

Table I. List of notations

1. Sets

Alt_n—alternating permutations, §2.1.
F_n—the set of all trees, §2.2.
U_n—the set of increasing trees, §2.2.
B_n—the set of binary increasing trees, §2.2.
V_n—the set of 0-2 binary trees, §2.2.
G_n—the Foata group, §2.3.
L_n—the set of 0-1-2 increasing trees, §2.3.
LB_n—the set of 0-1-2 binary increasing trees, $LB_n = \varphi(L_n)$, §2.3.
O_t—the set of trees in the orbit of the Foata group corresponding to a given $t \in L_n$, §2.3.
E_n—the set of even trees, §3.1.
R_n—the set of integer sequences $(z_1, \ldots, z_n)$, in which $1 \leqslant z_i \leqslant n+1-i$, §3.2.
Z_n—the set of elements of R_n such that $z_1 \leqslant z_2 > z_3 \leqslant \ldots$, §3.2.
$\widetilde{F}_n$—the set of rooted trees with an arbitrary root, §3.3.
$\mathfrak{p} = (P_1, P_2, \ldots)$—a collection of unordered subsets of the fixed set $[n] = \{1, \ldots, n\}$ such that $P_i \cap P_j = \varnothing$, $i \neq j$, $P_i \neq \varnothing$, $\bigcup_i P_i = [n]$, §3.3.
$\mathfrak{B}_n$—the set of such collections, §3.3.
$\mathfrak{B}_n^0$—the set of such collections in which $|P_i|$ is odd, §3.3.
FB_n—the set of binary trees, §3.5.
W_n—the set of trees with branchings at even levels, §4.1.
$WB_n = \varphi(W_n)$, §4.2.
$\overline{LB}_n = h(LB)$—trees symmetric to LB_n, §4.2.
T_n—the set of embedded trees, §5.1.
$TB_n = \varphi(T_n)$, §5.2.
Q_n—the set of weakly embedded trees, §5.3.

2. Bijections

$\psi : S_n \to B_n$, $\psi : \mathrm{Alt}_n \to V_n$, §2.2.
$\varphi : U_n \to B_n$, §2.2.
$\pi : B_n/G_n \to L_{n-1}$, §2.3.
$\lambda : \mathrm{Alt}_n \to B_n/G_n$, §2.3.
$\omega : S_n \to R_n$, $\omega : \mathrm{Alt}_n \to Z_n$, §3.2.
$\mu : Z_n \to E_n$, §3.2.
$\rho : W_n \to L_n$, §4.2.
$h : LB_n \to \overline{LB}_n$, §4.2.
$\eta : WB_n \to LB_n$, §4.2.
$\pi : TB_n \to L_n$, §5.2.
$\xi : L_n \to T_{n+1}$, §5.2.
$\zeta : Q_n \to W_n$, §5.3.
$\theta : H \to H$, where H is any geometric class of trees, §6.1.

$$\begin{tikzcd} V_n & \mathrm{Alt}_n \arrow[l, "\psi"'] \arrow[r, "\mu\circ\omega"] \arrow[d, "\pi\circ\lambda"] & E_n & \\ T_n & L_{n-1} \arrow[l, "\xi"'] & W_{n-1} \arrow[l, "\rho"'] & Q_{n-1}. \arrow[l, "\zeta"'] \end{tikzcd}$$

3. Statistics

$\varepsilon : S_n \to \mathbb{Z}_+$, $\varepsilon : \mathrm{Alt}_n \to \mathbb{Z}_+$, statistic of the first element, §2.2.
$\alpha = \varphi^{-1}(\psi(\varepsilon)) : U_n \to \mathbb{Z}_+$, statistic of the maximal vertex joined to the root, §2.2.
$\beta : U_n \to \mathbb{Z}_+$, statistic of the end vertex of a minimal descent, §2.3.
$\delta : U_n \to \mathbb{Z}_+$, statistic of the vertex joined to n by an edge, §2.3.
$\nu : U_n \to \mathbb{Z}_+$, statistic of the number of end vertices, §2.4.
$\mathrm{inv} : F_n \to \mathbb{Z}_+$, statistic of the number of inversions of a tree, §3.3.
$y : U_n \to a$ sequence, statistic of the minimal descent vector, §6.1.
$\kappa : S_n \to \mathbb{Z}_+$, $\kappa : \mathrm{Alt}_n \to \mathbb{Z}_+$, statistic of the number of extreme elements of a permutation, §6.2.

4. Sequences

a_n — the number of alternating permutations, §2.1.
$a_{n,k}$ — the Entringer number, §2.1.
$d_{n,k}$ — the number of 0-1-2 trees with k end vertices, §2.4.
b_n — the Bell number, §5.4.
c_n — the Catalan number, §5.4.

Table II. Some values of sequences

1) Numbers of alternating permutations.
$a_{2n} = E_n$ — the Euler numbers;
$a_{2n-1} = 2^{2n}(2^{2n} - 1)B_n/2n$, where B_n are the Bernoulli numbers.
2) the Euler–Bernoulli triangle, the Entringer numbers $a_{n,k}$.

$$\begin{array}{c}
1 \\
0 \quad 1 \\
1 \quad 1 \quad 0 \\
0 \quad 1 \quad 2 \quad 2 \\
5 \quad 5 \quad 4 \quad 2 \quad 0 \\
0 \quad 5 \quad 10 \quad 14 \quad 16 \quad 16 \\
61 \quad 61 \quad 56 \quad 46 \quad 32 \quad 16 \quad 0 \\
0 \quad 61 \quad 122 \quad 178 \quad 224 \quad 256 \quad 272 \quad 272 \\
1385 \quad 1385 \quad 1324 \quad 1202 \quad 1024 \quad 800 \quad 544 \quad 272 \quad 0
\end{array}$$

3) The numbers $d_{n,k}$.

n k	1	2	3	4
1	1			
2	1	1		
3	1	4		
4	1	11	4	
5	1	26	34	
6	1	57	180	34
7	1	120	768	496

4) The Catalan and Bell numbers.

n	1	2	3	4	5
c_n	1	2	5	14	42
b_n	1	2	5	15	52

Table III. The sets Alt(n) and V_n, $n \leqslant 5$

1) Alt$(n), n \leq 5$.

$n = 1$	(1) $\quad n = 2$ (12) $\quad n = 3$ (132), (231)
$n = 4$	(1324), (1423), (2314), (2431), (3412)
$n = 5$	(15243), (15342), (14253), (14352), (13254),
	(25143), (25341), (24153), (24351), (23154),
	(35142), (35241), (34152), (34251), (45132), (45231)

2) $V_n, n \leq 5$.

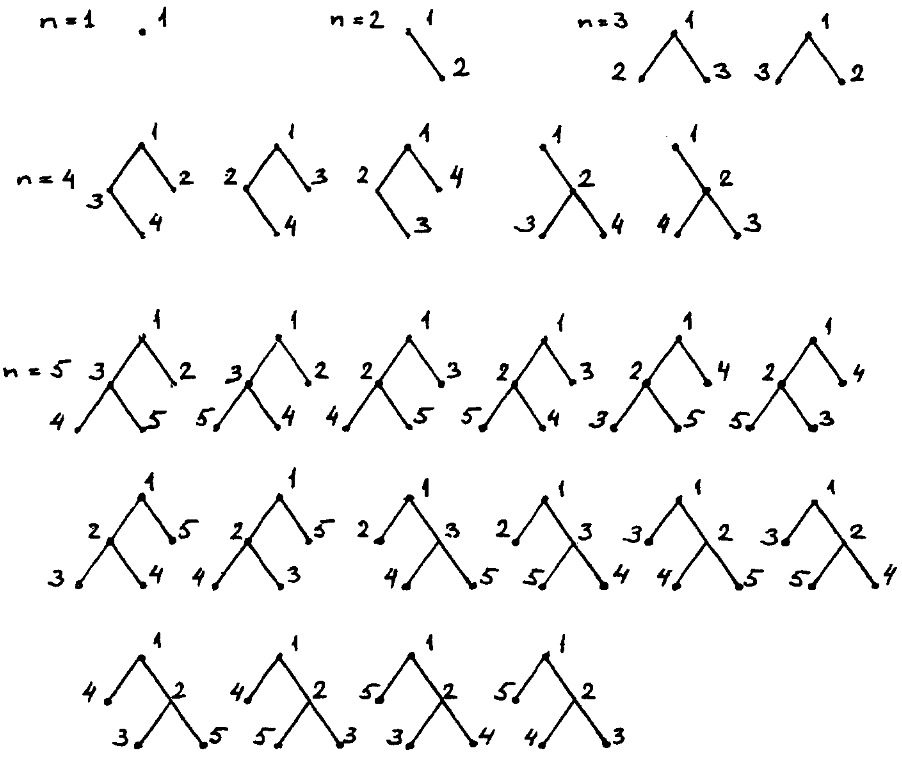
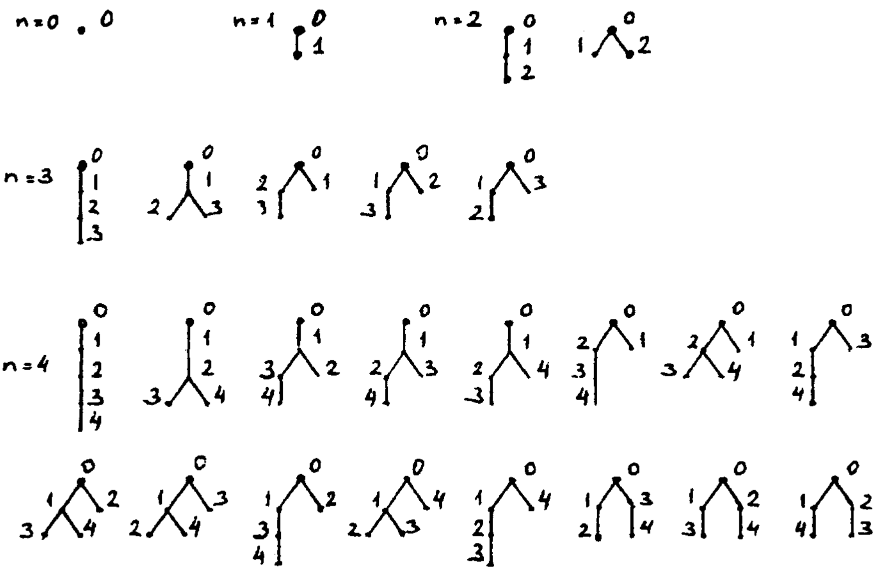


Table IV. The sets L_n, LB_n, $n \leqslant 4$

1) $L_n, n \leq 4$.



2) $LB_n, n \leq 4$.

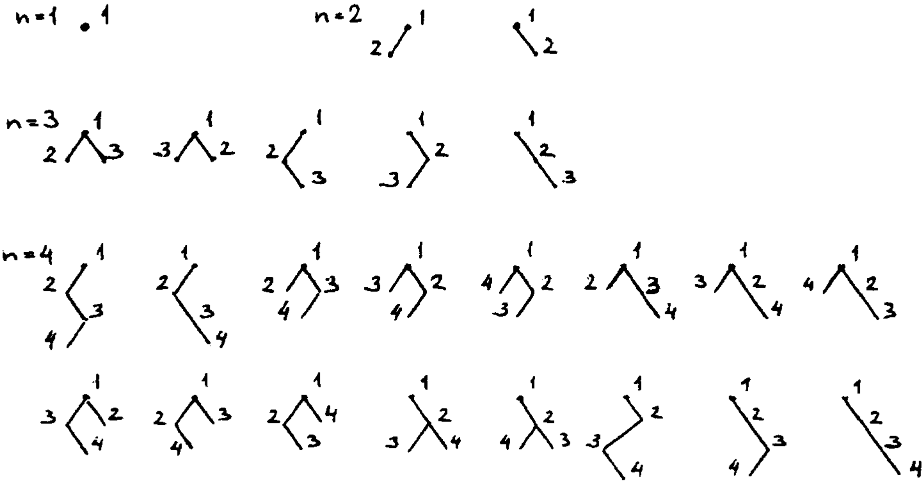


Table V. The sets E_n, $n \leqslant 5$

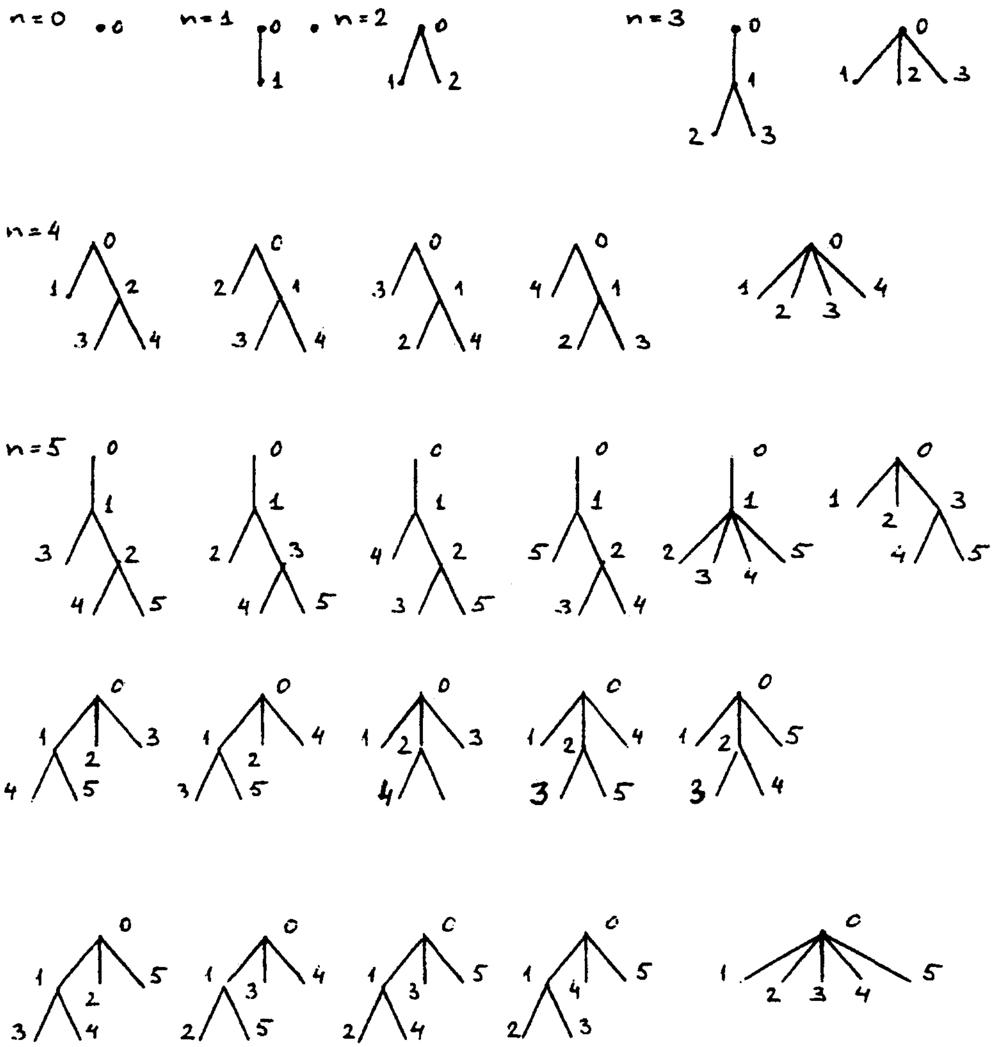
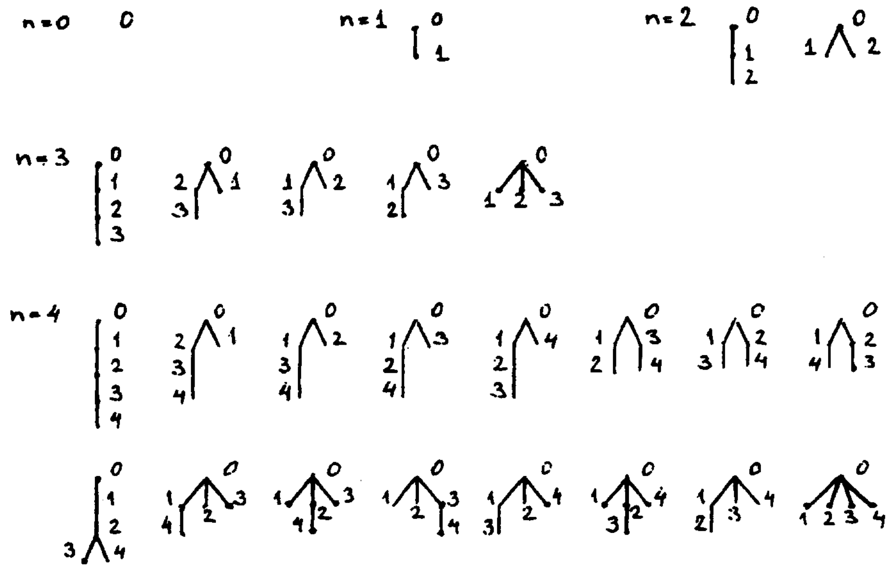


Table VI. The sets W_n and WB_n, $n \leqslant 4$

1) $W_n, n \leq 4$.



2) $WB_n, n \leq 4$.

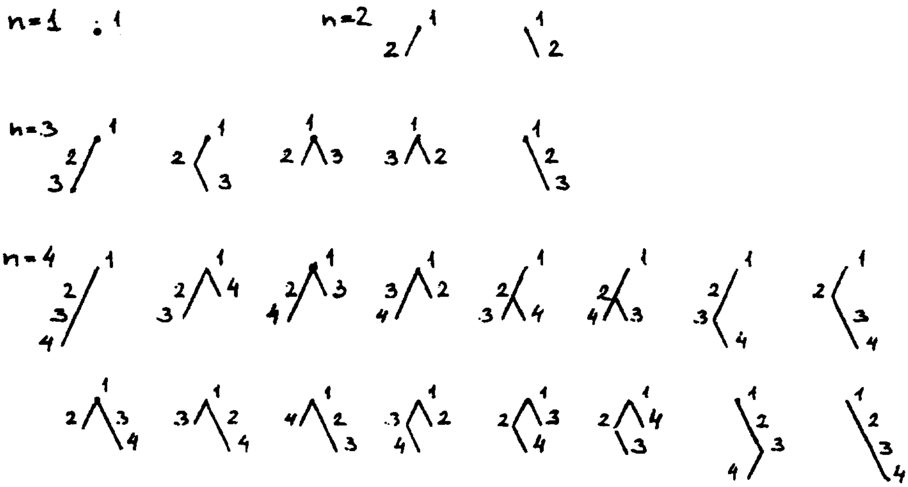
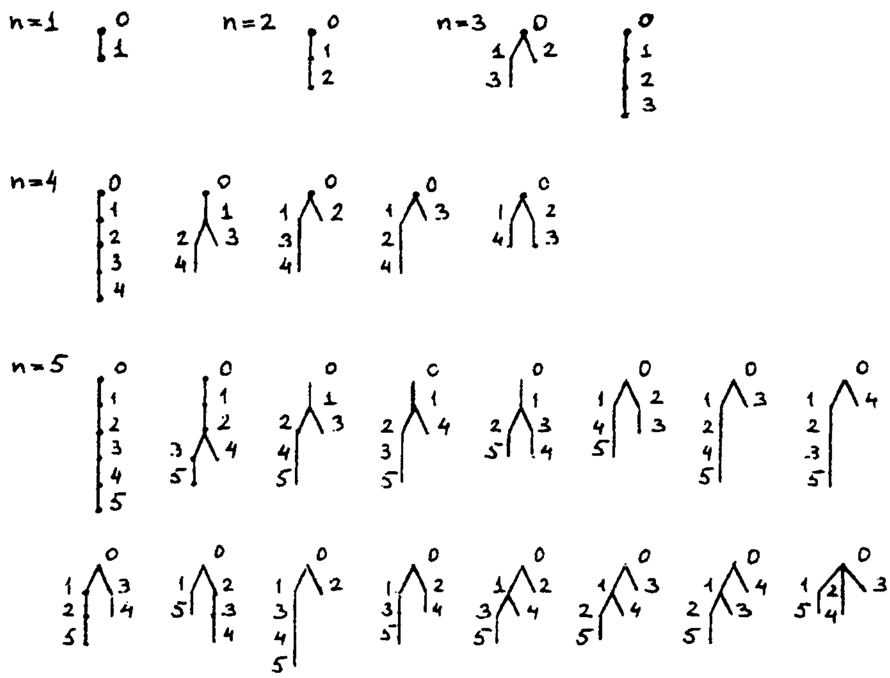
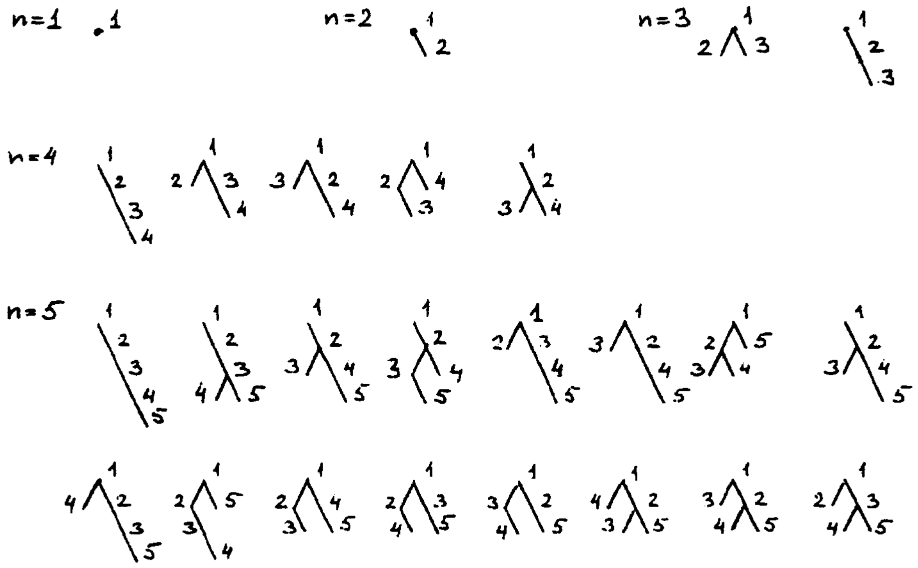


Table VII. The sets T_n and TB_n, Q_n

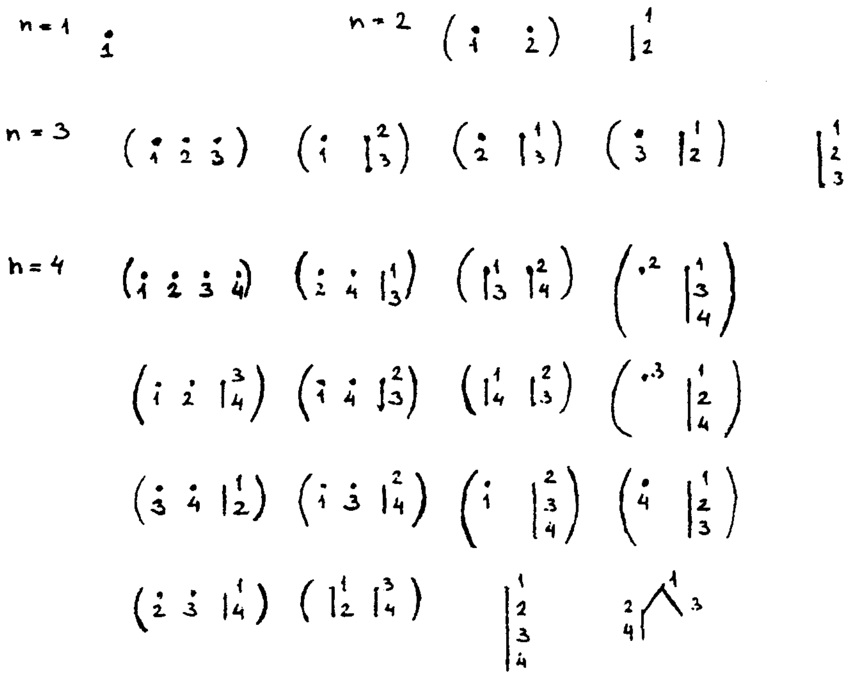
1) $T_n, n \leq 5$.



2) $TB_n, n \leq 5$.



3) $Q_n, n \leq 4$.



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